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A BARYCENTRIC PROJECTED-SUBGRADIENT ALGORITHM FOR EQUILIBRIUM PROBLEMS

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Abstract. This paper deals with an algorithm for approximating solutions of equilibrium problems in Hilbert spaces. We describe how to incorporate the diagonal subgradient and the projection methods, and then establish that the resulting algorithm is strongly convergent under mild conditions. To demonstrate the effectiveness and convergence of the algorithm, we provided numerical comparisons of the algorithm with four existing algorithms. The comparisons suggested that the algorithm is effective for solving equilibrium problems.

Keywords. Diagonal subgradient method; Equilibrium problem; Monotone bifunction; Pseudomonotone bifunction; Splitting algorithm.

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1. Introduction

Let H be a real Hilbert space with the inner product $\langle .,. \rangle$ and the induced norm $||.||$, and C be a nonempty closed convex subset of H. Our main purpose in this paper is to present a method for finding a solution of the following equilibrium problem (EP):

$$\text{Find } x^* \in C \text{ such that } f(x^*,y) \geq 0,\ \forall y \in C, \tag{EP}$$

where $f(x,y) := f_1(x,y) + f_2(x,y)$ and $f_i : C \times C \to \mathbb{R}$ $(i = 1,\ 2)$ are two given bifunctions. It is worth mentioning that problem (EP) is very general in the sense that it includes, as special cases, many mathematical models as minimization problems, variational inequality problems, saddle point problems, Nash equilibria in noncooperative games as well as certain fixed point problems, see for instance [4, 25]. In recent years, many methods have been proposed for solving problem (EP) as well as its special cases, see for example [1, 2, 9, 10, 11, 12, 13, 14, 15, 26, 27, 29, 30]. One of the popular methods for equilibrium problems is the proximal point method (PPM). This method was first introduced by Martinet [23] for minimization problems and further extended by Rockafellar [28] to null-point problem governed by maximal monotone operators. Moudafi [24] extended then the PPM to monotone equilibrium by proposing

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the following scheme: Given the current iterate $x_n \in C$, compute x_{n+1} by the following rule

$$x_{n+1} = J_r^f(x_n), \tag{PPM}$$

where J_r^f is the resolvent of f with $r > 0$, defined by

$$J_r^f(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle, \ \forall y \in C \right\}, \ x \in H.$$

Many mathematical works have been devoted to studying the convergence of algorithms based on this method for finding an approximate solution of (EP) with various types of conditions. At this stage, we would like to emphasize that for more general equilibrium problems involving for example pseudo-monotonicity, the strong convergence can not be obtained unless if the PPM is coupled with other techniques such as viscosity or hybrid concepts. To overcome this difficulty, another method based on the auxiliary problem principle was proposed and its convergence investigated in [9]. More recently, this method has been further extended and their convergence properties more analyzed in [27] under a pseudo-monotonicity assumption on the bifunctions with a Lipschitz-type condition. It should be noticed that the method in [27] is better than [9] where two optimization programs are required to be solved per each iteration. This method is also called the extragradient method (EGM) regarding its resemblance to the scheme introduced by Korpelevich [19] in the context of saddle point problems. More precisely, from any point $x_0 \in C$, the (EGM) generates two sequences $\{x_n\}$, $\{y_n\}$ by the following rule:

$$\begin{cases} y_n = \arg\min\{\rho f(x_n,y) + \frac{1}{2}||x_n - y||^2 : y \in C\}, \\ x_{n+1} = \arg\min\{\rho f(y_n,y) + \frac{1}{2}||x_n - y||^2 : y \in C\}, \end{cases} \tag{EGM}$$

where $\rho > 0$ is a suitable parameter. The sequences $\{x_n\}$ and $\{y_n\}$ were proved to be convergent to some solution of (EP) under mild assumptions in Euclidean spaces.

In this paper, we will focus our attention on splitting methods [2, 5, 22] for solving (EP) in Hilbert spaces for the bifunction $f := f_1 + f_2$. Since, in general, it is difficult to evaluate the resolvent operator of a given bifunction. One alternative is to decompose the given bifunction into the sum of two (or more) bifunctions whose resolvent are easier to evaluate than the resolvent of the original one. Following this approach, in 2009 Moudafi [22] introduced the following splitting proximal point method (SPPM), namely from any initial point x_0, it generates three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the following rules

$$\begin{cases} y_n = J_{r_n}^{f_1}(x_n), \\ z_n = J_{r_n}^{f_2}(x_n), \\ x_{n+1} = \frac{z_n + y_n}{2}. \end{cases} \tag{SPPM}$$

Under suitable assumptions and $\sum_{n=1}^{\infty} r_n = +\infty$, $\sum_{n=1}^{\infty} r_n^2 < +\infty$, the author proved that the sequence $\bar{x}_n = \frac{\sum_{i=1}^{n} r_i x_i}{\sum_{i=1}^{n} r_i}$ converges weakly to some solution of the equilibrium problem (EP). In 2012, following the same idea, and in order to have more numerical stabilities and applied abilities, Briceño-Arias [5] proposed the following Douglas-Rachford splitting method (DRSM),

$$\begin{cases} y_n = J_r^{f_2}(x_n) + b_n, \\ z_n = J_r^{f_1}(2y_n - x_n) + a_n, \\ x_{n+1} = x_n + \lambda_n(z_n - y_n), \end{cases} \tag{DRSM}$$

where $r > 0$, $\lambda_n \in (0,2)$, $\{a_n\} \subset H$, $\{b_n\} \subset H$ such that

$$b_n \rightharpoonup 0, \ \sum_{n=1}^{\infty} \lambda_n(2-\lambda_n) = +\infty, \ \sum_{n=1}^{\infty} \lambda_n(||a_n|| + ||b_n||) < +\infty.$$

The weak convergence of the sequence $\{x_n\}$ was also obtained. As this idea can lead to the development of very efficient methods, since one can treat each part of the original bifunction independently, very recently Anh and Hai [2] have considered the following parallel splitting method (PSM),

$$\begin{cases} y_n = \arg\min\{\lambda_n f_1(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\}, \\ z_n = \arg\min\{\lambda_n f_2(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\}, \\ x_{n+1} = \frac{y_n + z_n}{2}, \end{cases} \tag{PSM}$$

where f, f_i $(i = 1,2)$ and $\lambda_n > 0$ satisfy the following conditions.

