CLOSE-TO-CONVEXITY OF A CERTAIN FAMILY OF $q$-MITTAG-LEFFLER FUNCTIONS

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Abstract. In our present investigation, we study a certain family of $q$-Mittag-Leffler functions and find sufficient conditions under which it is close-to-convex in the open unit disk $U$. We consider various corollaries and consequences of our main results. We also point out relevant connections to some of the earlier known developments.

Keywords. Analytic and univalent functions; Taylor-Maclaurin series representation; Starlike and convex functions; Close-to-Convex functions; Mittag-Leffler and $q$-Mittag-Leffler functions.

2010 Mathematics Subject Classification. 30C45, 30C50, 33D15.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $A$ denote the class of all functions $f(z)$ normalized by

$$f(0) = 0 = f'(0) - 1,$$

which are analytic in the open unit disk

$$U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$

Thus, clearly, a function $f \in A$ has the following Taylor-Maclaurin series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U).$$

(1.1)

Also let $S$ denote the class of all functions $f \in A$ which are univalent in $U$.

A function $f \in A$ is called starlike (with respect to the origin 0), denoted by $f \in S^*$, if $\tau w \in f(U)$ whenever $w \in f(U)$ and $\tau \in [0,1]$. More generally, for a given parameter $\lambda$ ($0 \leq \lambda < 1$), a function $f \in A$ is called a starlike function of order $\lambda$ in $U$, denoted by $f \in S^*(\lambda)$, if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \lambda, \quad (z \in U; 0 \leq \lambda < 1).$$

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Received February 5, 2017; Accepted March 10, 2017.
It is well known that

\[ S^*(0) =: \mathcal{S}^*. \]

A function \( f \in \mathcal{A} \) is said to be convex of order \( \lambda \) \((0 \leq \lambda < 1)\) in \( \mathbb{U} \) if

\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \lambda, \quad (z \in \mathbb{U}; \ 0 \leq \lambda < 1). \]

We denote by \( \mathcal{K}(\lambda) \) the class of all functions \( f \in \mathcal{A} \) which are convex of order \( \lambda \) in \( \mathbb{U} \). As usual, we write

\[ \mathcal{K}(0) =: \mathcal{K}. \]

**Definition 1.1.** A function \( f \in \mathcal{A} \) is said to be close-to-convex in \( \mathbb{U} \) if the range \( f(\mathbb{U}) \) is close-to-convex, that is, if the complement of \( f(\mathbb{U}) \) can be written as the union of non-intersecting half-lines. Equivalently, a function \( f \in \mathcal{A} \) is said to be close-to-convex if there exists a starlike function \( g \) in \( \mathbb{U} \) (which is not necessarily normalized), denoted by \( f \in \mathcal{C} \), if

\[ \Re \left( \frac{zf'(z)}{g(z)} \right) > 0, \quad (z \in \mathbb{U}) \]

or, alternatively, if there exists a convex function \( g \) in \( \mathbb{U} \) (which is not necessarily normalized), denoted by \( f \in \mathcal{C} \), if

\[ \Re \left( \frac{f'(z)}{g'(z)} \right) > 0, \quad (z \in \mathbb{U}). \]

In fact, every close-to-convex function in \( \mathbb{U} \) is known to be univalent in \( \mathbb{U} \). Therefore, the set of all close-to-convex functions in \( \mathbb{U} \) forms a subclass of the normalized univalent function class \( \mathcal{S} \).

For further details about each of the above-defined function classes, the reader may refer to [5] (see also [26]).

In *Geometric Function Theory*, various subclasses of the normalized analytic function class \( \mathcal{A} \) have been studied from many different viewpoints. The \( q \)-calculus as well as the *fractional* \( q \)-calculus provide important tools that have been used in order to investigate several subclasses of \( \mathcal{A} \). Historically speaking, even though a \( q \)-analogue of the class \( \mathcal{S}^* \) of normalized starlike functions in \( \mathbb{U} \) was first introduced by Ismail *et al.* [9] by means of a \( q \)-difference operator \( D_q \), a firm footing of the usage of the \( q \)-calculus in the context of Geometric Function Theory was actually provided and the basic (or \( q \)-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [23, pp. 347 et seq.]).

