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EXISTENCE OF ANTI-PERIODIC SOLUTIONS FOR NONLINEAR IMPLICIT EVOLUTION EQUATIONS WITH TIME DEPENDENT PSEUDOMONOTONE OPERATORS

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Abstract. This paper is devoted to study the existence of anti-periodic solutions for implicit nonlinear differential equations associated to a time-dependent pseudomonotone (or quasimonotone) operator in the sense of Brézis. The method adopted in this paper is new and differs from the most used technics in literature, it is based on recent results on the theory of equilibrium problems. By this approach, we provide some new results which improve and unify most of the recent results obtained in this direction.

Keywords. Evolution equation; Equilibrium problem; Maximal monotone operator; Pseudomonotone operator; Quasimonotone operator.

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1. Introduction

We consider in a real Hilbert space V the following anti-periodic problem of implicit nonlinear differential equations

$$\begin{cases} \frac{d}{dt}(Bx(t)) + \mathscr{A}(t)x(t) = f(t), \quad \text{a.e. } t \in (0,T), \\ Bx(0) = -Bx(T), \end{cases} \tag{1.1}$$

where B is a linear bounded, symmetric and positive operator from V to V^* (the topological dual of V), $\mathscr{A}(t) : V \to V^*$ is a nonlinear time-dependent operator, and $f : [0,T] \to V^*$ is a functional.

Anti-periodic solutions arise naturally in the mathematical modeling of a variety of physical processes, see [5, 8, 19]. When B is the identity operator, the study of anti-periodic solutions for nonlinear evolution equations was initiated by Okochi [22, 23] in Hilbert spaces.

Many problems for nonlinear evolution equations related to problem (1.1) have been treated by using the theory of monotone operators (see [2]). Usually these equations are governed by the sum of a discontinuous monotone operator and a Nemytskij operator satisfying suitable smoothness and growth

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conditions. In contrast, the problem we study here is governed by a time-dependent pseudomonotone (or quasimonotone) operator in the sense of Brézis.

Recently, Liu and Liu [20], considered problem (1.1) when operator $\mathscr{A}$ is not time dependent, more precisely they considered the problem with $\mathscr{A} = A + G$ where A is a monotone operator and G is an operator which is both continuous and weak continuous. The method adopted in [20] is essentially based on a result related to pseudomonotone perturbations of maximal monotone operators, and it consists to use the maximal monotone property of the derivative operator with anti-periodic conditions and a convergent approximation procedure.

In this paper, we study problem (1.1) by a new approach based recent results on the theory of equilibrium problems, see the survey paper by Bigi-Castellani-Pappalardo-Passacantando [14] on this theory and the references therein. This new approach provides some new results which improve and unify most of the recent results obtained in this direction.

The paper is organized as the following. In the second section we give basic concepts related to operator theory and to equilibrium problems. We give also some preliminary results that will be used in the study of problem (1.1). In Section 3, we introduce an auxiliary problem and we prove a Hirano's Lemma for bifunctions. We give also several existence results for the auxiliary problem. In Section 4, we give the main results on the existence of solutions for problem (1.1). A conclusion and comparison with recent results is given at the end of the paper.

2. Preliminaries

Let V, H be real Hilbert spaces. V^* stands for the dual space of V and we suppose that the embeddings $V \subset H \cong H^* \subset V^*$ are dense and continuous, in this case we call (V,H,V^*) an evolution triple, also known as "Gelfand-Triple", see [26, Chapter 13]. The inner product in H is denoted by $(\cdot,\cdot)_H$. The norm of any Banach space U is denoted by $\|\cdot\|_U$. The duality pairing between U and U^* is denoted by $\langle\cdot,\cdot\rangle$. For each finite subset N of U, we denote by $co(N)$ the convex hull of N. The closure of a subset M of U will be denoted by $cl(M)$. For a multi-valued mapping $A : U \to 2^{U^*}$, we denote by $\mathscr{D}(A) := \{u \in U : A(u) \neq \emptyset\}$ the domain of A and by $G(A) := \{(u,u^*) : u \in D(T) \text{ and } u^* \in A(u)\}$ the graph of A. We denote by $\mathscr{J}$ the duality mapping from U into U^*, i.e., for each $u \in U$, $\mathscr{J}(u) = \{v \in U^* : \langle v,u\rangle = \|u\|_U^2 = \|v\|_{U^*}^2\}$. By using the Asplund's renorming theorem (see [3, Theorem 1.105]), we may assume that $\mathscr{J}$ is a single-valued monotone and demicontinuous mapping, see [4, Theorem 1.2]. Let p, p' and T be constants such that $T > 0$, $p \geq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $X = L^p(0,T;V)$, $X^* = L^{p'}(0,T;V^*)$ and $W^{1,p}([0,T];V,H) := \{u \in X : u' \in X^*\}$ be the Bochner-Sobolev space, where u' is the generalized derivative of u. The embedding $W^{1,p}([0,T];V,H) \subset C([0,T];H)$ is continuous, see [26, Proposition 23.23]. We use the standard notation "$\to$" to denote the strong convergence of a sequence, and "$\rightharpoonup$" to denote the weak convergence.

Now we formulate the assumptions needed in our study.

Assumptions: Let $B \in L(V,V^*)$, where $L(V,V^*)$ denotes the space of all bounded linear operators from V to V^*, and $\{\mathscr{A}(t) : t \in [0,T]\}$ be a family of operators from V to V^* with the following properties:

$[\mathbf{H}_1]$ $\langle Bu,u\rangle \geq 0$ for all $u \in V$ and B is symmetric;

[**H**$_2$] For $t \in [0,T]$ and $w \in V$, the mapping $u \in V \mapsto \langle \mathscr{A}(t)u, w-u\rangle$ is upper semicontinuous on $co(N)$ for each finite subset N of V;

[**H**$_3$] The function $t \mapsto \langle \mathscr{A}(t)u, w\rangle_V$ is measurable on [0,T] for all $u, w \in V$;

[**H**$_4$] There exists a constant $k_0 > 0$, as well a function $k_1 \in L^{p'}(]0,T[)$ such that $\|\mathscr{A}(t)u\|_{V^*} \leq k_0\|u\|_V^{p-1} + k_1(t)$ for all $u \in V$ and almost every $t \in [0,T]$;

[**H**$_5$] There exists a positive constant α_0 and a function $\alpha_1 \in L^1(]0,T[)$ such that $\langle \mathscr{A}(t)u, u\rangle_V \geq \alpha_0\|u\|_V^p - \alpha_1(t)$ for all $u \in V$ and almost every $t \in [0,T]$.

We recall the following concepts of mappings of monotone type which are presented here in the general setting of a Banach space U.

Definition 2.1. Let $A : \mathscr{D}(A) \subset U \to U^*$ be an operator with domain $\mathscr{D}(A)$. Then, A is said to be

(1) monotone if $\langle A(u) - A(v), u-v\rangle \geq 0$ for all $u, v \in \mathscr{D}(A)$;

(2) maximal monotone if A is monotone and $\langle v^* - A(u), v-u\rangle \geq 0$ for all $u \in \mathscr{D}(A)$ implies $A(v) = v^*$, i.e. A has no proper monotone extension;

(3) pseudomonotone in the sense of Brézis (in short B-PMO) if for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(A)$ with $u_n \rightharpoonup u$ in U and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$, we have

$$\liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle \geq \langle A(u), u-v\rangle, \quad \forall v \in U;$$

(4) quasimonotone in topological sense (in short T-QMO) if for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(A)$ with $u_n \rightharpoonup u$ in U, we have $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \geq 0$;

(5) demicontinuous if $u_n \to u$ in U implies $A(u_n) \rightharpoonup A(u)$ in U^*;

(6) hemicontinuous (respectively, upper hemicontinuous) if for all $u, v, w \in U$, the functional $t \in [0,1] \mapsto \langle T(u+tv), w\rangle$ is continuous (respectively, upper semicontinuous).

We recall in the following definition the well-known concept of generalized pseudomonotone maps with respect to the domain of a linear maximal monotone operator (see [24, 10, 11]).

Definition 2.2. Let $L : \mathscr{D}(L) \subset U \to U^*$ be a linear maximal monotone operator and $\mathscr{D}(L)$ densely embedded into U. A operator $A : U \to U^*$ is said to be L-generalized pseudomonotone (in short L-GPMO) if for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(L)$ with $u_n \rightharpoonup u$ and $Lu_n \rightharpoonup Lu$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$, we have

$$\liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle \geq \langle A(u), u-v\rangle, \quad \forall v \in U.$$

The definition of L-quasimonotone mapping (in short L-QMO) with respect to $\mathscr{D}(L)$ is given accordingly.

