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A GENERALIZED HYBRID STEEPEST DESCENT METHOD AND APPLICATIONS

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Abstract. The purpose of this paper is to investigate a generalized hybrid steepest descent method and develop a convergence theory for solving monotone variational inequality over the fixed point set of a mapping which is not necessarily Lipschitz continuous. Using this result, we consider the convex minimization problem for a continuously differentiable convex function whose gradient is not necessarily Lipschitzian.

Keywords. Convex minimization problem; Fixed point; Hybrid steepest descent method; Monotone variational inequality; Nearly asymptotically nonexpansive mapping.

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1. Introduction

Let X be a real Hilbert space with the norm and scalar product denoted, respectively, by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. Let C be a nonempty closed convex subset of X and $\mathscr{F}: C \to X$ a mapping. A variational inequality problem over a nonempty closed convex subset D of C is formulated as finding an element $x^* \in D$ such that

$$\langle \mathscr{F}(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in D. \tag{1.1}$$

The problem (1.1) is denoted by $\mathrm{VIP}_D(\mathscr{F}, C)$. The solution set of variational inequality problem (1.1) is denoted by $\Omega[\mathrm{VIP}_D(\mathscr{F}, C)]$, i.e.,

$$\Omega[\mathrm{VIP}_D(\mathscr{F}, C)] = \{u \in D : \langle \mathscr{F}(u), z - u \rangle \geq 0 \text{ for all } z \in D\}.$$

It is well known that convex minimization problem [4, 9, 11, 13] of a differentiable convex function subject to a closed convex set $D \subseteq X$ of the form:

$$\text{find a point } x^* \in D \text{ such that } \psi(x^*) = \min\{\psi(x) : x \in D\}, \tag{1.2}$$

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where $\psi : X \to \mathbb{R}$ is a differentiable convex function, can be casted into the variational inequality problem over D :

$$\text{find } u \in D \text{ such that } \langle \nabla\psi(u), z-u\rangle \geq 0 \text{ for all } z \in D,$$

where $\nabla\psi : X \to X$ is the gradient of ψ.

It is well known that if $\mathscr{F}$ is an η-strongly monotone and $\kappa-$Lipschitz continuous, then, for $\mu \in (0, 2\eta/\kappa^2)$, the mapping $P_C(I-\mu\mathscr{F})$ is contraction on C and hence $\mathrm{VIP}_C(\mathscr{F}, X)$ has a unique solution $x^* \in C$ and the projection gradient method:

$$x_{n+1} = P_C(I-\mu\mathscr{F})x_n, n \in \mathbb{N}, \tag{1.3}$$

converges strongly to x^* (see [27, Theorem 46.C]). Note that computation of the metric projection, P_C, onto C is not necessarily easy. In order to reduce such difficulty, which is caused by the metric projection P_C, in [26, Theorem 3.3, p. 486], Yamada introduced the following hybrid steepest descent method (for short, HSDM) for solving the variational inequality $\mathrm{VIP}_{\mathrm{Fix}(T)}(\mathscr{F}, C)$:

$$x_{n+1} = (I-\alpha_n\mu\mathscr{F})Tx_n, n \in \mathbb{N}, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ and T is a nonexpansive mapping from X into itself with a nonempty fixed point set $\mathrm{Fix}(T)$. Yamada [26, Theorem 3.3, p. 486] proved that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a unique solution of $\mathrm{VIP}_{\mathrm{Fix}(T)}(\mathscr{F}, X)$. There are many papers dealing with variational inequality problems when the constrained set D is a set of fixed point of a nonexpansive mapping or set of common fixed points of a family of nonexpansive mappings (see [7, 6, 8, 16, 18, 19, 23, 25, 28]).

It is well known that if $\psi : X \to \mathbb{R}$ is a differentiable convex function such that $\nabla\psi$ is L-Lipschitz continuous for some $L > 0$, then $\nabla\psi$ is $1/L$-inverse strongly monotone. Hence $I-\gamma\nabla\psi$ is nonexpansive for $\gamma \in (0, 2/L)$ (see [10]) and hence hybrid steepest descent method is applicable for solving the variational inequality problem $\mathrm{VIP}_{\mathrm{Fix}(I-\gamma\nabla\psi)}(\mathscr{F}, X)$. The variational inequality problem $\mathrm{VIP}_{\mathrm{Fix}(I-\gamma\nabla\psi)}(\mathscr{F}, X)$ is equally interesting when $\nabla\psi$ is uniformly continuous, but not necessarily L-Lipschitz continuous. In [12], Goldstein studied weak convergence of a hybrid steepest descent method when $\nabla\psi$ is uniformly continuous.

