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CLOSE-TO-CONVEXITY OF A CERTAIN FAMILY OF q-MITTAG-LEFFLER FUNCTIONS

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Abstract. In our present investigation, we study a certain family of q-Mittag-Leffler functions and find sufficient conditions under which it is close-to-convex in the open unit disk $\mathbb{U}$. We consider various corollaries and consequences of our main results. We also point out relevant connections to some of the earlier known developments.

Keywords. Analytic and univalent functions; Taylor-Maclaurin series representation; Starlike and convex functions; Close-to-Convex functions; Mittag-Leffler and q-Mittag-Leffler functions.

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1. Introduction, definitions and preliminaries

Let $\mathscr{A}$ denote the class of all functions $f(z)$ normalized by

$$f(0) = 0 = f'(0) - 1,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Thus, clearly, a function $f \in \mathscr{A}$ has the following Taylor-Maclaurin series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathbb{U}). \tag{1.1}$$

Also let $\mathscr{S}$ denote the class of all functions $f \in \mathscr{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathscr{A}$ is called starlike (with respect to the origin 0), denoted by $f \in \mathscr{S}^*$, if $\tau w \in f(\mathbb{U})$ whenever $w \in f(\mathbb{U})$ and $\tau \in [0,1]$. More generally, for a given parameter λ $(0 \leqq \lambda < 1)$, a function $f \in \mathscr{A}$ is called a starlike function of order λ in $\mathbb{U}$, denoted by $f \in \mathscr{S}^*(\lambda)$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \lambda, \qquad (z \in \mathbb{U};\ 0 \leqq \lambda < 1).$$

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It is well known that

$$\mathscr{S}^*(0) =: \mathscr{S}^*.$$

A function $f \in \mathscr{A}$ is said to be convex of order λ $(0 \leqq \lambda < 1)$ in $\mathbb{U}$ if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \lambda, \qquad (z \in \mathbb{U};\ 0 \leqq \lambda < 1).$$

We denote by $\mathscr{K}(\lambda)$ the class of all functions $f \in \mathscr{A}$ which are convex of order λ in $\mathbb{U}$. As usual, we write

$$\mathscr{K}(0) =: \mathscr{K}.$$

Definition 1.1. A function $f \in \mathscr{A}$ is said to be close-to-convex in $\mathbb{U}$ if the range $f(\mathbb{U})$ is close-to-convex, that is, if the complement of $f(\mathbb{U})$ can be written as the union of non-intersecting half-lines. Equivalently, a function $f \in \mathscr{A}$ is said to be close-to-convex if there exists a starlike function g in $\mathbb{U}$ (which is not necessarily normalized), denoted by $f \in \mathscr{C}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \qquad (z \in \mathbb{U})$$

or, alternatively, if there exists a convex function $\mathfrak{g}$ in $\mathbb{U}$ (which is not necessarily normalized), denoted by $f \in \mathscr{C}$, if

$$\Re\left(\frac{f'(z)}{\mathfrak{g}'(z)}\right) > 0, \qquad (z \in \mathbb{U}).$$

In fact, every close-to-convex function in $\mathbb{U}$ is known to be univalent in $\mathbb{U}$. Therefore, the set of all close-to-convex functions in $\mathbb{U}$ forms a subclass of the normalized univalent function class $\mathscr{S}$.

For further details about each of the above-defined function classes, the reader may refer to [5] (see also [26]).

In *Geometric Function Theory*, various subclasses of the normalized analytic function class $\mathscr{A}$ have been studied from many different viewpoints. The q-calculus as well as the *fractional* q-calculus provide important tools that have been used in order to investigate several subclasses of $\mathscr{A}$. Historically speaking, even though a q-analogue of the class $\mathscr{S}^*$ of normalized starlike functions in $\mathbb{U}$ was first introduced by Ismail *et al.* [9] by means of a q-difference operator D_q, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [23, pp. 347 *et seq.*]).