Since the approach developed in this paper is based on recent results obtained for some Ky Fan type minimax inequalities, or what is actually known in literature as "Equilibrium Problems" (see [7]), we recall for bifunctions the concepts introduced in Definition 2.1.

Definition 2.3. Let K be a nonempty closed convex subset of U. A real-valued bifunction $F : K \times K \to \mathbb{R}$ is said to be

(i) monotone if $F(u,v) + F(v,u) \leq 0$, $\forall u, v \in K$;

(ii) pseudomonotone in the sense of Brézis (in short, B-PMB) if for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset K$ such that $u_n \rightharpoonup u$ in K, we have $\liminf_{n\to\infty} F(u_n,u) \geq 0$ implies $\limsup_{n\to\infty} F(u_n,v) \leq F(u,v)$, $\forall v \in K$;

(iii) quasimonotone in topological sense (in short, T-QMB) if for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset K$ such that $u_n \rightharpoonup u$ in K, we have $\liminf_{n\to\infty} F(u_n,u) \leq 0$;

(iv) hemicontinuous (respectively, upper hemicontinuous) if for all $u,v \in K$, the functional $t \in [0,1] \mapsto F(tu+(1-t)v,u)$ is continuous (respectively, upper semicontinuous).

Remark 2.1. The pseudomonotonicity notion for bifunctions in the sense of Brézis has been considered first by Gwinner [15, 16], it is motivated by the concept of pseudomonotonicity in the sense of Brézis for operators (see [9]). The quasimonotonocity concept of bifunctions is considered here in a topological sense and represent an extension to bifunctions of the corresponding concept for nonlinear operators, see [6, 18, 25] where the definition for quasimonotone operators in topological sense is considered for some special subclasses. Some properties needed in the sequel and that can be obtained easily are listed below.

(a) If $A : U \to U^*$ is B-PMO, then the bifunction $F : U \times U \to \mathbb{R}$ defined by $F(u,v) = \langle Au, v-u\rangle$ is B-PMB.

(b) If the bifunction $F : U \times U \to \mathbb{R}$ is upper semicontinuous in the first argument with respect to the weak topology $\sigma(U,U^*)$, then it is B-PMB.

(c) If $F_1, F_2 : K \times K \to \mathbb{R}$ are B-PMB such that $F_1(u,u) \leq 0$ and $F_2(u,u) \leq 0$ for all $u \in K$, where K is a closed convex subset of U, then $F_1 + F_2$ is also B-PMB (see [12]).

Definition 2.4. Let $L : \mathscr{D}(L) \subset U \to U^*$ be a linear maximal monotone operator and $\mathscr{D}(L)$ densely embedded into U and K be a closed convex subset of U. A bifunction $F : K \times K \to \mathbb{R}$ is said to be L-generalized pseudomonotone (in short L-GPMB) if for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(L)$ with $u_n \rightharpoonup u$ and $Lu_n \rightharpoonup Lu$ and $\liminf_{n\to\infty} F(u_n,u) \geq 0$, we have $\limsup_{n\to\infty} F(u_n,v) \leq F(u,v)$, $\forall v \in K$.
The definition of L-quasimonotone bifunction (in short L-QMB) with respect to $\mathscr{D}(L)$ is given similarly.

In the following definition we introduce the concept of maximal monotonicity for bifunctions. This notion has been initiated by Blum-Oettli [7] in the aim to extend to bifunctions the notion of maximal monotonicity of operators.

Definition 2.5. [7] Let K be a nonempty closed convex subset of U and $\Phi : K \times K \to \mathbb{R}$ be a real-valued bifunction such that $\Phi(u,u) = 0$ for all $u \in K$. The bifunction Φ is said to be maximal monotone in the sense of Blum-Oettli (in short, BO-maximal monotone) if and only if for every $u \in K$ and for every convex function $\varphi : K \to \mathbb{R}$ with $\varphi(u) = 0$, we have

$$\Phi(v,u) \leq \varphi(v), \quad \forall v \in K \quad \Rightarrow \quad 0 \leq \Phi(u,v) + \varphi(v), \quad \forall v \in K.$$

Remark 2.2. For the properties of BO-maximal monotone bifunctions and their connection with maximal monotone operators, we refer to [13].

Next we will state some recent results concerning Ky Fan type minimax inequalities, which will be used in the following.

Lemma 2.1. [13] *Let U be a reflexive Banach space, K a nonempty closed convex subset of U and $\Phi, \Psi : K \times K \to \mathbb{R}$ two real-valued bifunctions such that $\Phi(u,u) = \Psi(u,u) = 0$ for all $u \in K$. Assume that*

(i) *Φ is monotone and BO-maximal monotone;*
(ii) *Φ is weakly lower semicontinuous with respect to the second argument;*
(iii) *Φ and Ψ are convex with respect to the second argument;*
(iv) *Ψ is B-PMB;*
(v) *For each finite subset N of K and each v in K fixed, the function $u \in K \mapsto \Psi(u,v)$ is upper semicontinuous on co(N);*
(vi) *(Coercivity) There exists a weakly compact subset D, such that for each $\lambda > 0$ (small enough) there exists a weakly compact convex subset B_λ of K satisfying*

$$\forall u \in K \setminus D,\ \exists v \in B_\lambda \text{ such that } \Psi(u,v) + \lambda \langle \mathscr{J} u, v-u \rangle < \Phi(v,u).$$

Then, there exists $\overline{u} \in K \cap D$ such that $\Phi(\overline{u}, v) + \Psi(\overline{u}, v) \geq 0$ for all $v \in K$.

Lemma 2.2. [13] *Let K be a nonempty, closed and convex subset of a reflexive Banach space U. Let $\Phi, \Psi : K \times K \longrightarrow \mathbb{R}$ be two real-valued bifunctions such that $\Phi(u,u) = \Psi(u,u) = 0$ for all $u \in K$. Let $\lambda > 0$ and suppose that*

(i) *Φ is monotone and BO-maximal monotone;*
(ii) *Φ is weakly lower semicontinuous with respect to the second argument;*
(iii) *Φ and Ψ are convex with respect to the second argument;*
(iv) *Ψ is T-QMB;*
(v) *Ψ is upper semicontinuous with respect to the first argument;*
(vi) *(Coercivity) There exists a weakly compact subset W_λ and a weakly compact convex subset B_λ of K satisfying*

$$\forall u \in K \setminus W_\lambda,\ \exists v \in B_\lambda \text{ such that } \Psi(u,v) + \lambda \langle \mathscr{J} u, v-u \rangle < \Phi(v,u).$$

Then, there exists $\overline{u}_\lambda \in K \cap W_\lambda$ such that $\Phi(\overline{u}_\lambda, v) + \Psi(\overline{u}_\lambda, v) + \lambda \langle \mathscr{J} \overline{u}_\lambda, v - \overline{u}_\lambda \rangle \geq 0$ for all $v \in K$.

Lemma 2.3. [13] *Let K be a nonempty, closed and convex subset of a reflexive Banach space U. Let $\Phi, \Psi, \Xi : K \times K \longrightarrow \mathbb{R}$ be real-valued bifunctions with $\Phi(u,u) = \Psi(u,u) = \Xi(u,u) = 0$ for all $u \in K$. Assume that*

(i) *Φ is monotone and BO-maximal monotone;*
(ii) *Ψ is B-PMB;*
(iii) *Ξ is T-QMB;*
(iv) *Φ, Ψ and Ξ are convex with respect to the second argument;*
(v) *Φ is weakly lower semicontinuous with respect to the second argument;*
(vi) *Ξ is upper semicontinuous with respect to the first argument;*
(vii) *For each finite subset N of K and each fixed $v \in K$, the function $u \in K \longmapsto \Psi(u,v)$ is upper semicontinuous on $co(N)$;*

(viii) *(Coercivity) There exists a weakly compact subset D, such that for each $\lambda > 0$ (small enough) there exists a weakly compact convex subset B_λ of K satisfying*

$$\forall u \in K \setminus D,\ \exists v \in B_\lambda \text{ such that } \Psi(u,v) + \Xi(u,v) + \lambda\langle \mathscr{J} u, v-u\rangle < \Phi(v,u).$$

Then, there exists $\overline{u} \in K \cap D$ such that $\Phi(\overline{u},v) + \Psi(\overline{u},v) + \Xi(\overline{u},v) \geq 0$ for all $v \in K$.

Remark 2.3. (i) If K is compact, the coercivity condition in the previous theorems can be dropped.