(a) $f = f_1 + f_2$ is γ - strongly pseudomonotone;

(b) f_i is partially τ_i - Hölder continuous on C $(i = 1,2)$, i.e., there exist $L > 0$ and $\tau_i \in (0,1]$ such that, for all $x, y, z \in C$,

$$|f_i(x,y) - f_i(z,y)| \leq L||x-z||^{\tau_i} \text{ or } |f_i(x,y) - f_i(x,z)| \leq L||y-z||^{\tau_i};$$

(c) $\sum_{n=1}^{\infty} \lambda_n = +\infty$, $\sum_{n=1}^{\infty} (\lambda_n)^{\frac{2}{2-\tau}} = +\infty$, $\tau = \min\{\tau_1, \tau_2\}$.

The strong convergence of the sequence $\{x_n\}$ to the unique solution x^* of the equilibrium problem (EP) was then obtained. The advantages of (PSM) are that it can be applied for pseudomonotone bifunctions and be numerically solved more easily than (SPPM) and (DRSM). However, the problem of solving two optimization programs in (PSM) can be still costly if the structures of the bifunctions and the feasible sets have complex structures.

It then seems natural to investigate other approches. The purpose of this paper is to propose an alternative approach which leads to a novel splitting algorithm for solving the equilibrium problem (EP) where $f = f_1 + f_2$ in Hilbert spaces. The algorithm is builds around both the diagonal subgradient method and the projection scheme and the computations are performed on each component f_i of the original bifunction f independently. The use of the metric projection leads to simpler and faster numerical computations when the projection can be computed efficiently, i.e. when the closed convex set C is simple enough so that the projection can be easily implemented. This is the case, for example, when C is a closed ball, a closed cone or a half-space. Under some suitable assumptions, the strong convergence of the resulting algorithm is established. Numerical experiments are also provided. These permit to compare numerically the proposed algorithm with four existing algorithms and to show its effectiveness and fast convergence. Finally, we would like to mention that the main result obtained in this paper is still valid when the function f satisfies condition (a) without assuming the partially Hölder continuity (b) or the Lipschitz-type conditions used in [2, 26].

The organization of the rest of the paper is as follows. In Sect. 2, some definitions and preliminary results which will be needed throughout the paper are stated. Sect. 3 is devoted to designing the algorithm and analyzing its convergence properties. In Sect. 4, further remarks on the proposed algorithm and the assumptions are suggested. Finally, in Sect. 5 several numerical results ensuring that the Algorithm converges to a solution of problem (EP) faster than some existing algorithms are given.

2. Preliminaries

Let C be a nonempty closed convex subset of H. For each $x \in H$, there exists an unique element in C, denoted by $P_C(x)$, such that

$$||P_C(x) - x|| = \min\{||y - x|| : y \in C\}.$$

The mapping $P_C : H \to C$ is called the metric projection from H onto C. Its characteristic properties are summarized in the following lemma, for more details see for example [20].

Lemma 2.1. (i) $\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2,\ \forall x, y \in H.$

(ii) $\|x - P_C(y)\|^2 + \|P_C(y) - y\|^2 \leq \|x - y\|^2, \forall x \in C, y \in H.$

(iii) $z = P_C(x) \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$

Now, let us recall some monotonicity notions of a bifunction, see [3, 4].

Definition 2.1. A bifunction $f : C \times C \to \mathbb{R}$ is said to be:

(i) strongly monotone on C, if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma||x - y||^2,\ \forall x, y \in C;$$

(ii) monotone on C, if $f(x, y) + f(y, x) \leq 0,\ \forall x, y \in C$;

(iii) strongly pseudomonotone on C, if there exists a constant $\gamma > 0$ such that

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq -\gamma||x - y||^2,\ \forall x, y \in C;$$

(iv) pseudomonotone on C, if $f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0,\ \forall x, y \in C$.

It is clear that the following implications hold true:

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iv) and (i)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(iv)}.$$

The converses are not true in general.
Remember also that a function $\varphi : C \to \mathbb{R}$ is said to be convex on C, if for all $x, y \in C$, $t \in [0, 1]$ one has $\varphi(tx + (1 - t)y) \leq t\varphi(x) + (1 - t)\varphi(y)$ and that its subdifferential at $x \in C$ is defined by $\partial\varphi(x) = \{w \in H : \varphi(y) - \varphi(x) \geq \langle w, y - x\rangle,\ \forall y \in C\}$. An enlargement of the subdifferential is the so-called ε-subdifferential defined at $x \in C$ and for $\varepsilon \geq 0$ by

$$\partial_\varepsilon\varphi(x) = \{w \in H : \varphi(y) - \varphi(x) + \varepsilon \geq \langle w, y - x\rangle,\ \forall y \in C\}.$$

It is clear that when $\varepsilon = 0$, ε-subdifferential reduces to the subdifferential.
Now, let $f : C \times C \to \mathbb{R}$ be a bifunction, throughout this paper, $\partial_\varepsilon f(x, .)(x)$ will be called the ε-diagonal

subdifferential of f at $x \in C$.

The following technical lemma will be needed in the sequel.

Lemma 2.2. [31] *Let $\{\nu_n\}$ and $\{\delta_n\}$ be two sequences of positive real numbers such that $\nu_{n+1} \le \nu_n + \delta_n$ for all $n \ge 1$ with $\sum_{n\ge1} \delta_n < +\infty$. Then, the sequence $\{\nu_n\}$ is convergent.*

3. A Barycentric Projected-subgradient Algorithm

In this section, we introduce a splitting algorithm for solving the equilibrium problem (EP) when $f = f_1 + f_2$. Given a current iterate x_n, pick an ε_n-diagonal subdifferential w_n^i of f_i, $i = 1, 2$, at x_n and select a suitable stepsize α_n. Then, compute x_n^i for $i = 1, 2$ as the projections of $x_n - \alpha_n w_n^i$ on C. Finally, the next iterate x_{n+1} is obtained as the average of x_n^i, $i = 1, 2$. More precisely, let $\{\rho_n\}$, $\{\beta_n\}$, $\{\varepsilon_n\}$ be three parameter sequences satisfying the following conditions:

A1. $\rho_n \ge \rho > 0$, $\beta_n > 0$, $\varepsilon_n \ge 0$,

A2. $\sum\limits_{n=1}^{\infty} \frac{\beta_n}{\rho_n} = +\infty, \qquad \sum\limits_{n=1}^{\infty} \left(\frac{\beta_n \varepsilon_n}{\rho_n} + \beta_n^2\right) < +\infty,$

we propose the algorithm below:

Algorithm 3.1 (Splitting algorithm for EPs)**.** .