The above-mentioned q-difference operator D_q acting on a function $f \in \mathscr{A}$ is defined by

$$(D_q f)(z) := \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, & (z \in \mathbb{U} \setminus \{0\};\ 0 < q < 1), \\[2ex] f'(0), & (z = 0;\ 0 < q < 1). \end{cases} \tag{1.2}$$

One can clearly see from the definition (1.2) that

$$\lim_{q \to 1-} \left\{(D_q f)(z)\right\} = f'(z), \qquad (z \in \mathbb{U}).$$

Definition 1.2. A function $f \in \mathscr{A}$ is said to belong to the class $\mathscr{S}_q^*$ if

$$\left| \frac{z}{f(z)}(D_q f)(z) - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}, \qquad (z \in \mathbb{U};\ 0 < q < 1). \tag{1.3}$$

Clearly, when $q \to 1-$, the function class $\mathscr{S}_q^*$ defined by (1.3) will coincide with the normalized starlike function class $\mathscr{S}^*$.

Definition 1.3. A function $f \in \mathscr{A}$ is said to belong to the class $\mathscr{C}_q$ if there exists a function $g \in \mathscr{S}^*$ such that

$$\left| \frac{z}{g(z)}(D_q f)(z) - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}, \qquad (z \in \mathbb{U};\ 0 < q < 1). \tag{1.4}$$

It is easily seen that, in the limit when $q \to 1-$, the function class $\mathscr{C}_q$ defined by (1.4) reduces to the normalized close-to-convex function class $\mathscr{C}$ given by Definition 1.1.

In the year 2012, Raghavendar and Swaminathan [17] investigated some basic properties of functions that are in the class $\mathscr{C}_q$. Since $(D_q f)(z) \to f'(z)$ in the limit when $q \to 1-$, we observe that, in the limiting sense, the closed disk:

$$\left| w - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}$$

becomes the right half-plane given by

$$\Re\left(\frac{zf'(z)}{g(z)} \right) > 0 \qquad (z \in \mathbb{U}).$$

Hence, clearly, the class $\mathscr{C}_q$ reduces to $\mathscr{C}$. Throughout this paper, we refer to the functions in the class $\mathscr{C}_q$ as the q-close-to-convex functions in $\mathbb{U}$. It is easy to see that

$$\mathscr{S}_q^* \subset \mathscr{C}_q, \qquad (0 < q < 1).$$

Moreover, it follows from the above discussion that

$$\bigcap_{0<q<1} \mathscr{C}_q \subset \mathscr{C} \subset \mathscr{S}.$$

Recently, several researchers studied various classes of analytic functions involving such families of special functions as (for example) $\mathfrak{F} \subset \mathscr{A}$ in order to find different conditions such that the members of the special function class $\mathfrak{F}$ possess certain geometric properties like univalence, starlikeness or convexity in $\mathbb{U}$. In this context, many results are available in the literature regarding the hypergeometric functions [6, 7, 11, 13, 14, 18, 23], the Bessel functions [2, 3, 4, 15], the Fox-Wright function [10, 16, 24] and the Mittag-Leffler function [1]. In this paper, we study several geometric properties of the normalized q-Mittag-Leffler function $\mathfrak{M}_{\alpha,\beta}(z;q)$ which is defined by Eq. (1.10) below.

In our present investigation, we shall need the following notations and definitions. First of all, for $q \in (0,1)$, $\kappa, \mu \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ($\mathbb{N}$ being the set of positive integers), the q-shifted factorial $(\kappa;q)_\mu$ is defined by

$$(\kappa;q)_\mu := \prod_{j=0}^{\infty} \left(\frac{1-\kappa q^j}{1-\kappa q^{\mu+j}} \right), \qquad (\kappa, \mu \in \mathbb{C}),$$

so that

$$(\kappa;q)_n := \begin{cases} 1, & (n=0), \\ \prod_{j=0}^{n-1} \left(1-\kappa q^j\right), & (n \in \mathbb{N}), \end{cases}$$

and

$$(\kappa;q)_\infty := \prod_{j=0}^{\infty} \left(1-\kappa q^j\right), \qquad (\kappa \in \mathbb{C}).$$