(ii) If U is a reflexive Banach space endowed with the weak topology $\sigma(U,U^*)$, then the coercivity condition (vi) in Lemma 2.1 and Lemma 2.2 is satisfied if the following condition holds

$$\text{(vi)'} \begin{cases} \text{there exists } v_0 \in K \text{ such that} \\ \dfrac{\Psi(u,v_0) + \lambda\langle \mathscr{J} u, v_0 - u\rangle}{\|u - v_0\|_U} \to -\infty \text{ when } \|u\|_U \to +\infty \text{ uniformly in } \lambda. \end{cases}$$

Furthermore, the family of solutions $\{\overline{u}_\lambda\}_{\lambda>0}$ generated by Lemma 2.2 is contained in a compact set D independent from $\lambda > 0$, see [13].

3. An auxiliary problem

3.1. **Formulation of the auxiliary problem.** Let $\varepsilon > 0$ be given and $\Gamma : V \to V^*$ be the canonical isomorphism. Under the assumptions on B, we see that $(\varepsilon\Gamma + B)$ is an isomorphism from V to V^*. Since B is symmetric, we can consider on V^* the inner product defined by $\langle u, v\rangle := \langle u, (\varepsilon\Gamma + B)^{-1} v\rangle_V$ for all $u, v \in V^*$. We denote V^* endowed with this inner product by $W := (V^*, \langle\cdot,\cdot\rangle_W)$ where $\langle u, v\rangle_W := \langle u, v\rangle$. We can easily see that W is a Hilbert space where the norm is denoted by $\|\cdot\|_W$. Furthermore, the two norms on V^* are equivalent since

$$\|(\varepsilon\Gamma + B)^{-1}\|^{-1/2}\|v\|_W \leq \|v\|_{V^*} \leq \|(\varepsilon\Gamma + B)\|^{1/2}\|v\|_W, \quad \text{for all } v \in V^*.$$

Let $Z = L^p(0,T;W)$. Since W is a Hilbert space, we can identify W with its dual. So, we may write $Z^* = L^{p'}(0,T;W)$. We denote the pairing between $Z = L^p(0,T;W)$ and $Z^* = L^{p'}(0,T;W)$ by $\langle\langle\cdot,\cdot\rangle\rangle$. Let $\mathscr{W} = \{v \in Z : v' \in Z^*\}$. Here v' stands for the generalized derivative of v, i.e.

$$\int_0^T v'(t)\phi(t)\, dt = -\int_0^T v(t)\phi'(t)\, dt, \quad \text{for all } \phi \in C_0^\infty([0,T]).$$

The generalized derivative $Lv = v'$ restricted to the subset $\mathscr{D}(L) = \{v \in \mathscr{W} : v(0) = -v(T)\}$ defines a linear operator $L : \mathscr{D}(L) \subset Z \to Z^*$ given for $v, z \in Z$ by $\langle\langle Lv, z\rangle\rangle = \int_0^T \langle v'(t), z(t)\rangle_W\, dt$. Note that the space $\mathscr{W}$ is a real, separable and reflexive Banach space with the norm $\|v\|_{\mathscr{W}} = \|v\|_Z + \|v'\|_{Z^*}$, the embedding $\mathscr{W} \subset C([0,T];W)$ is continuous and $\mathscr{D}(L)$ is a linear closed subspace of $\mathscr{W}$. $\mathscr{D}(L)$ equipped with the graph norm $\|v\|_L = \|v\|_Z + \|v'\|_{Z^*}$ is a reflexive Banach space. The operator $L : \mathscr{D}(L) \subset Z \to Z^*$ is densely defined, closed and maximal monotone, see [21, Proposition 1]. We denote by $\mathscr{J}$ the duality mapping from Z^* into Z, i.e., for each $v \in Z^*$,

$$\mathscr{J}(v) = \{z \in Z : \langle\langle z, v\rangle\rangle = \|z\|_Z^2 = \|v\|_{Z^*}^2\}.$$

By using the Asplund's renorming theorem (see [3, Theorem 1.105]), we may assume that $\mathscr{J}$ is a single-valued monotone and demicontinuous mapping, see [4, Theorem 1.2].

For $\varepsilon > 0$, consider the following auxiliary equation:

$$\begin{cases} ((\varepsilon\Gamma + B)x(t))' + \mathscr{A}(t)x(t) = f(t), & \text{a.e. } t \in (0,T), \\ (\varepsilon\Gamma + B)x(0) = -(\varepsilon\Gamma + B)x(T). \end{cases} \tag{3.1}$$

Define $\mathscr{A}_\varepsilon(t) : W \to W^*$ as

$$\mathscr{A}_\varepsilon(t)u = \mathscr{A}(t)((\varepsilon\Gamma + B)^{-1}u), \quad \text{for all } u \in W.$$

By considering $u(t) = (\varepsilon\Gamma + B)x(t)$, i.e. $u(0) = -u(T)$, we can write (3.1) as

$$\begin{cases} u(t)' + \mathscr{A}_\varepsilon(t)u(t) = f(t), & \text{a.e. } t \in (0,T), \\ u(0) = -u(T). \end{cases} \tag{3.2}$$

Define an operator $\widehat{\mathscr{A}_\varepsilon}$ related to $\mathscr{A}_\varepsilon$ by

$$\widehat{\mathscr{A}_\varepsilon}(u)(t) = \mathscr{A}_\varepsilon(t)u(t), \quad t \in [0,T],$$

which may be considered as the associated Nemytskij operator generated by the operator-valued function $t \mapsto \mathscr{A}_\varepsilon(t)$. By means of the operator L, we can write (3.2) as the following

$$\text{(AuxP)} \qquad \text{Find } u \in \mathscr{D}(L) \quad \text{such that} \quad Lu + \widehat{\mathscr{A}_\varepsilon}u = f. \tag{3.3}$$

Before studying the existence of solutions for the auxiliary problem, we need to show some fundamental results which are presented in the following.

3.2. **Hirano's Lemma.** In this section, we prove a generalized version of Hirano's Lemma [17]. A first version of this lemma was obtained with the assumption that V is compactly embedded in H, see [17, Proposition 2].

For $\varepsilon > 0$, consider the bifunction $\Theta_\varepsilon : \mathscr{D}(L) \times \mathscr{D}(L) \to \mathbb{R}$ defined by $\Theta_\varepsilon(u,v) = \langle\langle \widehat{\mathscr{A}_\varepsilon}u, v-u\rangle\rangle$ for $u,v \in \mathscr{D}(L)$. Here $\mathscr{D}(L)$ is a reflexive Banach space equipped with the graph norm $\|v\|_L = \|v\|_Z + \|v'\|_{Z^*}$. The bifunction Θ_ε can be written as the following: $\Theta_\varepsilon(u,v) = \int_0^T \phi_{t,\varepsilon}(u(t),v(t))dt$ where $\phi_{t,\varepsilon}$ is the real-valued bifunction defined on $W \times W$ by $\phi_{t,\varepsilon}(x,y) = \langle \mathscr{A}_\varepsilon(t)x, y-x\rangle_W$. Note that, from assumption [H_3], the function $t \in [0,T] \mapsto \phi_{t,\varepsilon}(x,y)$ is measurable.

We need the following preliminary result.

Lemma 3.1. *Suppose that $\{\mathscr{A}(t) : t \in [0,T]\}$ satisfy assumptions [H_4] and [H_5]. Then for each $x,y \in W$, there exists a function $\tau \in L^1([0,T])$ independent from x such that $\phi_{t,\varepsilon}(x,y) \le \tau(t)$ for almost every $t \in [0,T]$.*

Proof. Let $x,y \in W$ and $t \in [0,T]$. Then

$$\begin{aligned} \phi_{t,\varepsilon}(x,y) &= \langle \mathscr{A}_\varepsilon(t)x, y-x\rangle_W = \langle \mathscr{A}(t)((\varepsilon\Gamma + B)^{-1}x), (\varepsilon\Gamma + B)^{-1}(y-x)\rangle_V \\ &= \langle \mathscr{A}(t)((\varepsilon\Gamma + B)^{-1}x), (\varepsilon\Gamma + B)^{-1}y\rangle_V - \langle \mathscr{A}(t)((\varepsilon\Gamma + B)^{-1}x), (\varepsilon\Gamma + B)^{-1}x\rangle_V. \end{aligned}$$

Taking account of assumptions [H_4] and [H_5], we deduce that for almost every $t \in [0,T]$

$$\phi_{t,\varepsilon}(x,y) \le \|(\varepsilon\Gamma + B)^{-1}x\|_V^{p-1}\left[k_0\|(\varepsilon\Gamma + B)^{-1}y\|_V - \alpha_0\|(\varepsilon\Gamma + B)^{-1}x\|_V\right] + k_2(t)\|(\varepsilon\Gamma + B)^{-1}y\|_V + \alpha_1(t).$$

- If $k_0\|(\varepsilon\Gamma+B)^{-1}y\|_V-\alpha_0\|(\varepsilon\Gamma+B)^{-1}x\|_V\leq 0$, it follows

$$\phi_t(x,y)\leq\alpha_1(t)+k_2(t)\|(\varepsilon\Gamma+B)^{-1}y\|_V.$$

- If $k_0\|(\varepsilon\Gamma+B)^{-1}y\|_V-\alpha_0\|(\varepsilon\Gamma+B)^{-1}x\|_V>0$, it follows

$$\|(\varepsilon\Gamma+B)^{-1}y\|_V\|(\varepsilon\Gamma+B)^{-1}x\|_V^{p-1}<(k_0/\alpha_0)^{p-1}\|(\varepsilon\Gamma+B)^{-1}y\|_V^p.$$

Hence,

$$\phi_{t,\varepsilon}(x,y)\leq k_0(k_0/\alpha_0)^{p-1}\|(\varepsilon\Gamma+B)^{-1}y\|_V^p+k_2(t)\|(\varepsilon\Gamma+B)^{-1}y\|_V+\alpha_1(t).$$

□

Now, we show the following generalized version of Hirano's Lemma.