Initialization: Take parameter sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\varepsilon_n\}$ and choose $x_0 \in C$ arbitrarily.

Iterative Steps:

Step 1. Given $x_n \in C$, for each $i = 1,\ 2$ select $w_n^i \in \partial_{\varepsilon_n} f_i(x_n, .)(x_n)$ and compute x_n^i and α_n by

$$x_n^i = P_C(x_n - \alpha_n w_n^i) \quad \text{and} \quad \alpha_n = \frac{\beta_n}{\max\{\rho_n, ||w_n^1||, ||w_n^2||\}}.$$

Step 2. Compute x_{n+1} as

$$x_{n+1} = \frac{x_n^1 + x_n^2}{2}.$$

The main task of Algorithm 3.1 is to compute the two projection x_n^i, $i = 1, 2$. In comparison with (SPPM), (DRSM) and (PSM), these computations can be simpler, specially when C has a closed form expression. This is the case when C is, for example, a closed ball, a closed cone, a half-space or a polyhedral convex set, while subproblems in the other splitting methods are in general still costly. Furthermore, where C is a level set, which is the case in many real-world optimization problems, we can make use of the subgradient projection operator which has been playing important roles as a low complexity approximation of the metric projection onto level sets in many scenarios or the idea introduced in [32] where we just need projections onto half-spaces, thus making the algorithm again implementable.

In order to establish the convergence of Algorithm 3.1, we assume in what follows that f and f_i $(i=1,2)$ verify the following hypotheses:

B1. f is pseudomonotone and $f(x,x)=0,\ \forall x\in C$;

B2. $f_i(x,.)$ is convex and lower semicontinuous on C for each $i=1,2$ and $f(.,y)$ is weakly upper semicontinuous C;

B3. The ε-diagonal subdifferential of f_i is bounded on each bounded subset of C for $i=1,2$;

B4. f satisfies the following paramonotone condition

$$x\in EP(f,C),\ y\in C,\ f(y,x)=0\Longrightarrow y\in EP(f,C). \tag{PC}$$

Observe that under the assumptions B1 and B2, the solution set $EP(f,C)$ of the equilibrium problem (EP) is closed and convex and throughout the paper we will assume that $EP(f,C)$ is nonempty. Several remarks on the above assumptions can be found in [29]. Moreover, it is worth mentioning that assumption B3 has been considered in the optimization context in [16, 21] and in the equilibrium case in [17, 29, 33]. Furthermore, in Euclidean spaces setting this condition holds true if f_i is continuous on $\Omega\times\Omega$, where Ω is an open set containing C, see [17, Proposition 4.3] and in infinite dimensional Hilbert spaces if continuity is replaced by the jointly weakly continuity [30, Proposition 4.1]. This assumption ensures that $\left\{w_n^i\right\},\ i=1,2$ occurring in Step 1 of Algorithm 3.1 are bounded sequences, provided that $\{x_n\}$ is bounded. Note that in [16, 29] the sequences $\left\{w_n^i\right\},\ i=1,2$ are assumed directly to be bounded and so we can do the same. Finally, condition (PC) in B4 can be considered as an extension, from variational inequalities to equilibrium problems, of the cut property given in [8]. In section 4 we provide an example showing that without condition (PC) Algorithm 3.1 can not converge.

Based on [21, 29] we investigate now the convergence of Algorithm 3.1.

Theorem 3.1. *Assume that Problem (EP) admits a solution and Conditions A, B are satisfied. Then, the whole sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to some solution $x^\dagger$ of Problem (EP). Moreover, $x^\dagger=\lim_{n\to\infty}P_{EP(f,C)}(x_n)$.*

Proof. We divide the proof of Theorem 3.1 into four steps.

Claim 1. *For each $n\ge 0$ and $x^*\in EP(f,C)$ we have the following estimate,*

$$||x_{n+1}-x^*||^2\le ||x_n-x^*||^2+\alpha_n f(x_n,x^*)+\delta_n,$$

where $\delta_n=2\frac{\beta_n\varepsilon_n}{\rho_n}+2\beta_n^2$.

The proof of Claim 1. We can write

$$\begin{aligned}||x_n-x^*||^2 &= ||(x^*-x_n^i)-(x_n-x_n^i)||^2\\ &= ||x^*-x_n^i||^2+||x_n-x_n^i||^2-2\left\langle x^*-x_n^i,x_n-x_n^i\right\rangle\\ &\ge ||x_n^i-x^*||^2-2\left\langle x^*-x_n^i,x_n-x_n^i\right\rangle.\end{aligned}$$

Thus

$$\frac{||x_n^i - x^*||^2}{2} \leq \frac{||x_n - x^*||^2}{2} + \left\langle x^* - x_n^i, x_n - x_n^i \right\rangle, \; i = 1, \; 2,$$

which combined with the definition of x_{n+1} and the convexity of the square of the norm yield

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||\frac{x_n^1 + x_n^2}{2} - x^*)||^2 = ||\frac{x_n^1 - x^*}{2} + \frac{x_n^2 - x^*}{2}||^2 \\ &\leq \frac{||x_n^1 - x^*||^2}{2} + \frac{||x_n^2 - x^*||^2}{2} \\ &\leq ||x_n - x^*||^2 + \left\langle x^* - x_n^1, x_n - x_n^1 \right\rangle + \left\langle x^* - x_n^2, x_n - x_n^2 \right\rangle. \end{aligned} \tag{3.1}$$

Since $x_n^i = P_C(x_n - \alpha_n w_n^i)$ and according to property (iii)-Lemma 2.1, we obtain

$$\left\langle z - x_n^i, x_n - \alpha_n w_n^i - x_n^i \right\rangle \leq 0, \; \forall z \in C. \tag{3.2}$$

Setting $z = x^* \in C$ in relation (3.2) and using the Cauchy-Schwarz inequality, we successively get

$$\begin{aligned} \left\langle x^* - x_n^i, x_n - x_n^i \right\rangle &\leq \alpha_n \left\langle w_n^i, x^* - x_n^i \right\rangle \\ &= \alpha_n \left\langle w_n^i, x^* - x_n \right\rangle + \alpha_n \left\langle w_n^i, x_n - x_n^i \right\rangle \\ &\leq \alpha_n \left\langle w_n^i, x^* - x_n \right\rangle + \alpha_n ||w_n^i|| ||x_n - x_n^i||. \end{aligned} \tag{3.3}$$