The q-Gamma function $\Gamma_q(z)$ is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}\,(1-q)^{1-z}, \qquad (0<q<1;\ z\in\mathbb{C}), \tag{1.5}$$

so that

$$\lim_{q\to 1-} \{\Gamma_q(z)\} = \Gamma(z)$$

in terms of the familiar (Euler's) Gamma function $\Gamma(z)$. The q-Gamma function $\Gamma_q(z)$ satisfies the following functional equation:

$$\Gamma_q(z+1) = \left(\frac{1-q^z}{1-q}\right)\Gamma_q(z), \qquad (0<q<1;\ z\in\mathbb{C}). \tag{1.6}$$

For further details about the q-calculus, one may refer to the books by Gasper and Rahman [8] and by Srivastava and Karlsson [28, pp. 346–351].

We now turn to the familiar Mittag-Leffler function $E_\alpha(z)$ (see [12]) and its two-parameter extension $E_{\alpha,\beta}(z)$ (see [29, 30]), which are defined (as usual) by means of the following series:

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} =: E_{\alpha,1}(z) \qquad \left(z\in\mathbb{C};\ \Re(\alpha)>0\right) \tag{1.7}$$

and

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad \left(z,\beta\in\mathbb{C};\ \Re(\alpha)>0\right), \tag{1.8}$$

respectively. For a detailed investigation of the Mittag-Leffler type functions and their applications, the interested reader may refer to a recent work by Srivastava [25] (see also [21] and many other related developments which are cited in [25]).

The above-defined Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are *natural* extensions of the exponential, hyperbolic and trigonometric functions, since it is easily verified that

$$E_1(z) = e^z, \qquad E_2\left(z^2\right) = \cosh z, \qquad E_2\left(-z^2\right) = \cos z,$$

$$E_{1,2}(z) = \frac{e^z-1}{z} \qquad \text{and} \qquad E_{2,2}(z^2) = \frac{\sinh z}{z}.$$

The q-Mittag-Leffler function defined by (see [20])

$$E_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n+\beta)}, \qquad \left(z,\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0\right) \tag{1.9}$$

happens to be a very specialized case of the q-Fox-Wright function ${}_r\Phi_s(z;q)$ which, in turn, corresponds to the obvious *one-variable* version of Srivastava's q-Fox-Wright function in several variables (see, for details, [22]; see also [28, p. 350, Eq. 9.4(284)]). Since the q-Mittag-Leffler function $E_{\alpha,\beta}(z;q)$ defined by (1.9) does not belong to the *normalized* analytic function class $\mathscr{A}$, it is natural to consider the following normalization of this q-Mittag-Leffler function:

$$\mathfrak{M}_{\alpha,\beta}(z;q) = z\Gamma_q(\beta)E_{\alpha,\beta}(z;q) := \sum_{n=0}^{\infty} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)}\, z^{n+1} \tag{1.10}$$

$$\big(z \in \mathbb{U};\ \Re(\alpha) > 0;\ \beta \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}\big).$$

The function $\mathfrak{M}_{\alpha,\beta}(z;q)$ contains many known functions as its special cases. For example, we have

$$\begin{cases} \mathfrak{M}_{0,\beta}(z;q) = \dfrac{z}{1-z}, \qquad \mathfrak{M}_{1,1}(z;q) = ze_q^z, \\ \mathfrak{M}_{1,2}(z;q) = e_q^z - 1, \qquad \mathfrak{M}_{1,3}(z;q) = \dfrac{(e_q^z - z - 1)(q+1)}{z}, \\ \mathfrak{M}_{1,4}(z;q) = \dfrac{(1+q)(1+q+q^2)}{z^2}\left(e_q^z - 1 - z - \dfrac{z^2}{1+q}\right), \end{cases} \tag{1.11}$$

where e_q^z is one of the q-analogues of the classical exponential function e^z given by (see [27, p. 488, Eq. 6.3(7)])

$$\begin{aligned} e_q^z &:= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+1)} = \sum_{n=0}^{\infty} \frac{[(1-q)z]^n}{(q;q)_n} =: e_q(z) \\ &= \frac{1}{(z;q)_\infty}. \end{aligned} \tag{1.12}$$