These observations lead naturally to the following problem:

Problem 1.1. Let X be a real Hilbert space and $\psi : X \to \mathbb{R}$ a continuously differentiable convex function such that $\nabla\psi$ is not necessarily Lipschitzian. Is it possible to develop a hybrid steepest descent method which converges strongly to a solution of the variational inequality problem $\mathrm{VIP}_{\mathrm{Fix}(I-\gamma\nabla\psi)}(\mathscr{F}, C)$ for some $\gamma > 0$?

The main purpose of this paper is to investigate a hybrid steepest descent method for solving the variational inequality problem $\mathrm{VIP}_D(\mathscr{F}, C)$ when constrained set D is a set of fixed points of a self-mapping on C which more general than nonexpansive. It is important and actually quite surprising that we are able to do so for nonlinear operators which are not necessarily Lipschitz continuous. A strong convergence theorem for nearly asymptotically nonexpansive mapping is established in Section

3. In Section 4, we discuss a computational result for minimization problem (1.2) when $\nabla\psi$ is not necessarily Lipschitzian. Our results are definitive and settle Problem 1.1 in the mathematical theory of nearly Lipschitzian mappings and also improve several known results for the class of Lipschitzian type mappings that have appeared in Hilbert spaces.

2. Preliminaries

Let C be a nonempty subset of a normed space X and $T : C \to X$ a mapping. T is called an L-Lipschitz mapping if there exists $L \in [0,\infty)$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \text{ for all } x, y \in C.$$

The L-Lipschitz mapping T is called a non-expansive operator if $L = 1$ and contraction if $L \in [0,1)$. We denote by $B_r[x]$ the closed ball with center $x \in X$ and radius $r > 0$ and by $Fix(T)$ the set of fixed points of T.

A sequence $\{x_n\}$ in C is said to be *an approximating fixed point sequence* if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Recall that T is called a demiclosed mapping if

$$(\{x_n\} \text{ in } C,\ x_n \rightharpoonup x \text{ and } Tx_n \to y \text{ for some } x, y \in X) \Longrightarrow (x \in C \text{ and } Tx = y).$$

The following technical lemmas will be needed in the sequel.

Lemma 2.1. *([24]) Let $\{a_n\}$ be a sequence of nonnegative real numbers and let $\{b_n\}$ be a sequence in $\mathbb{R}$ satisfying the following condition:*

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$. If $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty}(b_n/\alpha_n) \leq 0$, then $\{a_n\}$ converges to zero.

Lemma 2.2. [22] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1-\beta_n)x_n + \beta_n z_n \text{ for all } n \in \mathbb{N}$$

and that $\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0$. Then $\lim_{n\to\infty} ||z_n - x_n|| = 0$.

2.1. Monotone operators.

Let C be a nonempty subset of a real Hilbert space X and $T : C \to C$ a nonlinear operator. Then T is said to be

(i) pseudocontractive if $\langle Tx - Ty, x - y\rangle \leq ||x - y||^2$ for all $x, y \in C$;

(ii) asymptotically pseudocontractive ([21]) if there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \leq k_n ||x - y||^2 \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

A set-valued operator $A : X \to 2^X$ with domain $D(A) = \{x \in X : Ax \neq \emptyset\}$ is said to be monotone, if

$$\langle u - v, x - y\rangle \geq 0 \text{ for all } x, y \in D(A) \text{ with } u \in Ax, v \in Ay.$$

A monotone operator $A : X \to 2^X$ is said to be maximal monotone, if $\mathscr{R}(I + rA) = X$ for each $r > 0$, where $\mathscr{R}(A)$ denotes the range of A. It is well known that if $A : X \to X$ is a nonlinear operator, then

$$A \text{ is monotone} \iff T := I - A \text{ is pseudocontractive.}$$

Let C be a nonempty subset of X and $T : C \to X$ a mapping. Then T is said to be

(i) η-strongly monotone if there exists a constant $\eta > 0$ such that $\langle Tx - Ty, x - y\rangle \geq \eta\|x - y\|^2$ for all $x, y \in C$;

(ii) ν-inverse strongly monotone (ν-ism) if there exists a constant $\nu > 0$ such that $\langle Tx - Ty, x - y\rangle \geq \nu\|Tx - Ty\|^2$ for all $x, y \in C$.