Lemma 3.2. *Suppose that the assumptions [H_3], [H_4] and [H_5] are satisfied. If $\mathscr{A}(t):V\to V^*$ is B-PMO for all $t\in[0,T]$, then the bifunction $\Theta_\varepsilon:\mathscr{D}(L)\times\mathscr{D}(L)\to\mathbb{R}$ is L-GPMB.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathscr{D}(L)$ such that $u_n\rightharpoonup u$ in Z, $Lu_n\rightharpoonup Lu$ in Z^* and $\liminf\Theta_\varepsilon(u_n,u)\geq 0$, let us verify that $\limsup\Theta_\varepsilon(u_n,v)\leq\Theta_\varepsilon(u,v)$ for all $v\in\mathscr{D}(L)$. Let us set $z_n=(\varepsilon\Gamma+B)^{-1}u_n$ and $z=(\varepsilon\Gamma+B)^{-1}u$. Note that $(\varepsilon\Gamma+B)^{-1}$ is weakly continuous since it is bounded. Therefore, from $u_n\rightharpoonup u$ in Z and $Lu_n\rightharpoonup Lu$ in Z^* it follows that $z_n\rightharpoonup z$ in X and $z_n'\rightharpoonup z'$ in X^*. From the evolution triple (see [26, Chapter 23]), we can write for any $n\in\mathbb{N}$, $z_n(t)=z_n(0)+\int_0^t z_n'(s)ds$, where $z_n:[0,T]\to V^*$ is absolutely continuous and $z_n\in C([0,T];H)$. Furthermore, since the embedding $W^{1,p}([0,T];V,H)\subset C([0,T];H)$ is continuous, we deduce that $\{z_n\}$ is bounded in C([0,T];H), and hence we may assume $z_n(0)\rightharpoonup z(0)$ in H. For each $w\in V\subset H$ we have

$$\langle z_n(t),w\rangle_V=(z_n(t),w)_H=\left(z_n(0)+\int_0^t z_n'(s)ds,w\right)_H=(z_n(0),w)_H+\int_0^t\left\langle z_n'(s),w\right\rangle_V ds.$$

Since $z_n'\rightharpoonup z'$ in X^*, we obtain

$$\begin{aligned}\lim\langle z_n(t),w\rangle_V=\lim(z_n(0),w)_H+\lim\textstyle\int_0^t\langle z_n'(s),w\rangle_V ds &= (z(0),w)_H+\textstyle\int_0^t\langle z'(s),w\rangle_V ds\\ &= \left(z(0)+\textstyle\int_0^t z'(s)ds,w\right)_H\\ &= \langle z(t),w\rangle_V.\end{aligned}\tag{3.4}$$

Hence, $z_n(t)\rightharpoonup z(t)$ in V^* and therefore $u_n(t)\rightharpoonup u(t)$ in W, for all $t\in[0,T]$.

Now, consider $\rho_n(t)=\phi_{t,\varepsilon}(u_n(t),u(t))$ for $t\in[0,T]$. First we prove that

$$\limsup\int_0^T\rho_n(t)dt\leq 0.\tag{3.5}$$

Indeed, from Lemma 3.1, there exists a non negative function $\tau\in L^1([0,T])$ such that

$$\rho_n(t)\leq\tau(t),\quad\text{for almost every } t\in[0,T].\tag{3.6}$$

By using Fatou's lemma, we deduce that

$$\limsup\int_0^T\rho_n(t)dt\leq\int_0^T\limsup\rho_n(t)dt.\tag{3.7}$$

Now suppose by contradiction that $\limsup \rho_n(t_0) > 0$ for some $t_0 \in [0,T]$. Let $\{\rho_{n_k}(t_0)\}_{k\in\mathbb{N}}$ be a subsequence of $\{\rho_n(t_0)\}_{n\in\mathbb{N}}$ such that $\lim \rho_{n_k}(t_0) > 0$. It follows, by using assumptions [H$_4$] and [H$_5$], that $\{u_{n_k}(t_0)\}_{k\in\mathbb{N}}$ is bounded in W. Hence, for a subsequence of $\{u_{n_k}(t_0)\}_{k\in\mathbb{N}}$ also denoted $\{u_{n_k}(t_0)\}_{k\in\mathbb{N}}$, we have $u_{n_k}(t_0) \rightharpoonup u^*$ in W, $u^* \in W$. We can verify, by using the same approach as in relation (3.4), that $u^* = u(t_0)$. On the other hand, we have that the operator $\mathscr{A}_\varepsilon(t_0) : W \to W^*$ is B-PMO since $\mathscr{A}(t_0)$ is B-PMO. Therefore

$$\limsup \phi_{t_0,\varepsilon}(u_{n_k}(t_0), v) \leq \phi_{t_0,\varepsilon}(u(t_0), v), \quad \text{for all } v \in W. \tag{3.8}$$

By considering $v = u(t_0)$ in relation (3.8), we obtain $\limsup \rho_{n_k}(t_0) \leq 0$, which leads to a contradiction. Hence,

$$\limsup \rho_n(t) \leq 0, \quad \text{for all } t \in [0,T]. \tag{3.9}$$

Consequently, relation (3.5) follows from (3.7) and (3.9). Thus, relation (3.5) and the assumption that $\liminf \Theta_\varepsilon(u_n, u) \geq 0$, lead us to obtain

$$\lim \Theta_\varepsilon(u_n, u) = \lim \int_0^T \phi_{t,\varepsilon}(u_n(t), u(t))dt = \lim \int_0^T \rho_n(t)dt = 0.$$

Consider $\rho_n^+(t) = \max\{\rho_n(t), 0\}$ and $\rho_n^-(t) = \rho_n^+(t) - \rho_n(t)$. From relation (3.6), we have that $0 \leq \rho_n^+(t) \leq \tau(t)$ for all $t \in [0,T]$. On the other hand, relation (3.9) permit us to have $\lim \rho_n^+(t) = 0$ for all $t \in [0,T]$. Hence, by the Lebesgue's dominated convergence theorem we get $\lim \int_0^T \rho_n^+(t)dt = 0$. Since

$$\limsup \int_0^T |\rho_n(t)|dt = \limsup \int_0^T \rho_n^+(t) + \rho_n^-(t)dt = \limsup \int_0^T 2\rho_n^+(t) - \rho_n(t)dt = 2\limsup \int_0^T \rho_n^+(t)dt,$$

it follows that $\limsup \int_0^T |\rho_n(t)|dt = 0$, i.e. $\rho_n \to 0$ in $L^1([0,T])$. Thus, there exists a measurable subset N of $[0,T]$ with $meas(N) = 0$, and a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ of $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\rho_{n_k}(t) \to 0$ for all $t \in [0,T] \setminus N$. Let $s \in [0,T] \setminus N$, then from $\rho_{n_k}(s) = \phi_{s,\varepsilon}(u_{n_k}(s), u(s)) \to 0$ and from assumptions [H$_4$] and [H$_5$], we deduce that $\{u_{n_k}(s)\}_{k\in\mathbb{N}}$ is bounded in W. Hence for a subsequence also denoted $\{u_{n_k}(s)\}_{k\in\mathbb{N}}$, we have $u_{n_k}(s) \rightharpoonup u(s)$. Since $\mathscr{A}_\varepsilon(s)$ is B-PMO, we conclude that

$$\limsup \phi_{s,\varepsilon}(u_{n_k}(s), v) \leq \phi_{s,\varepsilon}(u(s), v), \quad \text{for all } v \in W.$$

Therefore, for any $w \in \mathscr{D}(L)$, we have

$$\Theta_\varepsilon(u, w) = \int_0^T \phi_{t,\varepsilon}(u(t), w(t))dt \geq \int_0^T \limsup \phi_{t,\varepsilon}(u_{n_k}(t), w(t))dt.$$

From Lemma 3.1 and Fatou's lemma, we obtain

$$\Theta_\varepsilon(u, w) \geq \limsup \int_0^T \phi_{t,\varepsilon}(u_{n_k}(t), w(t))dt.$$

Thus, $\Theta_\varepsilon(u, w) \geq \limsup \Theta_\varepsilon(u_n, w)$ for any $w \in \mathscr{D}(L)$, which completes the proof. □

We end this section by the following Hirano's type Lemma considered for an operator quasimonotone in the topological sense.