Another widely-studied q-analogue of the classical exponential function e^z is given by (see [27, p. 488, Eq. 6.3(8)])

$$\begin{aligned} E_q^z &:= \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{\Gamma_q(n+1)} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{[(1-q)z]^n}{(q;q)_n} =: E_q\big((1-q)z\big) \\ &= \big(-(1-q)z;q\big)_\infty. \end{aligned} \tag{1.13}$$

Indeed, for the q-exponential functions defined by the equations (1.12) and (1.13), respectively, the notations $e_q(z)$ and $E_q(z)$ are used more commonly in the q-literature than the notations e_q^z and E_q^z.

2. THE MAIN RESULT AND ITS CONSEQUENCES

In order to prove our main theorem in this paper, we require the following lemma (see [17, Lemma 2.1]; see also [19, Theorem 2.2]).

Lemma 2.1. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers and define another sequence $\{B_n\}_{n\in\mathbb{N}}$ by*

$$B_n := \left(\frac{1-q^n}{1-q}\right) A_n, \qquad (n \in \mathbb{N}).$$

Suppose that

$$1 = B_1 \geqq B_2 \geqq B_3 \geqq \cdots \geqq B_n \geqq \cdots \geqq 0$$

or

$$1 = B_1 \leqq B_2 \leqq B_3 \leqq \cdots \leqq B_n \leqq \cdots \leqq 2.$$

Then

$$f(z)=z+\sum_{n=2}^{\infty}A_n\,z^n\in\mathscr{C}_q$$

with respect to

$$g(z)=\frac{z}{1-z},\qquad (z\in\mathbb{U}).$$

Making use of Lemma 2.1, we now prove the following theorem.

Theorem 2.1. *For each* $\alpha\geqq 1$ *and* $\beta\geqq 1$ *satisfying the following inequality*:

$$\Gamma_q(\alpha+\beta)\geqq(1+q)\Gamma_q(\beta),\qquad (0<q<1),$$

the normalized q-Mittag-Leffler function $\mathfrak{M}_{\alpha,\beta}(z;q)$ *is q-close-to-convex in* $\mathbb{U}$ *with respect to*

$$g(z)=\frac{z}{1-z},\qquad (z\in\mathbb{U}).$$

Proof. Let

$$\mathfrak{M}_{\alpha,\beta}(z;q)=z+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta)}{\Gamma_q\big(\alpha(n-1)+\beta\big)}z^n=z+\sum_{n=2}^{\infty}A_n\,z^n,$$

so that

$$A_n=\frac{\Gamma_q(\beta)}{\Gamma_q\big(\alpha(n-1)+\beta\big)},\qquad (n\in\mathbb{N})$$

and

$$B_n:=\left(\frac{1-q^n}{1-q}\right)A_n=\frac{(1-q^n)\Gamma_q(\beta)}{(1-q)\Gamma_q\big(\alpha(n-1)+\beta\big)}.\tag{2.1}$$

It is easily seen that $B_1=1$ and that $B_n\geqq 0$ for all $n\in\mathbb{N}$. Furthermore, in view of the hypothesis of the above theorem, we have

$$B_2=\frac{(1+q)\Gamma_q(\beta)}{\Gamma_q\big(\alpha+\beta\big)}\leqq 1=B_1.$$

We next show that

$$B_{n+1}\leqq B_n,\qquad (n\in\mathbb{N}\setminus\{1\}),$$

that is, that

$$\frac{(1-q^{n+1})\Gamma_q(\beta)}{(1-q)\Gamma_q(\alpha n+\beta)}\leqq\frac{(1-q^n)\Gamma_q(\beta)}{(1-q)\Gamma_q\big(\alpha(n-1)+\beta\big)},\qquad (n\in\mathbb{N}\setminus\{1\}),$$

which is equivalent to

$$(1-q^{n+1})\Gamma_q\big(\alpha(n-1)+\beta\big)\leqq(1-q^n)\Gamma_q(\alpha n+\beta),\qquad (n\in\mathbb{N}\setminus\{1\}).\tag{2.2}$$