Lemma 2.3. [26] *Let X be a real Hilbert space and $\mathscr{F} : X \to X$ an η-strongly monotone and $L-$Lipschitz continuous. Then, for each $\lambda \in (0,1)$ and fixed $\mu \in (0, 2\eta/L^2)$, the $I - \lambda\mu\mathscr{F} : X \to X$ is a contraction mapping with Lipschitz constant $1 - \tau\lambda$, where $\tau := \sqrt{1 - \mu(2\eta - \mu L^2)}$.*

The following are well known facts on convex functions.

Lemma 2.4. *(see [3]) Let X be a real Hilbert space, f a continuously Fréchet differentiable, convex functional on X and ∇f the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse-strongly-monotone.*

Example 2.1. Let X be a real Hilbert space, $b \in X$ and $Q\colon X \to X$ a self-adjoint bounded linear operator and strongly positive; that is, there exists $\alpha > 0$ such that $\langle Q(x), x\rangle \geq \alpha||x||^2$ for all $x \in X$. Define a quadratic functional $f\colon X \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}\langle Q(x), x\rangle + \langle x, b\rangle \text{ for all } x \in X.$$

Then $\nabla f(\cdot) = Q(\cdot) + b$ is α-strongly monotone and $||Q||$-Lipschitz continuous.

When $X = \mathbb{R}^N$ is finite dimensional, the above operator Q coincides with a positive definite matrix. Then $\nabla^2 f(x) = Q$ and $\lambda_{min}||x||^2 \leq \langle Q(x), x\rangle \leq \lambda_{max}||x||^2$ for all $x \in \mathbb{R}^N$, where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of Q, respectively. Hence $\alpha = \lambda_{min} \leq \lambda_{max} = ||Q||$.

2.2. Lipschitzian type mappings.

We begin with the following:

Definition 2.1. Let C be a nonempty subset of a Banach space X. The modulus of continuity of a continuous mapping $T : C \to C$ is a function $\omega_T : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\omega_T(t) := \sup\{\|Tx - Ty\| : x, y \in C; \|x - y\| \leq t\}, t \in \mathbb{R}^+.$$

If T is uniformly continuous, then

(i) for all $x,y \in C$, we have

$$\|Tx - Ty\| \leq \omega_T(\|x-y\|),$$

(ii) ω_T is nonnegative, nondecreasing, continuous on $(0,\infty)$, and $\omega_T(0) = 0$.

Definition 2.2. Let C be a nonempty subset of a normed space X. The mapping $T : C \to C$ is said to be:

(i) uniformly *Lipschitzian* if , there exists a constant $L \in [0,\infty)$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\| \text{ for all } x,y \in C \text{ and } n \in \mathbb{N};$$

(ii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x-y\| \text{ for all } x,y \in C \text{ and } n \in \mathbb{N};$$

(iii) *asymptotically quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n\|x-p\| \quad \text{for all } x \in C,\ p \in \mathrm{Fix}(T) \text{ and } n \in \mathbb{N}.$$

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called asymptotically nonexpansive in the intermediate sense ([5]) provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \leq 0. \tag{2.1}$$

Set $c_n := \max\{\sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|), 0\}$. One can see that (2.1) reduces to relation

$$\|T^n x - T^n y\| \leq \|x-y\| + c_n \text{ for all } x,y \in C \text{ and } n \in \mathbb{N}. \tag{2.2}$$

The class of mappings of asymptotically nonexpansive in the intermediate sense which is essentially wider than that of asymptotically nonexpansive mappings was introduced by Bruck, Kuczumow and Reich [5]. The class of asymptotically κ-strict pseudocontractive mappings in the intermediate sense which are not necessarily Lipschitzian is introduced and studied in Sahu, Yao and Xu [17].

On the other hand, Alber, Chidume and Zegeye [2] introduced a more general class of asymptotically nonexpansive mappings called total asymptotically nonexpansive mappings and studied methods of approximation of fixed points of mappings belonging to this class:

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be total asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty}\mu_n = \lim_{n\to\infty} c_n = 0$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x-y\| + \mu_n\phi(\|x-y\|) + c_n \tag{2.3}$$

for all $x,y \in C$ and $n \in \mathbb{N}$.