Lemma 3.3. *Suppose that the assumptions [H$_3$], [H$_4$] and [H$_5$] are satisfied. If $\mathscr{A}(t) : V \to V^*$ is T-QMO for all $t \in [0,T]$, then the bifunction $\Theta_\varepsilon : \mathscr{D}(L) \times \mathscr{D}(L) \to \mathbb{R}$ is L-QMB.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(L)$ satisfying $u_n \rightharpoonup u$ in Z and $Lu_n \rightharpoonup Lu$ in Z^*. Let us set $z_n = (\varepsilon\Gamma + B)^{-1}u_n$ and $z = (\varepsilon\Gamma + B)^{-1}u$. We have that $z_n(t) \rightharpoonup z(t)$ in V for all $t \in [0,T]$, and hence $u_n(t) \rightharpoonup u(t)$ in W. Furthermore, for every $t \in [0,T]$ the bifunction $\phi_{t,\varepsilon} : W \times W \to \mathbb{R}$ is T-QMB since $\mathscr{A}(t) : V \to V^*$ is T-QMO. On the other hand, by Lemma 3.1, there exists a non negative function $\tau \in L^1([0,T])$ such that

$$\phi_{t,\varepsilon}(u_n(t),u(t)) \leq \tau(t), \quad \text{for all } t \in [0,T].$$

From Fatou's lemma, we derive that

$$\limsup \int_0^T \phi_{t,\varepsilon}(u_n(t),u(t))dt \leq \int_0^T \limsup \phi_{t,\varepsilon}(u_n(t),u(t))dt. \tag{3.10}$$

In order to conclude, we verify that $\limsup \phi_{t,\varepsilon}(u_n(t),u(t)) \leq 0$ for all $t \in [0,T]$. To this aim, suppose by contradiction that there exists $t_0 \in [0,T]$ such that $\limsup \phi_{t_0,\varepsilon}(u_n(t_0),u(t_0)) > 0$. Hence, there exists a subsequence $\{u_{n_k}(t_0)\}_{k\in\mathbb{N}}$ of $\{u_n(t_0)\}_{n\in\mathbb{N}}$ such that

$$\lim \phi_{t_0,\varepsilon}(u_{n_k}(t_0),u(t_0)) > 0. \tag{3.11}$$

It follows, by using assumptions [H4] and [H5], that $\{z_{n_k}(t_0)\}_{k\in\mathbb{N}}$ is bounded in V, and hence $\{u_{n_k}(t_0)\}_{k\in\mathbb{N}}$ is bounded in W. Therefore, for a subsequence of $\{u_{n_k}(t_0)\}_{k\in\mathbb{N}}$ also denoted $\{u_{n_k}(t_0)\}_{k\in\mathbb{N}}$, we have $u_{n_k}(t_0) \rightharpoonup u(t_0) \in W$. Since the bifunction $\phi_{t_0,\varepsilon}$ is T-QMB, we derive that $\liminf \phi_{t_0,\varepsilon}(u_{n_k}(t_0),u(t_0)) \leq 0$, which contradicts relation (3.11). Therefore, $\limsup \phi_{t,\varepsilon}(u_n(t),u(t)) \leq 0$ for all $t \in [0,T]$. Consequently, relation (3.10) permits us to obtain

$$\liminf \Theta_\varepsilon(u_n,u) = \liminf \int_0^T \phi_{t,\varepsilon}(u_n(t),u(t))dt \leq \limsup \int_0^T \phi_{t,\varepsilon}(u_n(t),u(t))dt \leq 0.$$

This completes the proof. □

3.3. **Existence of solutions for the auxiliary problem.** Now, we shall study the existence of solutions for the auxiliary problem (3.3) where the operator $\mathscr{A}(t) : V \to V^*$ is B-PMO (respectively T-QMO) for any $t \in [0,T]$. To this aim, we consider the following mixed equilibrium problem (or Ky Fan minimax type inequality problem) defined on $\mathscr{D}(L)$:

$$\text{Find } u \in \mathscr{D}(L) \text{ such that } \Phi(u,v) + \Psi_\varepsilon(u,v) \geq 0, \quad \text{for all } v \in \mathscr{D}(L), \tag{3.12}$$

where $\mathscr{D}(L)$ is considered here as a reflexive Banach space equipped with the graph norm $\|v\|_L = \|v\|_Z + \|v'\|_{Z^*}$, and the bifunctions Φ and Ψ are defined for $u,v \in \mathscr{D}(L)$ by

$$\Phi(u,v) = \langle\langle Lu, v-u\rangle\rangle \text{ and } \Psi_\varepsilon(u,v) = \Theta_\varepsilon(u,v) + \Psi(u,v),$$

with $\Psi(u,v) = \langle\langle f, u-v\rangle\rangle$ and $\Theta_\varepsilon(u,v) = \langle\langle \widehat{\mathscr{A}_\varepsilon} u, v-u\rangle\rangle = \int_0^T \phi_{t,\varepsilon}(u(t),v(t))dt$, here $\phi_{t,\varepsilon}$ is the real-valued bifunction defined on $W \times W$ by $\phi_{t,\varepsilon}(x,y) = \langle \mathscr{A}_\varepsilon(t)x, y-x\rangle_W$.

We shall study the problem (3.12) in $\mathscr{D}(L)$ considered here as a reflexive Banach space equipped with the graph norm $\|v\|_L = \|v\|_Z + \|v'\|_{Z^*}$. Hence, on $\mathscr{D}(L)$ equipped with the graph norm $\|v\|_L = \|v\|_Z + \|v'\|_{Z^*}$, the concept of pseudomonotone bifunctions in the sense of Brézis introduced in Definition 2.3 is traduced for an arbitrary bifunction $F : \mathscr{D}(L) \times \mathscr{D}(L) \to \mathbb{R}$ by the following: If for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(L)$ satisfying $u_n \rightharpoonup u$ in Z, $Lu_n \rightharpoonup Lu$ in Z^* and $\liminf F(u_n,u) \geq 0$, we have that $\limsup F(u_n,v) \leq \Psi(u,v)$ for all $v \in \mathscr{D}(L)$. This leads to the concept of L-generalized pseudomonotone bifunction introduced in Definition 2.4. In an obvious similar way, we obtain the L-quasimonotonicity concept for a bifunction with respect to $\mathscr{D}(L)$.

We need to show the following preliminary results.

Lemma 3.4. *Let the assumption [H_2] be satisfied. Then, for each finite subset N of $\mathscr{D}(L)$ the bifunction Θ_ε is upper semicontinuous with respect to the first argument on $co(N)$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset co(N)$ such that $u_n \rightharpoonup u$ in Z. Since on $co(N)$ the weak and strong convergence coincide, it follows that $u_n \to u \in co(N)$, i.e. $\lim \int_0^T \|u_n(t) - u(t)\|_W^p dt = 0$. Consequently, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ such that $u_{n_k}(t) \to u(t)$ for a.e. $t \in [0,T]$. Hence, by assumption [H_2], we have for all $z \in Z$ and a.e. $t \in [0,T]$, $\limsup \phi_{t,\varepsilon}(u_{n_k}(t), z(t)) \leq \phi_{t,\varepsilon}(u(t), z(t))$. Taking account of Lemma 3.1, we obtain by Fatou's lemma

$$\limsup \int_0^T \phi_{t,\varepsilon}(u_{n_k}(t), z(t))dt \leq \int_0^T \limsup \phi_{t,\varepsilon}(u_{n_k}(t), z(t))dt \leq \int_0^T \phi_{t,\varepsilon}(u(t), z(t))dt.$$

By using a contradiction argument, we obtain that the inequality follows for the whole sequence. □

Remark 3.1. The result of Lemma 3.4 is obtained in the particular case when the operator $\mathscr{A}(t) : V \to V^*$ is monotone, hemicontinuous and satisfies assumption [H_4]. Indeed, it is well known that an operator which is monotone and hemicontinuous is B-PMO. Therefore, if it is locally bounded then it is demicontinous and hence the assumption [H_2] is verified.