The above-mentioned \( q \)-difference operator \( D_q \) acting on a function \( f \in \mathcal{A} \) is defined by

\[
(D_qf)(z) := \begin{cases} 
\frac{f(z) - f(qz)}{z(1 - q)}, & (z \in \mathbb{U} \setminus \{0\}; \ 0 < q < 1), \\
f'(0), & (z = 0; \ 0 < q < 1).
\end{cases}
\]  

(1.2)

One can clearly see from the definition (1.2) that

\[
\lim_{q \to 1^-} \{(D_qf)(z)\} = f'(z), \quad (z \in \mathbb{U}).
\]
**Definition 1.2.** A function \( f \in \mathcal{S} \) is said to belong to the class \( \mathcal{S}_{q}^{*} \) if

\[
\left| \frac{z}{f(z)} (D_{q}f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad (z \in U; \ 0 < q < 1). \tag{1.3}
\]

Clearly, when \( q \to 1- \), the function class \( \mathcal{S}_{q}^{*} \) defined by (1.3) will coincide with the normalized starlike function class \( \mathcal{S}^{*} \).

**Definition 1.3.** A function \( f \in \mathcal{S} \) is said to belong to the class \( \mathcal{C}_{q} \) if there exists a function \( g \in \mathcal{S}^{*} \) such that

\[
\left| \frac{z}{g(z)} (D_{q}f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad (z \in U; \ 0 < q < 1). \tag{1.4}
\]

It is easily seen that, in the limit when \( q \to 1- \), the function class \( \mathcal{C}_{q} \) defined by (1.4) reduces to the normalized close-to-convex function class \( \mathcal{C} \) given by Definition 1.1.

In the year 2012, Raghavendar and Swaminathan [17] investigated some basic properties of functions that are in the class \( \mathcal{C}_{q} \). Since \( (D_{q}f)(z) \to f'(z) \) in the limit when \( q \to 1- \), we observe that, in the limiting sense, the closed disk:

\[
\left| w - \frac{1}{1-q} \right| \leq \frac{1}{1-q}
\]

becomes the right half-plane given by

\[
\Re \left( \frac{zf'(z)}{g(z)} \right) > 0 \quad (z \in U).
\]

Hence, clearly, the class \( \mathcal{C}_{q} \) reduces to \( \mathcal{C} \). Throughout this paper, we refer to the functions in the class \( \mathcal{C}_{q} \) as the \( q \)-close-to-convex functions in \( U \). It is easy to see that

\[
\mathcal{S}_{q}^{*} \subset \mathcal{C}_{q}, \quad (0 < q < 1).
\]

Moreover, it follows from the above discussion that

\[
\bigcap_{0 < q < 1} \mathcal{C}_{q} \subset \mathcal{C} \subset \mathcal{S}.
\]

Recently, several researchers studied various classes of analytic functions involving such families of special functions as (for example) \( \mathcal{F} \subset \mathcal{A} \) in order to find different conditions such that the members of the special function class \( \mathcal{F} \) possess certain geometric properties like univalence, starlikeness or convexity in \( U \). In this context, many results are available in the literature regarding the hypergeometric functions [6, 7, 11, 13, 14, 18, 23], the Bessel functions [2, 3, 4, 15], the Fox-Wright function [10, 16, 24] and the Mittag-Leffler function [1]. In this paper, we study several geometric properties of the normalized \( q \)-Mittag-Leffler function \( \mathcal{M}_{\alpha, \beta}(z; q) \) which is defined by Eq. (1.10) below.