If $\mu_n = 0$, then (2.3) reduces to (2.2). We remark that the corresponding Lipschitzian type mappings (for instance, contraction type mappings) concerning asymptotically nonexpansive mapping in the intermediate sense and total asymptotically nonexpansive mapping are not defined in Bruck, Kuczumow and Reich [5] and Alber, Chidume and Zegeye [2].

The class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian mappings and was introduced by Sahu in [14]. The extensions of the classical Banach contraction principle and Browder-Gӧhde-Kirk theorem for the class of nearly Lipschitzian mappings can be found in [1, 14] and recent results for nearly nonexpansive mappings can be found in [1, 15, 16]. Let C be a nonempty subset of a normed space X and fix a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$. A mapping $T : C \to C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x-y\| + a_n) \text{ for all } x, y \in C. \tag{2.4}$$

The infimum of constants k_n for which (2.4) holds is denoted by $\eta(T^n)$ and called *the nearly Lipschitz constant*.

A nearly Lipschitzian mapping T with sequence $\{(\eta(T^n), a_n\}$ is said to be

(i) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,

(ii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$,

(iii) *nearly uniformly* k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$ and for some $k \in [0,\infty)$,

(iv) *nearly uniform k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

By the definitions, we have the following implication:

$$\text{contraction } \Rightarrow \text{nearly uniformly } k\text{-contraction} \Rightarrow \text{nearly nonexpansive.}$$

Definition 2.3. Let C be a nonempty subset of a normed space X and fix a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$. A mapping $T : C \to C$ is said to be nearly asymptotically quasi-nonexpansive with respect to $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$ if $\mathrm{Fix}(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n(\|x-p\| + a_n) \text{ for all } x \in C,\ p \in \mathrm{Fix}(T) \text{ and n} \in \mathbb{N}.$$

Remark 2.1. Nearly asymptotically quasi-nonexpansive mappings are also called generalized asymptotically quasi-nonexpansive mappings (see [20]).

Example 2.2. Let $X = \mathbb{R}$, $C = [0,1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} kx & if \quad x \in [0,1/2], \\ 0 & if \quad x \in (1/2,1], \end{cases}$$

where $k \in (0,1)$. Noticing that $T : C \to C$ is discontinuous at $x = 1/2$. Hence T is not Lipschitzian and hence it is not asymptotically nonexpansive. Following [17], we have.

$$\|T^n x - T^n y\| \leq \|x-y\| + k^n \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

It means that T is nearly nonexpansive with sequence $\{k^n\}$.

Example 2.3. Let $X=\mathbb{R}$, $C=[-1,1]$ and $T:C\to C$ be a mapping defined by

$$Tx=\begin{cases}\frac{x}{2} & if \quad x\in[-1,0],\\ 0 & if \quad x\in[0,1].\end{cases}$$

It is obvious that T is uniformly continuous mapping. However, it is a nearly nonexpansive with sequence $\{\frac{1}{2^n}\}$. Indeed,

$$\|T^nx-T^ny\|\le\|x-y\|+\frac{1}{2^n}\ \text{ for all }\ x,y\in C \text{ and } n\in\mathbb{N}.$$

Remark 2.2. Every nearly asymptotically nonexpansive mapping with nonempty fixed point set is nearly asymptotically quasi-nonexpansive mapping.

Lemma 2.5. *(see* [14, Corollary 4.3] *) Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T:C\to C$ a demicontinuous nearly asymptotically nonexpansive mapping. Then $Fix(T)$ is closed and convex.*

3. Main results

We begin with the following proposition:

Proposition 3.1. *Let C be a nonempty convex subset of a real Hilbert space X and $T:C\to C$ a mapping. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into itself defined by*

$$U_n=(1-\lambda_n)I+\lambda_nT^n, n\in\mathbb{N}.$$

Then we have the following:

(a) If T is asymptotically pseudocontractive mapping with sequence $\{k_n\}$, then

$$\langle U_nx-U_ny,x-y\rangle\le k_n\|x-y\|^2 \textit{ for all } x,y\in C \textit{ and } n\in\mathbb{N}.$$