Inequality (3.2) with $z = x_n \in C$ leads to

$$\left\langle x_n - x_n^i, x_n - \alpha_n w_n^i - x_n^i \right\rangle \leq 0.$$

Clearly $||x_n - x_n^i||^2 \leq \alpha_n \left\langle x_n - x_n^i, w_n^i \right\rangle \leq \alpha_n ||w_n^i|| ||x_n - x_n^i||$ and implies $||x_n - x_n^i|| \leq \alpha_n ||w_n^i||$, which combined to the definition of α_n yields

$$\alpha_n ||w_n^i|| ||x_n - x_n^i|| \leq (\alpha_n ||w_n^i||)^2 = \beta_n^2 \left(\frac{||w_n^i||}{\max\{\rho_n, ||w_n^1||, ||w_n^2||\}} \right)^2 \leq \beta_n^2. \tag{3.4}$$

Now, since $w_n^i \in \partial_{\varepsilon_n} f_i(x_n, .)(x_n)$, using the definition of ε_n-diagonal subdifferential, we can write

$$f_i(x_n, z) - f_i(x_n, x_n) + \varepsilon_n \geq \left\langle w_n^i, z - x_n \right\rangle, \text{ for all } z \in C.$$

By setting $z = x^* \in C$ we obtain

$$\left\langle w_n^i, x^* - x_n \right\rangle \leq f_i(x_n, x^*) - f_i(x_n, x_n) + \varepsilon_n. \tag{3.5}$$

Combining relations (3.3)-(3.5), we get

$$\left\langle x^* - x_n^i, x_n - x_n^i \right\rangle \leq \alpha_n \{f_i(x_n, x^*) - f_i(x_n, x_n)\} + \alpha_n \varepsilon_n + \beta_n^2, \; i = 1, 2,$$

which together with inequality (3.1), $f = f_1 + f_2$ and the fact that $f(x_n, x_n) = 0$ imply

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 + \alpha_n \{f_1(x_n, x^*) - f_1(x_n, x_n)\} + \alpha_n \varepsilon_n + \beta_n^2 \\ &\quad + \alpha_n \{f_2(x_n, x^*) - f_2(x_n, x_n)\} + \alpha_n \varepsilon_n + \beta_n^2 \\ &= ||x_n - x^*||^2 + \alpha_n \{f(x_n, x^*) - f(x_n, x_n)\} + 2\alpha_n \varepsilon_n + 2\beta_n^2 \\ &= ||x_n - x^*||^2 + \alpha_n f(x_n, x^*) + 2\alpha_n \varepsilon_n + 2\beta_n^2. \end{aligned} \tag{3.6}$$

From the definiton of α_n, we clearly have

$$\alpha_n = \frac{\beta_n}{\max\{\rho_n, ||w_n^1||, ||w_n^2||\}} \leq \frac{\beta_n}{\rho_n}. \tag{3.7}$$

The latter combined with (3.6) and the definition of δ_n lead to the desired inequality.

Claim 2. *For any $x^* \in EP(f,C)$, the sequence $\left\{||x_n - x^*||^2\right\}$ is convergent and thus the sequence $\{x_n\}$ is bounded.*

The proof of Claim 2. Since $x^* \in EP(f,C)$, we have $f(x^*, x_n) \geq 0$ and by pseudomonotonicity of f we can then write $f(x_n, x^*) \leq 0$. This combined with Claim 1 and the fact that $\alpha_n > 0$ leads to the following inequalilty,

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + \delta_n. \tag{3.8}$$

Hypothesis A2 and the definition of δ_n ensure that $\sum_{n\geq 1} \delta_n < +\infty$. In view of (3.8) and by applying Lemma 2.2, we deduce that the sequence $\left\{||x_n - x^*||^2\right\}$ is convergent which in turn implies the boundedness of the sequence $\{x_n\}$.

Claim 3. *For any $x^* \in EP(f,C)$, we have $\limsup_{n\to\infty} f(x_n, x^*) = 0$.*

The proof of Claim 3. Since $f(x_n, x^*) \leq 0$ and $\alpha_n > 0$, from Claim 1 we can write

$$0 \leq -\alpha_n f(x_n, x^*) \quad \leq \quad ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \delta_n.$$

Summing the last inequality from $n = 1$ to ∞, we obtain

$$0 \leq \sum_{n\geq 1} \alpha_n \left[-f(x_n, x^*)\right] \leq ||x_1 - x^*||^2 + \sum_{n\geq 1} \delta_n < +\infty. \tag{3.9}$$

On the other hand, since $\{x_n\}$ is bounded so are the sequences $\{w_i\},\ i = 1,2$ by hypothesis B3. Consequently, there exists a number $L \geq \rho$ such that $||w_n^i|| \leq L$ for all $n \geq 0$ and $i = 1,2$. This together with $\rho_n \geq \rho > 0$ implies that

$$\max\left\{\rho_n, ||w_n^1||, ||w_n^2||\right\} = \rho_n \max\left\{1, \frac{||w_n^1||}{\rho_n}, \frac{||w_n^2||}{\rho_n}\right\} \leq \frac{\rho_n L}{\rho}.$$

From the definition of α_n, we also have

$$\alpha_n = \frac{\beta_n}{\max\{\rho_n, ||w_n^1||, ||w_n^2||\}} \geq \frac{\rho}{L}\frac{\beta_n}{\rho_n},$$

which combined with (3.9) leads to

$$0 \leq \frac{\rho}{L} \sum_{n\geq 1} \frac{\beta_n}{\rho_n} \left[-f(x_n, x^*)\right] \leq \sum_{n\geq 1} \alpha_n \left[-f(x_n, x^*)\right] < +\infty.$$

Thanks to assumption A2 and the fact that $\rho/L > 0$, we infer $\liminf_{n\to\infty} [-f(x_n, x^*)] = 0$, from which follows the desired result.

Claim 4. $\lim_{n\to\infty} x_n = x^\dagger \in EP(f,C)$.