This last inequality (2.2) is verified by the fact that

$$\begin{aligned}(1-q^n)\Gamma_q(\alpha n+\beta)&=(1-q^n)\Gamma_q\big(\alpha(n-1)+\alpha+\beta\big)\\&\geqq(1-q^n)\Gamma_q\big(\alpha(n-1)+\beta+1\big)\qquad(\alpha\geqq 1;\ \beta\geqq 1)\\&=(1-q^n)\left(\frac{1-q^{\alpha(n-1)+\beta}}{1-q}\right)\Gamma_q\big(\alpha(n-1)+\beta\big).\end{aligned}\tag{2.3}$$

Now, in view of the following inequality:

$$\Gamma_q(\alpha+\beta)\geqq(1+q)\Gamma_q(\beta),$$

it can be seen that $\alpha+\beta \geqq 3$, which implies that

$$(n-1)\alpha+\beta \geqq n+1, \qquad (n \in \mathbb{N}\setminus\{1\}).$$

We thus obtain

$$(1-q^n)\Gamma_q(\alpha n+\beta) \geqq \left(1-q^{n+1}\right)\Gamma_q\big(\alpha(n-1)+\beta\big), \qquad (n \in \mathbb{N}\setminus\{1\}),$$

which establishes the desired inequality (2.2). By applying Lemma 2.1, we get the result asserted by the theorem. □

If we set $\alpha = 1$ in Theorem 2.1, then we get the following inequality:

$$\Gamma_q(\alpha+\beta) \geqq (1+q)\Gamma_q(\beta),$$

which, on simplification, yields

$$\frac{1-q^\beta}{1-q} \geqq 1+q,$$

which holds true only if $\beta \geqq 2$. This leads us to Corollary 2.1 below.

Corollary 2.1. *The normalized q-Mittag-Leffler function* $\mathfrak{M}_{1,\beta}(z;q) \in \mathscr{C}_q$ *for* $\beta \geqq 2$.

Example 2.1. By applying Corollary 2.1, we readily deduce that

$$\mathfrak{M}_{1,2}(z;q) = e_q^z - 1 \in \mathscr{C}_q, \qquad \mathfrak{M}_{1,3}(z;q) = \frac{(1+q)(e_q^z - z - 1)}{z} \in \mathscr{C}_q$$

and

$$\mathfrak{M}_{1,4}(z;q) = \frac{(1+q)(1+q+q^2)}{z^2}\left(e_q^z - 1 - z - \frac{z^2}{1+q}\right) \in \mathscr{C}_q.$$

If we take $\alpha = 2$ in Theorem 2.1, then the inequality:

$$\Gamma_q(\alpha+\beta) \geqq (1+q)\Gamma_q(\beta)$$

can be simplified to the following form:

$$\left(\frac{1-q^{\beta+1}}{1-q}\right)\left(\frac{1-q^\beta}{1-q}\right) \geqq 1+q,$$

which holds true for $\beta \geqq 1$. We are thus led to Corollary 2.2 below.

Corollary 2.2. *The normalized q-Mittag-Leffler function* $\mathfrak{M}_{2,\beta}(z;q) \in \mathscr{C}_q$ *for* $\beta \geqq 1$.

3. Concluding remarks and observations

Our main objective in this paper has been to introduce and investigate a certain family of q-Mittag-Leffler functions. For this family of q-Mittag-Leffler functions, we have successfully found sufficient conditions under which it is close-to-convex in the open unit disk $\mathbb{U}$. We then have considered various corollaries and consequences of our main theorem. We have also pointed out relevant connections to some of the earlier known developments.