Lemma 3.5. *Let the assumptions [H_1], [H_4] and [H_5] be satisfied. Further assume that for all $t \in [0,T]$ the operator $\mathscr{A}(t) : V \to V^*$ is demicontinuous. Then, the bifunction Θ_ε is upper semicontinuous with respect to the first argument.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in Z such that $u_n \to u$. Hence, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ such that $u_{n_k}(t) \to u(t)$ for almost all $t \in [0,T]$. By assumption [H_1] and the demicontinuity of the operator $\mathscr{A}(t)$, we deduce that $\mathscr{A}_\varepsilon(t)(u_{n_k}(t)) \rightharpoonup \mathscr{A}_\varepsilon(t)(u(t))$ a.e. $t \in [0,T]$. It follows, for any $w \in Z$ and almost every $t \in [0,T]$, that

$$\phi_{t,\varepsilon}(u_{n_k}(t), w(t)) \to \phi_{t,\varepsilon}(u(t), w(t)).$$

By Lemma 3.1 and the Lebesgue's dominated convergence theorem, we deduce that $\Theta_\varepsilon(u_{n_k}, w) \to \Theta_\varepsilon(u, w)$ for any $w \in Z$. By a contradiction argument, we can verify the convergence for the all sequence. Therefore, for any fixed $w \in Z$, the function $u \in Z \mapsto \Theta_\varepsilon(u, w)$ is continuous and therefore it is upper semicontinuous. □

Now, we are in position to give an existence result for the auxiliary problem (3.3) when the operator $\mathscr{A}(t) : V \to V^*$ is B-PMO for $t \in [0,T]$.

Theorem 3.1. *Let the assumptions [H_1]-[H_5] be satisfied and $\mathscr{A}(t) : V \to V^*$ is B-PMO for all $t \in [0,T]$. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in C([0,T]; W) \cap Z$ such that $u'_\varepsilon \in Z^*$ and*

$$\begin{cases} u'_\varepsilon(t) + \mathscr{A}_\varepsilon(t)(u_\varepsilon(t)) = f(t), & a.e. \ \ t \in (0,T), \\ u_\varepsilon(0) = -u_\varepsilon(T). \end{cases} \tag{3.13}$$

Proof. First, we show that problem (3.12) has at least one solution. To this aim, we shall apply Lemma 2.1. Since the operator L is maximal monotone, it follows, by Lemma 2.3 in [13] and Proposition 3.8 in [1], that the bifunction Φ is monotone and BO-maximal monotone, and hence condition (i) of Lemma 2.1 is satisfied. Conditions (ii) and (iii) of Lemma 2.1 are easy to obtain and condition (v) is a direct consequence of Lemma 3.4. For condition (iv) of Lemma 2.1, we have, by Lemma 3.2, that the bifunction Θ_ε is B-PMB with respect to $\mathscr{D}(L)$; and since the bifunction Ψ is continuous in the first argument, it follows, by taking account of Remark 2.1 (b)-(c), that the bifunction $\Psi_\varepsilon = \Theta_\varepsilon + \Psi$ is B-PMB with respect to $\mathscr{D}(L)$. For the proof of condition (vi) of Lemma 2.1, we shall use Remark 2.3 (ii). To this end, consider $v_0 = 0 \in \mathscr{D}(L)$. From assumptions [H$_4$] and [H$_5$], and since the two norms $\|\cdot\|_{V^*}$ and $\|\cdot\|_W$ are equivalent in V^*, there exists constants $\gamma_1 > 0$, $\gamma_2 > 0$ such that

$$\begin{aligned}\Psi_\varepsilon(u, v_0) + \lambda\langle \mathscr{J} u, v_0 - u\rangle &\leq -\gamma_1\|u\|_Z^p + \gamma_2\|u\|_Z^{p-1} + \|f\|_{Z^*}\|u\|_Z - \lambda\|u\|_Z^2 \\ &\leq -\gamma_1\|u\|_Z^p + \gamma_2\|u\|_Z^{p-1} + \|f\|_{Z^*}\|u\|_Z.\end{aligned}$$

It follows

$$\frac{\Psi_\varepsilon(u, v_0) + \lambda\langle \mathscr{J} u, v_0 - u\rangle}{\|u - v_0\|_Z} \to -\infty \text{ when } \|u\|_Z \to +\infty \text{ uniformly in } \lambda.$$

Hence problem (3.12) has a solution. Since $\mathscr{D}(L)$ is dense in Z and $\mathscr{W}$ is continuously embedded in $C([0,T];W)$, we obtain the desired result. □

When for $t \in [0,T]$, the operator $\mathscr{A}(t) : V \to V^*$ is T-QMO, we obtain the following results on the existence of solutions for the auxiliary problem (3.3).

Theorem 3.2. *Let the assumptions [H$_1$], [H$_3$], [H$_4$] and [H$_5$] be satisfied. Suppose that $\mathscr{A}(t) : V \to V^*$ is T-QMO and demicontinuous for all $t \in [0,T]$. Then for $f \in X^*$, $\varepsilon > 0$ and $\lambda > 0$, there exists $u_\varepsilon \in C([0,T];W) \cap Z$ such that $u_\varepsilon' \in Z^*$ and*

$$\begin{cases} u_\varepsilon'(t) + \mathscr{A}_\varepsilon(t)(u_\varepsilon(t)) + \lambda \mathscr{J}(u_\varepsilon(t)) = f(t), \quad a.e.\ t \in (0,T), \\ u_\varepsilon(0) = -u_\varepsilon(T). \end{cases}$$

Proof. By following the same approach used in the proof of the previous theorem, we shall show that the problem (3.12) has at least one solution under the considered assumptions. This will be treated by using Lemma 2.2. To this aim, we need to verify that all the assumptions of Lemma 2.2 are satisfied. Conditions (i), (ii) and (iii) of Lemma 2.2 are obtained similarly as in the proof of the previous theorem. By Lemma 3.3, we have that the bifunction Θ_ε is L-QMB, hence it is T-QMB with respect to $\mathscr{D}(L)$. Since Ψ is linear and continuous with respect to the second argument, it follows that the bifunction $\Psi_\varepsilon = \Theta_\varepsilon + \Psi$ is T-QMB with respect to $\mathscr{D}(L)$, and hence condition (iv) of Lemma 2.2 is satisfied. On the other hand, since the operator $\mathscr{A}(t) : V \to V^*$ is demicontinuous, it follows by Lemma 3.5 that the bifunction Θ_ε is upper semicontinuous with respect to the first argument in $\mathscr{D}(L)$. Therefore, the bifunction Ψ_ε is upper semicontinuous with respect to the first argument in $\mathscr{D}(L)$, and hence condition (v) of Lemma 3.5 is satisfied. The coercivity assumption (vi) of Lemma 3.5 is obtained similarly as in the proof of the previous theorem. Therefore, for $\varepsilon > 0$ and $\lambda > 0$ there exists $u_\varepsilon \in \mathscr{D}(L)$ such that

$$\Phi(u_\varepsilon, v) + \Psi_\varepsilon(u_\varepsilon, v) + \lambda\langle\langle \mathscr{J} u_\varepsilon, v - u_\varepsilon\rangle\rangle \geq 0 \text{ for all } v \in \mathscr{D}(L). \tag{3.14}$$

Consequently, from the density of $\mathscr{D}(L)$ in Z and since $\mathscr{W}$ is continuously embedded in $C([0,T];W)$, we obtain the conclusion of the theorem. $\square$

Remark 3.2. The previous theorem gives in fact an approximated solution for the auxiliary problem associated to a time-dependent operator which is T-QMO. In the next theorem, we examine a situation in which it is possible to obtain exact solutions instead of approximated ones for the auxiliary problem associated to a T-QMO time-dependent operator.