The proof of Claim 4. Since $\{x_n\}$ is bounded, there exists a subsequence, says $\{x_m\}$, of $\{x_n\}$ that weakly converges to some $x^\dagger \in C$ such that

$$\lim_{n\to\infty} \sup f(x_n, x^*) = \lim_{m\to\infty} f(x_m, x^*) \text{ for any } x^* \in EP(f,C). \tag{3.10}$$

Using weakly upper semicontinuity of $f(.,x^*)$ and Claim 3, we can write

$$f(x^\dagger, x^*) \geq \lim_{m\to\infty} \sup f(x_m, x^*) = \lim_{m\to\infty} f(x_m, x^*) = \lim_{n\to\infty} \sup f(x_n, x^*) = 0. \tag{3.11}$$

We also have $f(x^*, x^\dagger) \geq 0$, because $x^* \in EP(f,C)$ which implies, by the pseudomonotonicity of f, that $f(x^\dagger, x^*) \leq 0$ and by (3.11) we hence infer that $f(x^\dagger, x^*) = 0$. Now, by virtue of hypothesis B4, we then obtain $x^\dagger \in EP(f,C)$. Consequently, we obtain thanks to Claim 2 the convergence of the sequence $\left\{||x_n - x^\dagger||^2\right\}$. Observe that $x^\dagger$ is a weak cluster point of the sequence $\{x_n\}$ and hence the whole sequence $\{x_n\}$ converges strongly to $x^\dagger \in EP(f,C)$.

To complete the proof of Theorem 3.1, it remains to show that $x^\dagger = \lim_{n\to\infty} P_{EP(f,C)}(x_n)$. Remember that from relation (3.8) we have, for any $x^* \in EP(f,C)$, that

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + \delta_n. \tag{3.12}$$

By substituting $x^* = P_{EP(f,C)}(x_n) \in EP(f,C)$ in the inequality (3.12), we get

$$||x_{n+1} - P_{EP(f,C)}(x_n)||^2 \leq ||x_n - P_{EP(f,C)}(x_n)||^2 + \delta_n. \tag{3.13}$$

Now, the definition of the metric projection gives

$$||x_{n+1} - P_{EP(f,C)}(x_{n+1})||^2 \leq ||x_{n+1} - z||^2, \ \forall z \in EP(f,C),$$

which with $z = P_{EP(f,C)}(x_n) \in EP(f,C)$ implies that

$$||x_{n+1} - P_{EP(f,C)}(x_{n+1})||^2 \leq ||x_{n+1} - P_{EP(f,C)}(x_n)||^2. \tag{3.14}$$

Combining relations (3.13) and (3.14), we obtain

$$||x_{n+1} - P_{EP(f,C)}(x_{n+1})||^2 \leq ||x_n - P_{EP(f,C)}(x_n)||^2 + \delta_n,$$

or in other words $a_{n+1} \leq a_n + \delta_n$ by setting $a_n = ||x_n - P_{EP(f,C)}(x_n)||^2$.
Clearly, the sequence $\{a_n\}$ is convergence by taking into account the fact that $\sum_{n=1}^{\infty} \delta_n < \infty$ and by a direct application of Lemma 2.2. Now, let $u_n = P_{EP(f,C)}(x_n)$ and having in hand (ii)-Lemma 2.1 and relation

(3.12), we can successively write for all $m > n$

$$\begin{array}{rcl} ||u_n - u_m||^2 & = & ||P_{EP(f,C)}(x_n) - P_F(x_m)||^2 \\ & \leq & ||x_m - P_{EP(f,C)}(x_n)||^2 - ||x_m - P_{EP(f,C)}(x_m)||^2 \\ & \leq & \left(||x_{m-1} - P_{EP(f,C)}(x_n)||^2 + \delta_{m-1}\right) - ||x_m - P_{EP(f,C)}(x_m)||^2 \\ & \leq & \left(||x_{m-2} - P_{EP(f,C)}(x_n)||^2 + \delta_{m-2} + \delta_{m-1}\right) - ||x_m - P_{EP(f,C)}(x_m)||^2 \\ & \leq & \ldots \\ & \leq & \left(||x_n - P_{EP(f,C)}(x_n)||^2 + \sum_{l=n}^{m-1} \delta_l\right) - ||x_m - P_{EP(f,C)}(x_m)||^2 \\ & = & a_n - a_m + \sum_{l=n}^{m-1} \delta_l. \end{array}$$

By passing to the limit in the last inequality as $m,n \to \infty$ and having in mind the fact that $\sum_{n=1}^{\infty} \delta_n < \infty$, we infer that $\lim_{m,n\to\infty} ||u_n - u_m||^2 = 0$. Hence, $\{u_n\}$ is a Cauchy sequence and consequently converges to some $u \in EP(f,C)$, namely

$$\lim_{n\to\infty} u_n = u \in EP(f,C).$$

Since $u_n = P_{EP(f,C)}(x_n)$, using property (iii)-Lemma 2.1 and the fact that $x^\dagger \in C$, we get

$$\langle x^\dagger - u_n, x_n - u_n \rangle \leq 0.$$

Finaly, by passing to the limit in the last inequality as $n \to \infty$, we obtain

$$||x^\dagger - u||^2 = \langle x^\dagger - u, x^\dagger - u \rangle \leq 0.$$

Thus $x^\dagger = u = \lim_{n\to\infty} P_{EP(f,C)}(x_n)$, which completes the proof of Theorem 3.1. □

4. Further remarks

Remark 4.1. Algorithm 3.1 can be extended to the case where f is of the following form $f = \sum_{i=1}^N f_i$. The resulting barycentric parallel algorithm is given by

$$\begin{cases} x_0 \in C, \\ w_n^i \in \partial_{\varepsilon_n} f_i(x_n,.)(x_n), \; i = 1, \ldots, N, \\ \alpha_n = \frac{\beta_n}{\max\{\rho_n, ||w_n^1||, \ldots, ||w_n^N||\}}, \; x_n^i = P_C(x_n - \alpha_n w_n^i), \; i = 1, \ldots, N, \\ x_{n+1} = \frac{1}{N} \sum_{i=1}^N x_n^i. \end{cases} \tag{4.1}$$

Under the same assumptions as in Theorem 3.1, the sequence $\{x_n\}$ generated by (4.1) strongly converges to a solution $x^\dagger$ of problem (EP) and $x^\dagger = \lim_{n\to\infty} P_{EP(f,C)}(x_n)$.

Remark 4.2. The main convergence result of this paper still holds true if we replace assumptions B1 and B4 by the strong pseudomonotonicity of f. The latter was used in establishing the convergence of the splitting methods introduced in [2]. This assumption is weaker than the strong monotonicity considered in [26] and our convergence result is valid without assuming the partially Hölder continuity and the Lipschitz-type hypotheses used in [2].