In our present investigation, we shall need the following notations and definitions. First of all, for \( q \in (0, 1), \ \kappa, \mu \in \mathbb{C} \) and \( n \in \mathbb{N}_{0} = \mathbb{N} \cup \{0\} \) (\( \mathbb{N} \) being the set of positive integers), the \( q \)-shifted factorial \( (\kappa; q)_{\mu} \) is defined by

\[
(\kappa; q)_{\mu} := \prod_{j=0}^{\mu} \left( 1 - \frac{\kappa q^{j}}{1 - \kappa q^{\mu+j}} \right), \quad (\kappa, \mu \in \mathbb{C}),
\]
so that

\[
(\kappa; q)_n := \begin{cases} 
1, & (n = 0), \\
\prod_{j=0}^{n-1} (1 - \kappa q^j), & (n \in \mathbb{N}), 
\end{cases}
\]

and

\[
(\kappa; q)_\infty := \prod_{j=0}^{\infty} (1 - \kappa q^j), \quad (\kappa \in \mathbb{C}).
\]

The \(q\)-Gamma function \(\Gamma_q(z)\) is defined by

\[
\Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z}, \quad (0 < q < 1; \ z \in \mathbb{C}),
\]

so that

\[
\lim_{q \to 1} \{\Gamma_q(z)\} = \Gamma(z)
\]

in terms of the familiar (Euler’s) Gamma function \(\Gamma(z)\). The \(q\)-Gamma function \(\Gamma_q(z)\) satisfies the following functional equation:

\[
\Gamma_q(z+1) = \left(\frac{1-q^z}{1-q}\right) \Gamma_q(z), \quad (0 < q < 1; \ z \in \mathbb{C}).
\]

For further details about the \(q\)-calculus, one may refer to the books by Gasper and Rahman [8] and by Srivastava and Karlsson [28, pp. 346–351].

We now turn to the familiar Mittag-Leffler function \(E_\alpha(z)\) (see [12]) and its two-parameter extension \(E_{\alpha,\beta}(z)\) (see [29, 30]), which are defined (as usual) by means of the following series:

\[
E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} =: E_{\alpha,1}(z) \quad (z \in \mathbb{C}; \ \Re(\alpha) > 0)
\]

and

\[
E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (z, \beta \in \mathbb{C}; \ \Re(\alpha) > 0),
\]

respectively. For a detailed investigation of the Mittag-Leffler type functions and their applications, the interested reader may refer to a recent work by Srivastava [25] (see also [21] and many other related developments which are cited in [25]).

The above-defined Mittag-Leffler functions \(E_\alpha(z)\) and \(E_{\alpha,\beta}(z)\) are natural extensions of the exponential, hyperbolic and trigonometric functions, since it is easily verified that

\[
E_1(z) = e^z, \quad E_2(z^2) = \cosh z, \quad E_2(-z^2) = \cos z,
\]

\[
E_{1,2}(z) = \frac{e^z - 1}{z} \quad \text{and} \quad E_{2,2}(z^2) = \frac{\sinh z}{z}.
\]

The \(q\)-Mittag-Leffler function defined by (see [20])

\[
E_{\alpha,\beta}(z; q) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C}; \ \Re(\alpha) > 0)
\]

happens to be a very specialized case of the \(q\)-Fox-Wright function \(\Phi_5(z; q)\) which, in turn, corresponds to the obvious one-variable version of Srivastava’s \(q\)-Fox-Wright function in several variables (see, for
Indeed, for the $q$-exponential function $E_{\alpha, \beta}(z; q)$ defined by (1.9) does not belong to the normalized analytic function class $\mathcal{A}$, it is natural to consider the following normalization of this $q$-Mittag-Leffler function:

$$\mathcal{M}_{\alpha, \beta}(z; q) = z\Gamma_q(\beta)E_{\alpha, \beta}(z; q) := \sum_{n=0}^{\infty} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n + \beta)} z^{n+1}$$

where $\alpha > 0$; $\beta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

The function $\mathcal{M}_{\alpha, \beta}(z; q)$ contains many known functions as its special cases. For example, we have

$$\begin{cases}
\mathcal{M}_{0,0}(z; q) = \frac{z}{1-z}, & \mathcal{M}_{1,0}(z; q) = ze^q,
\mathcal{M}_{1,1}(z; q) = e_q - z - 1)(q + 1),
\mathcal{M}_{1,2}(z; q) = e_q^z - 1, & \mathcal{M}_{1,3}(z; q) = \left(e_q^z - 1 - z - \frac{z^2}{1+q}\right),
\mathcal{M}_{1,4}(z; q) = \frac{(1+q)(1+q+q^2)}{z^2} \left(e_q^z - 1 - z - \frac{z^2}{1+q}\right),
\end{cases}$$