Theorem 3.3. *Let the assumptions [H_1], [H_3], [H_4] and [H_5] be satisfied. Suppose that $\mathscr{A}(t): V \to V^*$ is T-QMO and weakly continuous for all $t \in [0,T]$. Then for $f \in X^*$ and $\varepsilon > 0$, there exists $u_\varepsilon \in C([0,T];W) \cap Z$ such that $u'_\varepsilon \in Z^*$ and*

$$\begin{cases} u'_\varepsilon(t) + \mathscr{A}_\varepsilon(t)(u_\varepsilon(t)) = f(t), & a.e.\ \ t \in (0,T), \\ u_\varepsilon(0) = -u_\varepsilon(T). \end{cases}$$

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of real positive numbers such that $\lambda_n \to 0$. Note that $\mathscr{A}(t)$ is demicontinuous for all $t \in [0,T]$ since it is weakly continuous, it follows from the previous theorem, that for each $\lambda_n > 0$, there exists $u_{\varepsilon,n} \in \mathscr{D}(L)$ satisfying relation (3.14). From assumptions [H_4] and [H_5], we deduce, by taking account of Remark 2.3 (ii), that the sequence $\{u_{\varepsilon,n}\}_{n\in\mathbb{N}}$ is bounded. Hence, for a subsequence of $\{u_{\varepsilon,n}\}_{n\in\mathbb{N}}$ also denoted $\{u_{\varepsilon,n}\}_{n\in\mathbb{N}}$, we have $u_{\varepsilon,n} \rightharpoonup u_\varepsilon \in \mathscr{D}(L)$. On the other hand, since $u_{\varepsilon,n}$ satisfies relation (3.14), it follows by the monotonicity of Φ that for each $v \in \mathscr{D}(L)$

$$\Theta_\varepsilon(u_{\varepsilon,n}, u_\varepsilon) + \Psi(u_{\varepsilon,n}, v) + \langle\langle \widehat{\mathscr{A}_\varepsilon} u_{\varepsilon,n}, v - u_\varepsilon \rangle\rangle + \lambda_n \langle\langle \mathscr{J} u_{\varepsilon,n}, v - u_{\varepsilon,n} \rangle\rangle \geq \Phi(v, u_{\varepsilon,n}). \tag{3.15}$$

On the other hand, by assumption [H_4] and since $\mathscr{A}(t)$ is weakly continuous for all $t \in [0,T]$, we deduce by using the Lebesgue's dominated convergence theorem, that $\widehat{\mathscr{A}_\varepsilon}: Z \to Z^*$ is weakly continuous. Hence, by considering the upper limit in the inequality (3.15) and taking account of the fact that the bifunction Θ_ε is T-QMB, it follows

$$\Theta_\varepsilon(u_\varepsilon, v) + \Psi(u_\varepsilon, v) \geq \Phi(v, u_\varepsilon), \quad \text{for all } v \in \mathscr{D}(L).$$

Since Φ is BO-maximal monotone, we deduce that

$$\Phi(u_\varepsilon, v) + \Theta_\varepsilon(u_\varepsilon, v) + \Psi(u_\varepsilon, v) \geq 0, \quad \text{for all } v \in \mathscr{D}(L).$$

Therefore, from the density of $\mathscr{D}(L)$ in Z and since $\mathscr{W}$ is continuously embedded in $C([0,T];W)$, we obtain the conclusion of the theorem. $\square$

Now, we consider the case where the auxiliary problem is associated to the operator $\mathscr{A}(t)$ defined as the following

$$\mathscr{A}(t) = \mathscr{B}(t) + \mathscr{C}(t)$$

with $\mathscr{B}(t): V \to V^*$ is B-PMO and $\mathscr{C}(t): V \to V^*$ is T-QMO. The operators $\mathscr{B}_\varepsilon(t), \mathscr{C}_\varepsilon(t): W \to W^*$ are defined in the same way as the operator $\mathscr{A}_\varepsilon(t)$, the same applies to operators $\widehat{\mathscr{B}}_\varepsilon, \widehat{\mathscr{C}}_\varepsilon: Z \to Z^*$.

Theorem 3.4. *Suppose that for all $t \in [0,T]$, the operator $\mathscr{B}(t) : V \to V^*$ is B-PMO and satisfies conditions [H_2]-[H_5]. Suppose that $\mathscr{C}(t) : V \to V^*$ is T-QMO and weakly continuous for all $t \in [0,T]$. Furthermore, suppose that $\mathscr{C}(t) : V \to V^*$ satisfies conditions [H_3], [H_4] and the following condition*

$$[C] \qquad \langle \mathscr{C}(t)u, u\rangle \geq -\delta_1 \|u\|_V^p - \delta_2(t), \quad \text{for all } u \in V,\ t \in [0,T]$$

with some $\delta_1 > 0$ and $\delta_2 \in L^1(0,T)$. Then for $f \in X^*$ and $\varepsilon > 0$, there exists $u_\varepsilon \in C([0,T];W) \cap Z$ such that $u'_\varepsilon \in Z^*$ and

$$\begin{cases} u'_\varepsilon(t) + \mathscr{B}_\varepsilon(t)(u_\varepsilon(t)) + \mathscr{C}_\varepsilon(t)(u_\varepsilon(t)) = f(t), & \text{a.e. } t \in (0,T), \\ u_\varepsilon(0) = -u_\varepsilon(T). \end{cases}$$

Proof. We shall apply Lemma 2.3 on $\mathscr{D}(L)$ with the bifunctions $\Phi, \Psi_\varepsilon, \Xi_\varepsilon : \mathscr{D}(L) \times \mathscr{D}(L) \to \mathbb{R}$ are defined for $u, v \in \mathscr{D}(L)$ by

$$\Phi(u,v) = \langle\langle Lu, v-u\rangle\rangle,\ \Psi_\varepsilon(u,v) = \langle\langle \widehat{\mathscr{B}}_\varepsilon u, v-u\rangle\rangle \text{ and } \Xi_\varepsilon(u,v) = F_\varepsilon(u,v) + G(u,v),$$

where $F_\varepsilon(u,v) = \langle\langle \widehat{\mathscr{C}}_\varepsilon u, v-u\rangle\rangle$ and $G(u,v) = \langle\langle f, u-v\rangle\rangle$. We can write the bifunctions Ψ_ε and F_ε as the following:

$$\Psi_\varepsilon(u,v) = \int_0^T g_{t,\varepsilon}(u(t), v(t))dt,$$

where $g_{t,\varepsilon}$ is the bifunction defined for $x, y \in W$ by

$$g_{t,\varepsilon}(x,y) = \langle \mathscr{B}_\varepsilon(t)x, y-x\rangle_W,$$

and

$$F_\varepsilon(u,v) = \int_0^T h_{t,\varepsilon}(v(t), z(t))dt,$$

where $h_{t,\varepsilon}$ is the bifunction defined for $x, y \in W$ by $h_{t,\varepsilon}(x,y) = \langle \mathscr{C}_\varepsilon(t)x, y-x\rangle_W$. Since for any $t \in [0,T]$ the operator $\mathscr{C}(t) : V \to V^*$ is weakly continuous, it follows from assumption [H_4] and the Lebesgue's dominated convergence theorem that $\widehat{\mathscr{C}_\varepsilon} : Z \to Z^*$ is weakly continuous. Hence, the bifunction Ξ_ε is upper semicontinuous with respect to the first argument. By a similar procedure as in the previous theorems, we obtain that Φ is BO-maximal monotone, Ψ_ε is B-PMB and Ξ_ε is T-QMB. Condition (viii) of Lemma 2.3 is obtained from the assumptions [H_4], [H_5] and [C] by following the same development used in the proof of the previous theorems. The other assumptions (v), (vi) and (vii) of Lemma 2.3 are obtained easily. Therefore, there exists $u_\varepsilon \in \mathscr{D}(L)$ such that

$$\Phi(u_\varepsilon, v) + \Psi_\varepsilon(u_\varepsilon, v) + \Xi_\varepsilon(u_\varepsilon, v) \geq 0, \quad \text{for all } v \in \mathscr{D}(L).$$

We conclude by using the density of $D(L)$ in Z. □

4. Main results

By using the results obtained for the auxiliary problem introduced and studied in the previous section, we derive the existence of anti-periodic solutions for implicit differential equations associated to time-dependent pseudomonotone (or topological quasimonotone) operator in the sense of Brézis.

Theorem 4.1. *Let the assumptions [H_1]-[H_5] be satisfied and $\mathscr{A}(t): V \to V^*$ is B-PMO for all $t \in [0,T]$. Then, there exists $x \in X$ such that $Bx \in L^p([0,T];V^*)$, $(Bx)' \in L^{p'}([0,T];V^*)$ and*

$$\begin{cases} \frac{d}{dt}(Bx(t)) + \mathscr{A}(t)x(t) = f(t), \quad a.e. \;\; t \in (0,T), \\ Bx(0) = -Bx(T). \end{cases}$$

Proof. From Theorem 3.1, we have that for each $\varepsilon > 0$ there exists $u_\varepsilon \in \mathscr{D}(L)$ solution of the auxiliary problem (3.13). Hence, there exists $x_\varepsilon \in X$ such that $x'_\varepsilon \in X^*$ and

$$\begin{cases} ((\varepsilon\Gamma + B)x_\varepsilon(t))' + \mathscr{A}(t)x_\varepsilon(t) = f(t), \quad \text{a.e. } \; t \in (0,T), \\ x_\varepsilon(0) = -x_\varepsilon(T). \end{cases} \tag{4.1}$$

Therefore, we have, for a.e. $t \in [0,T]$

$$\frac{\varepsilon}{2}\frac{d}{dt}\langle \Gamma x_\varepsilon(t), x_\varepsilon(t)\rangle_V + \frac{1}{2}\frac{d}{dt}\langle Bx_\varepsilon(t), x_\varepsilon(t)\rangle_V + \langle \mathscr{A}(t)x_\varepsilon(t), x_\varepsilon(t)\rangle_V = \langle f(t), x_\varepsilon(t)\rangle_V. \tag{4.2}$$