Remark 4.3. Now, we provide an example showing that the convergence of Algorithm 3.1 can not be achieved without condition (PC) in B4. To that end, let us consider $H = C = \mathbb{R}^2$, $f(x,y) = f_1(x,y) = x_1y_2 - x_2y_1$ for all $x,\ y \in \mathbb{R}^2$ and $f_2 = 0$. The unique solution of problem (EP) here is $x^* = (0,0)^T$. The bifunction f is monotone and thus pseudomonotone, because $f(x,y) + f(y,x) = 0$ for all $x,\ y \in \mathbb{R}^2$. It is easily seen that assumptions B1-B3 are satisfied. However, $f(y,x^*) = 0$ for all $y \in C = \mathbb{R}^2$ and this implies that f do not satisfy the assumption (PC) of B4! Now taking $\varepsilon_n = 0$ and given $x_n = (x_{1n}, x_{2n})^T \in \mathbb{R}^2$, Algorithm 3.1 reduces to

$$\begin{cases} w_n \in \partial_{\varepsilon_n} f(x_n,.)(x_n) = \partial f(x_n,.)(x_n) = \left\{(-x_{2n}, x_{1n})^T\right\}. \\ x_{n+1} = x_n - \alpha_n w_n = (x_{1n} + \alpha_n x_{2n}, x_{2n} - \alpha_n x_{1n})^T. \end{cases} \tag{4.2}$$

In view of (4.2), we can write

$$||x_{n+1}||^2 = (x_{1n} + \alpha_n x_{2n})^2 + (x_{2n} - \alpha_n x_{1n})^2 = (1+\alpha_n^2)\left[(x_{1n})^2 + (x_{2n})^2\right] = (1+\alpha_n^2)||x_n||^2.$$

By induction, we then get

$$||x_{n+1}||^2 = \left(\prod_{k=0}^{n}(1+\alpha_k^2)\right)||x_0||^2 \geq ||x_0||^2,$$

from which follows that $\lim_{n\to\infty} ||x_{n+1}||^2 \neq 0$, provided that $x_0 \neq 0$. Thus, $\{x_n\}$ does not converge to the solution x^* of the considered problem.

Remark 4.4. Algorithm 3.1 proposed in this paper is a barycentric parallel algorithm. An open question, is then to design a sequential splitting algorithm like in [2, Algorithm 1] and in [5, Theorem 3.2] in the context of equilibrium problems and also in [22, Sect. 3] in optimization problem settings.

5. Numerical experiments

In this section, we consider a Nash-Cournot oligopolistic equilibrium model in electricity markets [6] with the affine price function and the nonsmooth fee and tax function. Suppose that there are m companies that produce a commodity. Let x denote the vector whose entry x_j stands for the quantity of the commodity producing by company j and C_j be the strategy set of company j. Then the strategy set of the model is $C := C_1 \times C_2 \times \ldots \times C_m$. Assume that the price function $p_j(s)$ is a decreasing affine function of s with $s = \sum_{j=1}^{m} x_j$, i.e.,

$$p_j(s) = \alpha_j - \beta_j s,$$

where $\alpha_j > 0$, $\beta_j > 0$. Moreover, also assume that the fee and tax function $h_j(x_j)$ for generating x_j is given by $h_j(x_j) = \max\left\{\bar{h}_j(x_j), \widehat{h}_j(x_j)\right\}$ where $\bar{h}_j(x_j), \widehat{h}_j(x_j)$ are given functions. Then the profit made by company j is given by

$$f_j(x) = p_j(s)x_j - h_j(x_j),\ j = 1,\ldots,m.$$

In fact, each company seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies is a parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept. We recall that a point $x^* \in C = C_1 \times C_2 \times \cdots \times C_m$ is an equilibrium point of the model if

$$f_j(x^*) \geq f_j(x^*[x_j]) \ \forall x_j \in C_j, \ \forall j = 1, 2, \ldots, m,$$

where the vector $x^*[x_j]$ stands for the vector obtained from x^* by replacing x_j^* with x_j. Define the bifunction f by

$$f(x,y) := \psi(x,y) - \psi(x,x)$$

with $\psi(x,y) := -\sum_{j=1}^m f_j(x[y_j]) + \sum_{j=1}^m h_j(y_j)$. The problem of finding a Nash equilibrium point of the model can be formulated as:

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0 \ \forall y \in C.$$

In that case, the bifunction $f(x,y)$ can be formulated in the following form

$$f(x,y) = \langle Px + Qy + q, y - x \rangle + h(y) - h(x),$$

where $h(x) = \sum_{j=1}^m h_j(x_j)$, q is a vector in $\mathbb{R}^m$ and P, Q are two matrices of order m. This model is equivalent to the equilibrium problem (EP) considered in this paper, by setting

$$f_1(x,y) = \langle Px + Qy + q, y - x \rangle \text{ and } f_2(x,y) = h(y) - h(x).$$

In what follows, we will study the convergence of Algorithm 3.1 and give a comparison with four algorithms, namely PSM considered in [2, Algorithm 3], SSM introduced in [2, Algorithm 1], EGM proposed in [27, Algorithm 1] for $G(x) = \frac{1}{2}||x||^2$ and GM presented in [26, Theorem 2.1]. In EGM and GM, we need to use the Lipschitz-type constant of f as $L_2 = ||Q - P||/2$. All the programs are implemented on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50GHz 2.50 GHz, RAM 2.00 GB.