REFERENCES

[1] D. Bansal, J. K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, Complex Var. Elliptic Equ. 61 (2016), 338-350.

[2] Á. Baricz, Generalized Bessel Functions of the First Kind, Lecture Notes in Mathematics, Vol. 1994, Springer-Verlag, Berlin, Heidelberg and New York, 2010.

[3] Á. Baricz, P. A. Kupán, R. Szász, The radius of starlikeness of normalized Bessel function of first kind, Proc. Amer. Math. Soc. 142 (2014), 2019-2025.

[4] Á. Baricz, R. Szász, The radius of convexity of normalized Bessel functions of the first kind, Anal. Appl. 12 (2014), 485-509.

[5] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[6] J. Dziok, H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999), 1-13.

[7] J. Dziok, H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct. 14 (2003), 7-18.

[8] G. Gasper, M. Rahman, Basic Hypergeometric Series (with a Foreword by Richard Askey), Encyclopedia of Mathematics and Its Applications, Vol. 35, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne and Sydney, 1990; Second edition, Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, London and New York, 2004.

[9] M. E. H. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, Complex Variables Theory Appl. 14 (1990), 77–84.

[10] V. Kiryakova, Criteria for univalence of the Dziok-Srivasava and the Srivastava-Wright operators in the class $\mathscr{A}$, Appl. Math. Comput. 218 (2011), 883-892.

[11] S. S. Miller, P. T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, Proc. Amer. Math. Soc. 110 (1990), 333-342.

[12] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, C. R. Acad. Sci. Paris Sér. II 137 (1903), 554-558.

[13] S. Owa, H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math. 39 (1987), 1057-1077.

[14] S. Ponnusamy, M. Vuorinen, Univalence and convexity properties for Gaussian hypergeometric functions, Rocky Mountain J. Math. 31 (2001), 327-353.

[15] J. K. Prajapat, Certain geometric properties of normalized Bessel functions, Appl. Math. Lett. 24 (2011), 2133-2139.

[16] J. K. Prajapat, Certain geometric properties of the Wright functions, Integral Transforms Spec. Funct. 26 (2015), 203-212.

[17] K. Raghavendar, A. Swaminathan, Close-to-convexity of basic hypergeometric functions using their Taylor coefficients, J. Math. Appl. 35 (2012), 53-67.

[18] St. Ruscheweyh, V. Singh, On the order of starlikeness of hypergeometric functions, J. Math. Anal. Appl. 113 (1986), 1-11.

[19] S. K. Sahoo, N. L. Sharma, On a generalization of close-to-convex functions, Ann. Polon. Math. 2426 (2015), 93-108

[20] S. K. Sharma, R. Jain, On some properties of generalized q-Mittag Leffler function, Math. Aeterna 4 (2014), 613-619.

[21] H. M. Srivastava, On an extension of the Mittag-Leffler function, Yokohama Math. J. 16 (1968), 77-88.

[22] H. M. Srivastava, Certain q-polynomial expansions fot functions of several variables. I and II, IMA J. Appl. Math. 30 (1983), 315-323; ibid. 33 (1984), 205-209.

[23] H. M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in Univalent Functions, Fractional Calculus, and Their Applications (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), pp. 329-354, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[24] H. M. Srivastava, Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators, Appl. Anal. Discrete Math. 1 (2007), 56-71.

[25] H. M. Srivastava, Some families of Mittag-Leffler type functions and associated operators of fractional calculus, TWMS J. Pure Appl. Math. 7 (2016), 123-145.

[26] H. M. Srivastava, S. Owa, Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[27] H. M. Srivastava, J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[28] H. M. Srivastava, P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[29] A. Wiman, Über den Fundamentalsatz in der Theorie der Funcktionen $E_\alpha(x)$, Acta Math. 29 (1905), 191-201.

[30] A. Wiman, Über die Nullstellen der Funktionen $E_\alpha(x)$, Acta Math. 29 (1905), 217-234.