From assumption [H_5] and Hölder's inequality, we obtain by integrating the inequality (4.2) on $[0,T]$

$$\alpha_0 \int_0^T \|x_\varepsilon(t)\|_V^p dt \le \left(\int_0^T \|f(t)\|_{V^*}^{p'} dt\right)^{1/p'} \left(\int_0^T \|x_\varepsilon(t)\|_V^p dt\right)^{1/p} + \|\alpha_1\|_{L^1(0,T)}.$$

It follows, by Youngs inequality, there exists a constant $C > 0$ depending on $\|f\|_{X^*}$ and T such that

$$\|x_\varepsilon\|_X^p \le C. \tag{4.3}$$

Now, let us consider the operator $\widehat{\mathscr{A}}: X \to X^*$ defined by $\widehat{\mathscr{A}(u)}(t) = \mathscr{A}(t)u(t)$ for all $t \in [0,T]$. From relation (4.3) and the assumption [H_4], we derive that $\{\widehat{\mathscr{A}}x_\varepsilon\}_{\varepsilon>0}$ is bounded in X^*. Hence, we obtain

$$\begin{array}{rcl} x_\varepsilon & \rightharpoonup & x \;\text{ in } X, \\ \widehat{\mathscr{A}}x_\varepsilon & \rightharpoonup & \eta \;\text{ in } X^*, \\ Bx_\varepsilon & \rightharpoonup & Bx \;\text{ in } X^*, \\ ((\varepsilon\Gamma + B)x_\varepsilon)' & \rightharpoonup & (Bx)' \;\text{ in } X^*. \end{array} \tag{4.4}$$

In order to conclude, we need to verify that $\eta = \widehat{\mathscr{A}}x$. To this aim, after multiplying relation (4.1) by $x(t) - x_\varepsilon(t)$ and integrating on $[0,T]$, we obtain

$$\langle\langle \widehat{\mathscr{A}}x_\varepsilon, x - x_\varepsilon\rangle\rangle_X = \langle\langle f, x - x_\varepsilon\rangle\rangle_X + \langle\langle [(\varepsilon\Gamma + B)(x_\varepsilon - x)]', x_\varepsilon - x\rangle\rangle_X + \langle\langle [(\varepsilon\Gamma + B)x]', x_\varepsilon - x\rangle\rangle_X. \tag{4.5}$$

Consider the bifunction $\Theta : X \times X \to \mathbb{R}$ defined by $\Theta(x,y) = \langle\langle \widehat{\mathscr{A}}x, y - x\rangle\rangle_X$. From relation (4.5), we deduce that

$$\liminf \Theta(x_\varepsilon, x) \ge 0. \tag{4.6}$$

On the other hand, since the operator $\mathscr{A}(t) : V \to V^*$ is B-PMO, it follows from Lemma 3.2 that the bifunction Θ is B-PMB. Hence,

$$\limsup \Theta(x_\varepsilon, y) \le \Theta(x,y), \quad \text{for all } y \in X. \tag{4.7}$$

Therefore, by taking account of (4.4), that $\langle\langle \widehat{\mathscr{A}}x, y - x\rangle\rangle_X \ge \langle\langle \eta, y - x\rangle\rangle_X$ for all $v \in X$. Thus, we obtain that $\eta = \widehat{\mathscr{A}}x$. $\square$

Theorem 4.2. *Let the assumptions [H_1], [H_3], [H_4] and [H_5] be satisfied. Suppose that $\mathscr{A}(t) : V \to V^*$ is T-QMO and weakly continuous for all $t \in [0,T]$. Then, there exists $x \in X$ such that $Bx \in L^p([0,T];V^*)$, $(Bx)' \in L^{p'}([0,T];V^*)$ and*

$$\begin{cases} \frac{d}{dt}(Bx(t)) + \mathscr{A}(t)x(t) = f(t), & \text{a.e. } t \in (0,T), \\ Bx(0) = -Bx(T). \end{cases}$$

Proof. We use Theorem 3.3 on the existence of solutions for the auxiliary problem and we conclude by using a similar approach to the one developed in the proof of the previous theorem. □

We end this section by the following theorem which proof can be obtained by using Theorem 4.3 and a similar procedure to what precede.

Theorem 4.3. *Suppose that for all $t \in [0,T]$, $\mathscr{A}(t) : V \to V^*$ is B-PMO and satisfies conditions [H_2]-[H_5]. Suppose that $\mathscr{B}(t) : V \to V^*$ is T-QMO and weakly continuous for all $t \in [0,T]$. Furthermore, suppose that $\mathscr{B}(t) : V \to V^*$ satisfies conditions [H_3], [H_4] and the following condition*

$$[C] \qquad \langle \mathscr{B}(t)x, x\rangle \geq -\delta_1 \|x\|_V^p - \delta_2(t), \quad \text{for all } x \in V,\ t \in [0,T]$$

with some $\delta_1 > 0$ and $\delta_2 \in L^1(0,T)$.
Then, there exists $x \in X$ such that $Bx \in L^p([0,T];V^*)$, $(Bx)' \in L^{p'}([0,T];V^*)$ and

$$\begin{cases} \frac{d}{dt}(Bx(t)) + \mathscr{A}(t)x(t) + \mathscr{B}(t)x(t) = f(t), & \text{a.e. } t \in (0,T), \\ Bx(0) = -Bx(T). \end{cases}$$

5. Conclusion

In this section, we give a comparison with recent results obtained in literature which are related to the problem studied in this paper, we give also a concrete example where the approach developed in this paper can be applied.

Liu and Liu in [20] studied the following problem

$$\begin{cases} \frac{d}{dt}(Bx(t)) + Ax(t) + Gx(t) = f(t), & \text{a.e. } t \in (0,T), \\ Bx(0) = -Bx(0), \end{cases}$$

where $A : V \to V^*$ is monotone, bounded and demicontinuous; and the operator $G : V \to V^*$ is T-QMO and both continuous and weakly continuous. Furthermore, it supposed that the operators A and G satisfy the assumptions [H_4] and [H_5]. The results obtained by our approach improve and generalize the results obtained in [20].

We end this section by an example illustrating the problem studied in this paper and which can be solved by the approach developed.

Example 5.1. Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary. Let $V = H_0^1(\Omega)$, $V^* = H^{-1}(\Omega)$ and $Q = \Omega \times [0,T]$. We consider for $t \in [0,T]$, $A(t) : V \to V^*$ a nonlinear elliptic operator on Ω satisfying assumptions [H_4] and [H_5]; and $B : V \to V^*$ be the multiplication operator $Bu = \beta(x)u$

for $u \in V$, where $\beta(\cdot)$ is a nonnegative function on Ω satisfying some appropriate conditions leading to $B \in L(V, V^*)$. For instance those conditions could be as the following

$$\begin{cases} \beta \in L^1(\Omega) & \text{if } n = 1, \\ \beta \in L^s(\Omega) & \text{if } n = 2, \text{ with } s > 1 \\ \beta \in L^{n/2}(\Omega) & \text{if } n > 2, \end{cases}$$

and by using the Sobolev embedding theorem we can verify that in this case $B \in L(V, V^*)$. We consider the following implicit nonlinear differential equation:

$$\begin{cases} \frac{d}{dt}(\beta(x)u(t)) + A(t)u(t) = f(t), \quad \text{a.e. } (x,t) \in \Omega \times (0,T), \\ u(0) = -u(T). \end{cases}$$

For instance, we take $A(t)u(x,t) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x,t,u(x,t), \nabla u(x,t))$ such that the coefficients $a_i : Q \times \mathbb{R} \times \mathbb{R}^n \to R$, for $i \in \{1, \cdots, n\}$, satisfy the following conditions:

[A] Carathéodory and growth conditions: Each $a_i(x,t,s,\xi)$ satisfies Carathéodory conditions, i.e., is measurable in $(x,t) \in Q$ for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and continuous in (s,ξ) for a.e. $(x,t) \in Q$. A constant $c_0 > 0$ and a function $\tau_0 \in L^{p'}(Q)$ exists so that

$$|a_i(x,t,s,\xi)| \leq \tau_0(x,t) + c_0 \left(|s|^{p-1} + \|\xi\|_{\mathbb{R}^n}^{p-1} \right)$$

for a.e. $(x,t) \in Q$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

[B] Coercivity type condition:

$$\sum_{i=1}^{n} a_i(x,t,s,\xi) \geq \mu \|\xi\|_{\mathbb{R}^n}^p - \alpha(x,t)$$

for a.e. $(x,t) \in Q$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^n$ with some constant $\mu > 0$ and some function $\alpha \in L^1(Q)$.

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