For experiments, we consider the problem in $\mathbb{R}^{10}$ $(m = 10)$ and the feasible set C being a box and a ball containing the original point. The functions $\bar{h}_j(x_j)$, $\widehat{h}_j(x_j)$ have the following forms,

$$\bar{h}_j(x_j) = \bar{a}_j x_j^2 + \bar{b}_j x_j + \bar{c}_j,$$
$$\widehat{h}_j(x_j) = \widehat{a}_j x_j^2 + \widehat{b}_j x_j + \widehat{c}_j,$$

where $\bar{a}_j$, $\bar{b}_j$, $\bar{c}_j$, $\widehat{a}_j$, $\widehat{b}_j$, $\widehat{c}_j$ are real numbers such that $\bar{a}_j > 0$, $\widehat{a}_j > 0$ for all $j = 1, \ldots, m$. The function $h(x)$ is nonsmooth, convex and subdifferentiable. The subdifferential of h at x is given by $\partial h(x) = (\partial h_1(x_1), \ldots, \partial h_m(x_m))^T$ where, for each $j = 1, \ldots, m$,

$$\partial h_j(x_j) = \begin{cases} 2\bar{a}_j x_j + \bar{b}_j & \text{if} \quad \bar{h}_j(x_j) > \widehat{h}_j(x_j), \\ \left[2\bar{a}_j x_j + \bar{b}_j, 2\widehat{a}_j x_j + \widehat{b}_j\right] & \text{if} \quad \bar{h}_j(x_j) = \widehat{h}_j(x_j), \\ 2\widehat{a}_j x_j + \widehat{b}_j, & \text{if} \quad \bar{h}_j(x_j) < \widehat{h}_j(x_j). \end{cases}$$

We also generate two datas as follows:

Data 1: We choose $q = 0$, $\bar{b}_j = \bar{c}_j = \widehat{b}_j = \widehat{c}_j = 0$ and $\bar{a}_j$, $\widehat{a}_j$ are generated randomly and uniformly in $[1, m]$ for all i, j. Two matrices P, Q are also generated randomly such that Q is symmetric positive semidefine and $Q-P$ is symmetric negative definite. The unique solution of the problem is $x^*=0\in \mathbb{R}^m$. It is easy to see that all the assumptions B1-B4 are satisfied. In this case, we use the sequence $D_n=||x_n-x^*||^2,\ n=0,1,2,\ldots$ to study the convergence of all the aforementioned algorithms. The convergence of D_n to 0 implies that $\{x_n\}$ converges to the solution of the problem.

Data 2: Two matrices P, Q and $\bar{a}_j$, $\widehat{a}_j$ are generated as in Data 1, and $\bar{b}_j,\bar{c}_j,\widehat{b}_j,\widehat{c}_j$, are in $[-m,m]$ for all i,j, the vector q is also generated randomly with its entries in $[-m,m]$. Since the solution of the problem is unknown, we use the sequence $F_n=||x_{n+1}-x_n||^2,\ n=0,1,2\ldots$ to study the convergence of all the algorithms.

The starting point is $x_0=(1,1,\ldots,1)^T\in\mathbb{R}^m$, $\varepsilon_n=0$ (exact subdifferential) and $\rho_n=1$ in all the experiments. Two following parameter groups are used in the first experiment:

(P1): $\beta_n=\lambda_n=\frac{1}{n+1}$ for Algorithm 3.1, PSM, SSM; $\rho=0.99/2L_2$ for EGM, GM.

(P2): $\beta_n=\lambda_n=\frac{1}{(n+1)^{0.51}}$ for Algorithm 3.1, PSM, SSM; $\rho=0.5/2L_2$ for EGM, GM.

Experiment 5.1. Consider the feasible set C of the following form,

$$C=\left\{x\in\mathbb{R}^m: -2\le x_j\le 5, j=1,\ldots,m\right\},$$

where optimization subproblems in PSM, SSM, EGM, EG can be effectively solved by the function *quadprog* in Matlab 7.0 Optimization Toolbox. The projections in Algorithm 3.1 are rewritten equivalently to optimization problems. Figs. 1 - 8 describe the numerical results for all the given datas and parameters. In these figures, the y-axes represent for value of D_n or F_n while the x-axes represent for number of iterations (# iterations (n)) or for elapsed execution time (in second - CPU time). In view of these figures, specially Figs. 1-4, we see that Algorithm 3.1 is faster in all cases. In the early iterations, D_n and F_n with Algorithm 3.1 are more slowly decreasing than other algorithms, but after that, they quickly decrease to zero. The error obtained for Algorithm 3.1 is better than those obtained for the other algorithms.

Experiment 5.2. We consider the feasible set C being a ball, namely $C=\left\{x\in\Re^m: ||x||^2\le 4\right\}$. In this case, all the optimization subproblems in SSM, PSM, GM, EGM seem more complex to solve numerically while the two projection steps in Algorithm 3.1 are inherenly explicit. Then, we only study the numerical behavior of Algorithm 3.1. All the datas are generated as in Experiment 5.1 and we consider five sequences of β_n as:

(I) $\beta_n=\frac{1}{(n+1)^{0.51}}$ (II) $\beta_n=\frac{1}{(n+1)^{0.7}}$ (III) $\beta_n=\frac{1}{n+1}$

(IV) $\beta_n=\frac{\log(n+3)}{n+1}$ (V) $\beta_n=\frac{1}{(n+1)\log(n+3)}$.

Note that all the above sequences of β_n have to satisfy conditions A1 and A2. Figs. 9 and 10 show the

Two matrices P, Q are generated randomly as follows: we randomly choose $\lambda_{1k}\in[0,m]$, $\lambda_{2k}\in[-m,-1]$, $k=1,\ldots,m$. Set $\widehat{Q}$, $\widehat{T}$ as two diagonal matrices with eigenvalues $\{\lambda_{1k}\}_{k=1}^m$ and $\{\lambda_{2k}\}_{k=1}^m$, respectively. Then, we define a positive semidefinite matrix Q and a negative semidefinite matrix T by using $\widehat{Q}$ and $\widehat{T}$ with two random orthogonal matrices, respectively. Finally, we set $P=Q-T$.

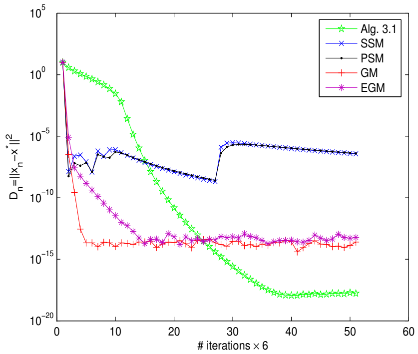

FIGURE 1. D_n and #iter. for Data 1 and (P1)

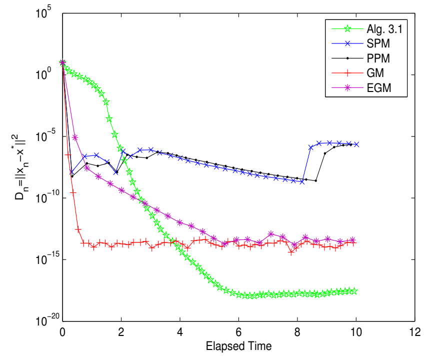

FIGURE 2. D_n and Time for Data 1 and (P1)