where $e_q^z$ is one of the $q$-analogue of the classical exponential function $e^z$ given by (see [27, p. 488, Eq. 6.3(7)])

$$e_q^z := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+1)} = \sum_{n=0}^{\infty} \frac{(1-q)z^n}{(q; q)_n} =: e_q(z)$$

$$= \frac{1}{(z; q)_\infty}. \quad (1.12)$$

Another widely-studied $q$-analogue of the classical exponential function $e^z$ is given by (see [27, p. 488, Eq. 6.3(8)])

$$E_q^z := \sum_{n=0}^{\infty} q(z)^n \frac{z^n}{\Gamma_q(n+1)} = \sum_{n=0}^{\infty} q(z)^n \frac{(1-q)z^n}{(q; q)_n} =: E_q((1-q)z)$$

$$= \left(- (1-q)z; q\right)_\infty. \quad (1.13)$$

Indeed, for the $q$-exponential functions defined by the equations (1.12) and (1.13), respectively, the notations $e_q(z)$ and $E_q(z)$ are used more commonly in the $q$-literature than the notations $e_q^z$ and $E_q^z$.

2. The Main Result and Its Consequences

In order to prove our main theorem in this paper, we require the following lemma (see [17, Lemma 2.1]; see also [19, Theorem 2.2]).

**Lemma 2.1.** Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers and define another sequence $\{B_n\}_{n \in \mathbb{N}}$ by

$$B_n := \left(\frac{1-q^n}{1-q}\right) A_n, \quad (n \in \mathbb{N}).$$

Suppose that

$$1 = B_1 \geq B_2 \geq B_3 \geq \cdots \geq B_n \geq \cdots \geq 0$$

or

$$1 = B_1 \leq B_2 \leq B_3 \leq \cdots \leq B_n \leq \cdots \leq 2.$$
Then
\[ f(z) = z + \sum_{n=2}^{\infty} A_n \, z^n \in \mathcal{G}_q \]
with respect to
\[ g(z) = \frac{z}{1-z}, \quad (z \in \mathbb{U}). \]

Making use of Lemma 2.1, we now prove the following theorem.

**Theorem 2.1.** For each \( \alpha \geq 1 \) and \( \beta \geq 1 \) satisfying the following inequality:
\[ \Gamma_q(\alpha + \beta) \geq (1 + q) \Gamma_q(\beta), \quad (0 < q < 1), \]
the normalized q-Mittag-Leffler function \( \mathcal{M}_{\alpha, \beta}(z; q) \) is q-close-to-convex in \( \mathbb{U} \) with respect to
\[ g(z) = \frac{z}{1-z}, \quad (z \in \mathbb{U}). \]

**Proof.** Let
\[ \mathcal{M}_{\alpha, \beta}(z; q) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(n-1) + \beta)} z^n = z + \sum_{n=2}^{\infty} A_n \, z^n, \]
so that
\[ A_n = \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(n-1) + \beta)}, \quad (n \in \mathbb{N}) \]
and
\[ B_n := \left( \frac{1-q^n}{1-q} \right) A_n = \frac{(1-q^n) \Gamma_q(\beta)}{(1-q) \Gamma_q(\alpha(n-1) + \beta)}. \]  
(2.1)

It is easily seen that \( B_1 = 1 \) and that \( B_n \geq 0 \) for all \( n \in \mathbb{N} \). Furthermore, in view of the hypothesis of the above theorem, we have
\[ B_2 = \frac{(1+q) \Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)} \leq 1 = B_1. \]