(b) If T is nearly asymptotically nonexpansive mapping with sequence $\{(\eta(T^n),a_n)\}$, then

$$\|U_nx-U_ny\|\le\eta(T^n)(\|x-y\|+a_n) \textit{ for all } x,y\in C \textit{ and } n\in\mathbb{N}.$$

Proof. (a) Suppose that T is asymptotically pseudocontractive mapping with sequence $\{k_n\}$. Then

$$\lambda_nk_n+1-\lambda_n=\lambda_n(k_n-1)+1\le k_n \text{ for all } n\in\mathbb{N}.$$

Let $x,y\in C$. Then

$$\begin{aligned}\langle U_nx-U_ny,x-y\rangle &= \langle(1-\lambda_n)(x-y)+\lambda_n(T^nx-T^ny),x-y\rangle\\ &= (1-\lambda_n)\|x-y\|^2+\lambda_n\langle T^nx-T^ny,x-y\rangle\\ &\le (1-\lambda_n)\|x-y\|^2+\lambda_nk_n\|x-y\|^2\\ &\le k_n\|x-y\|^2 \text{ for all } n\in\mathbb{N}.\end{aligned}$$

(b) T is nearly asymptotically nonexpansive mapping with sequence $\{(\eta(T^n), a_n)\}$. Let $x, y \in C$. Then

$$\begin{aligned}\|U_n x - U_n y\| &\leq (1-\lambda_n)\|x-y\| + \lambda_n \|T^n x - T^n y\| \\ &\leq (1-\lambda_n)\|x-y\| + \lambda_n \eta(T^n)(\|x-y\| + a_n) \\ &\leq \eta(T^n)(\|x-y\| + a_n) \text{ for all } n \in \mathbb{N}.\end{aligned}$$

□

First, we give demiclosed principle for class of continuous nearly asymptotically nonexpansive mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space X and $T : C \to C$ a continuous nearly asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero in the sense that if $\{y_n\}$ is a sequence in C with $y_n \rightharpoonup y$ such that $\limsup_{m\to\infty} \limsup_{n\to\infty} \|y_n - T^m y_n\| = 0$, then $(I-T)y = 0$.*

Proof. Define $\psi : X \to [0, \infty)$ by

$$\psi(z) = \limsup_{n\to\infty} \|y_n - z\|^2, z \in X.$$

Since $y_n \rightharpoonup y$, by [17, Lemma 2.4(c)], we have

$$\psi(z) = \psi(y) + \|y - z\|^2 \text{ for all } z \in X. \tag{3.1}$$

Now we show that y is a fixed point of T. Suppose T is a nearly asymptotically nonexpansive mapping with $\{(\eta(T^n), a_n)\}$. For $m \in \mathbb{N}$, we have

$$\begin{aligned}\psi(T^m y) &= \limsup_{n\to\infty} \|y_n - T^m y\|^2 \\ &\leq \limsup_{n\to\infty} (\|y_n - T^m y_n\| + \|T^m y_n - T^m y\|)^2 \\ &= \limsup_{n\to\infty} (\|y_n - T^m y_n\|^2 + \|T^m y_n - T^m y\|^2 + 2\|y_n - T^m y_n\| \, \|T^m y_n - T^m y\|) \\ &\leq \limsup_{n\to\infty} (\|y_n - T^m y_n\|^2 + \|T^m y_n - T^m y\|^2 + 2\|y_n - T^m y_n\| \, K') \\ &\leq \limsup_{n\to\infty} \|T^m y_n - T^m y\|^2 + \limsup_{n\to\infty} (\|y_n - T^m y_n\|^2 + 2\|y_n - T^m y_n\| \, K') \\ &\leq \limsup_{n\to\infty} \eta(T^m)^2 (\|y_n - y\| + a_m)^2 + \limsup_{n\to\infty} (\|y_n - T^m y_n\|^2 + 2\|y_n - T^m y_n\| \, K') \\ &= \limsup_{n\to\infty} \eta(T^m)^2 (\|y_n - y\|^2 + (2\|y_n - y\| + a_m) a_m) \\ &\quad + \limsup_{n\to\infty} (\|y_n - T^m y_n\|^2 + 2\|y_n - T^m y_n\| \, K') \\ &\leq \eta(T^m)^2 (\psi(y) + a_m K'') + \limsup_{n\to\infty} (\|y_n - T^m y_n\|^2 + 2\|y_n - T^m y_n\| \, K')\end{aligned}$$