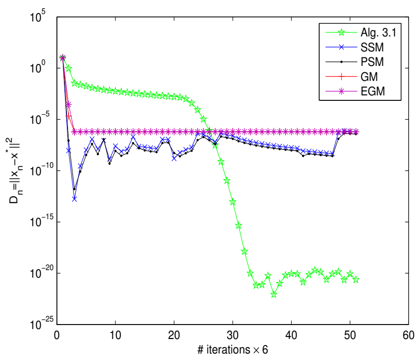

FIGURE 3. D_n and #iter. for Data 1 and (P2)

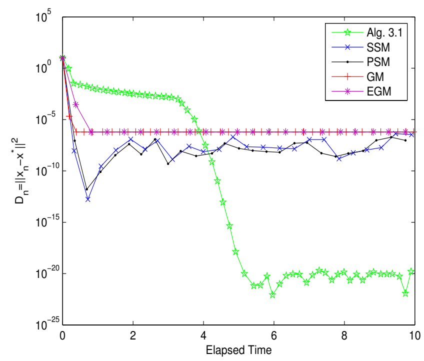

FIGURE 4. D_n and Time for Data 1 and (P2)

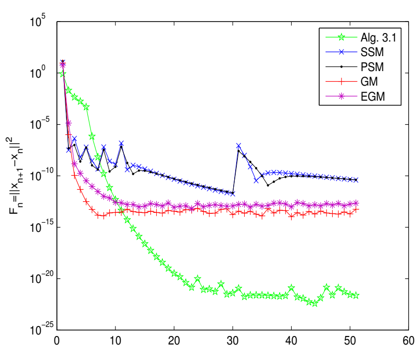

FIGURE 5. F_n and #iter. for Data 2 and (P1)

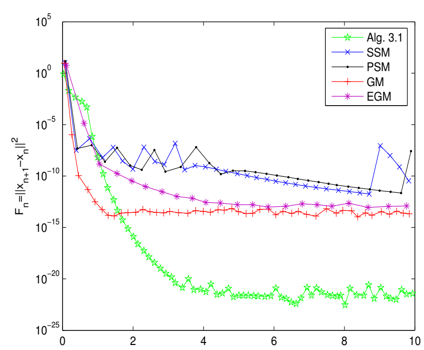

FIGURE 6. F_n and Time for Data 2 and (P1)

numerical results in this experiment. These figures show that Algorithm 3.1 is the best for the sequence $\{\beta_n\}$ in (I) which is slowly convergent to 0. The proposed algorithm with $\{\beta_n\}$ in (V) is the worst where D_n and F_n are very slowly decreasing to 0 and they seem to be stable in the first 300 iterations. While comparing Experiment 5.1 with Experiment 5.2 when C has a closed-form expression and the projection

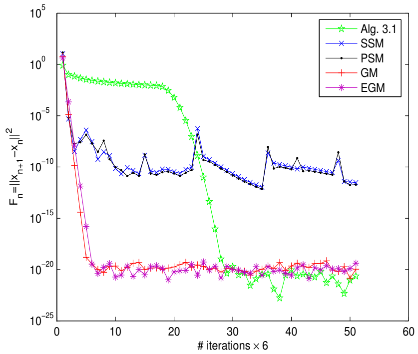

FIGURE 7. F_n and #iter. for Data 2 and (P2)

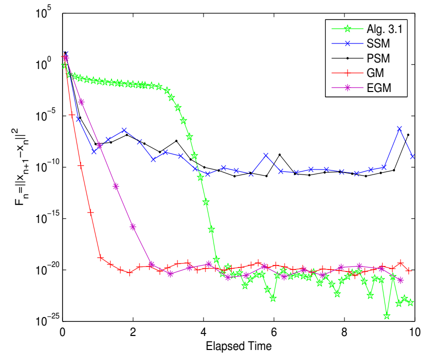

FIGURE 8. F_n and Time for Data 2 and (P2)

is explicit, the error obtained for Algorithm 3.1 is very much better. From figures 9 and 10 is also seen that the execution times for Algorithm 3.1 in the first 300 iterations are very small and the error can be obtained approximately equal to 10^{-150}.

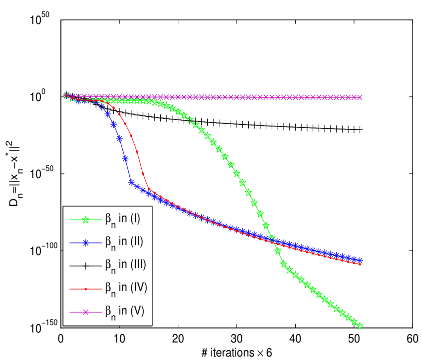

FIGURE 9. D_n of Algorithm 3.1 for Data 1. Execution times relatively are 0.016, 0.010, 0.015, 0.011 and 0.016s

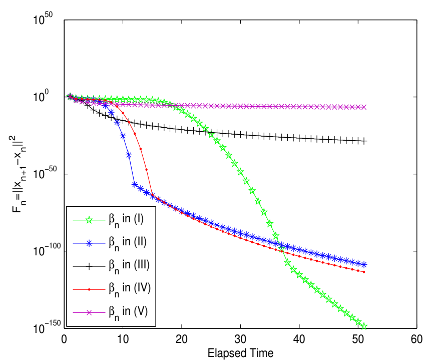

FIGURE 10. F_n of Algorithm 3.1 for Data 2. Execution times relatively are 0.024, 0.026, 0.026, 0.020 and 0.025s

6. CONCLUSIONS

In this paper, we present a splitting algorithm for solving equilibrium problems in Hilbert spaces. This barycentric projected-subgradient method uses the metric projection onto the convex set C, which when it can be computed efficiently brings simplicity in numerical experiments with respect to algorithms based on the resolvent operator. This is the case when C has a simple structure and the projection has a closed-form expression. The strong convergence of the algorithm is established under suitable assumptions. The numerical examples in this paper demonstrated that, for concrete equilibrium problems, the proposed algorithm converges to the desired solutions faster than the existing algorithms. It is worth mentioning that when the metric projection is hard to compute, we can make use of the subgradient projection operator which plays important roles as a low complexity approximation of the metric projection in many scenarios when C is a level set; e.g., in signal and image processing applications. This together with the idea introduced in [32] which suggests to replace the projection on C by projections onto half-spaces will be investigated in a forthcoming paper.

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