We next show that
\[ B_{n+1} \leq B_n, \quad (n \in \mathbb{N} \setminus \{1\}), \]
that is, that
\[ \frac{(1-q^{n+1}) \Gamma_q(\beta)}{(1-q) \Gamma_q(\alpha n + \beta)} \leq \frac{(1-q^n) \Gamma_q(\beta)}{(1-q) \Gamma_q(\alpha(n-1) + \beta)}, \quad (n \in \mathbb{N} \setminus \{1\}), \]
which is equivalent to
\[ (1-q^{n+1}) \Gamma_q(\alpha(n-1) + \beta) \leq (1-q^n) \Gamma_q(\alpha n + \beta), \quad (n \in \mathbb{N} \setminus \{1\}). \]  
(2.2)

This last inequality (2.2) is verified by the fact that
\[ (1-q^n) \Gamma_q(\alpha n + \beta) = (1-q^n) \Gamma_q(\alpha(n-1) + 1 + \beta) \]
\[ \geq (1-q^n) \Gamma_q(\alpha(n-1) + \beta + 1) \quad (\alpha \geq 1; \beta \geq 1) \]
\[ = (1-q^n) \left( \frac{1-q^{\alpha(n-1)+\beta}}{1-q} \right) \Gamma_q(\alpha(n-1) + \beta). \]  
(2.3)

Now, in view of the following inequality:
\[ \Gamma_q(\alpha + \beta) \geq (1+q) \Gamma_q(\beta), \]

\[ \Gamma_q(\alpha + \beta) \geq (1+q) \Gamma_q(\beta), \]
it can be seen that $\alpha + \beta \geq 3$, which implies that
$$ (n-1)\alpha + \beta \geq n + 1, \quad (n \in \mathbb{N} \setminus \{1\}). $$

We thus obtain
$$ (1 - q^n)\Gamma_q(\alpha n + \beta) \geq (1 - q^{n+1})\Gamma_q(\alpha(n-1) + \beta), \quad (n \in \mathbb{N} \setminus \{1\}), $$
which establishes the desired inequality (2.2). By applying Lemma 2.1, we get the result asserted by the theorem. \qed

If we set $\alpha = 1$ in Theorem 2.1, then we get the following inequality:
$$ \Gamma_q(\alpha + \beta) \geq (1 + q)\Gamma_q(\beta), $$
which, on simplification, yields
$$ \frac{1 - q^\beta}{1 - q} \geq 1 + q, $$
which holds true only if $\beta \geq 2$. This leads us to Corollary 2.1 below.

**Corollary 2.1.** The normalized $q$-Mittag-Leffler function $M_{1,\beta}(z; q) \in \mathcal{C}_q$ for $\beta \geq 2$.

**Example 2.1.** By applying Corollary 2.1, we readily deduce that
$$ M_{1,2}(z; q) = e^z_q - 1 \in \mathcal{C}_q, \quad M_{1,3}(z; q) = \frac{(1 + q)(e^z_q - z - 1)}{z} \in \mathcal{C}_q $$
and
$$ M_{1,4}(z; q) = \frac{(1 + q)(1 + q + q^2)}{z^2} \left( e^z_q - 1 - z - \frac{z^2}{1 + q} \right) \in \mathcal{C}_q. $$

If we take $\alpha = 2$ in Theorem 2.1, then the inequality:
$$ \Gamma_q(\alpha + \beta) \geq (1 + q)\Gamma_q(\beta) $$
can be simplified to the following form:
$$ \left( \frac{1 - q^{\beta+1}}{1 - q} \right) \left( \frac{1 - q^\beta}{1 - q} \right) \geq 1 + q, $$
which holds true for $\beta \geq 1$. We are thus led to Corollary 2.2 below.

**Corollary 2.2.** The normalized $q$-Mittag-Leffler function $M_{2,\beta}(z; q) \in \mathcal{C}_q$ for $\beta \geq 1$.

3. **Concluding Remarks and Observations**

Our main objective in this paper has been to introduce and investigate a certain family of $q$-Mittag-Leffler functions. For this family of $q$-Mittag-Leffler functions, we have successfully found sufficient conditions under which it is close-to-convex in the open unit disk $U$. We then have considered various corollaries and consequences of our main theorem. We have also pointed out relevant connections to some of the earlier known developments.
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