for some $K', K'' > 0$. From (3.1), we have

$$\begin{aligned}\psi(y) + \|y - T^m y\|^2 &= \psi(T^m y) \\ &\leq \eta(T^m)^2(\psi(y) + a_m K'') \\ &\quad + \limsup_{n\to\infty}(\|y_n - T^m y_n\|^2 + 2\|y_n - T^m y_n\|\ K'),\end{aligned}$$

which implies that

$$\begin{aligned}\|y - T^m y\|^2 &\leq (\eta(T^m)^2 - 1)\psi(y) + \eta(T^m)^2 a_m K'' \\ &\quad + \limsup_{n\to\infty}(\|y_n - T^m y_n\|^2 + 2\|y_n - T^m y_n\|\ K').\end{aligned}$$

Since $\limsup_{m\to\infty}\limsup_{n\to\infty}\|y_n - T^m y_n\| = 0$, we have

$$\limsup_{m\to\infty}\|y - T^m y\|^2 \leq 0.$$

It means that $T^m y \to y$ as $m \to \infty$. By the continuity of T, we conclude that $y = Ty$. □

Let C be a nonempty closed convex subset of a real Hilbert space X and let $T : C \to C$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(\eta(T^n), a_n)\}$ such that $\mathrm{Fix}(T) \neq \emptyset$. Let $\mathscr{F} : C \to X$ be η-strongly monotone and L-Lipschitz continuous such that $(I - \alpha\mu\mathscr{F})(C) \subseteq C$ for all $\alpha \in (0,1]$, where $\mu \in (0, 2\eta/L^2)$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into itself defined by

$$U_n = (1 - \lambda_n)I + \lambda_n T^n, n \in \mathbb{N}.$$

Remark 3.1. From Lemma 2.5 and [27, Theorem 46.C] we see that the variational inequality problem $\mathrm{VIP}_{\mathrm{Fix}(T)}(\mathscr{F}, C)$ has a unique solution $x^* \in (T)$.

We now introduce a hybrid steepest descent-like method for computation of unique solution $x^* \in F(T)$ of the variational inequality problem $\mathrm{VIP}_{\mathrm{Fix}(T)}(\mathscr{F}, C)$.

Algorithm 3.1. Hybrid steepest descent-like method

Step 0: Choose $x_1 \in C$ and $\alpha_1 \in (0,1]$ arbitrarily.
Step 1: Given $x_n \in C$, choose $\alpha_n \in (0,1]$ and define $x_{n+1} \in C$ by

$$\begin{cases} x_{n+1} := (1 - \lambda_n)y_n + \lambda_n T^n(y_n), \\ y_n = x_n - \alpha_n\mu\mathscr{F}(x_n) \text{ for all } n \in \mathbb{N}. \end{cases} \tag{3.2}$$

We study convergence analysis of Algorithm 3.1 under the following assumptions:

(A1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(A2) $a \leq \lambda_n \leq b$ for all $n \in \mathbb{N}$ and for some $a, b \in (0,1)$;

(A3) $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < infty$ and $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$ with a constant $K \in [0,\infty)$ such that $\frac{a_n}{\alpha_n} \leq K$ for all $n \in \mathbb{N}$.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space X and $T : C \to C$ a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(\eta(T^n), a_n)\}$ such that $\mathrm{Fix}(T) \neq \emptyset$. Let $\mathscr{F} : C \to X$ be η-strongly monotone and L-Lipschitz continuous for some positive constants η and L and let $\mu \in (0, 2\eta/L^2)$ such that $(I - \alpha\mu\mathscr{F})(C) \subseteq C$ for all $\alpha \in (0,1]$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C generated by Algorithm 3.1, where $\{\lambda_n\}$ is a sequence in $(0,1)$. Assume that assumptions (A1)-(A3) hold. Then we have the following:*

(a) $\Omega[\mathrm{VIP}_{\mathrm{Fix}(T)}(\mathscr{F}, C)] = \{x^*\}$.

(b) *The orbit $\{x_n\}$ of the Algorithm 3.1 is well defined in the closed convex set $B_r[x^*] \cap C$, where r is a positive constant such that*

$$\max\left\{\|x_1 - x^*\|, \frac{1}{\tau}(\mu\|\mathscr{F}(x^*)\| + K)\right\} \leq r.$$

(c) *If the following assumption holds:*

(A4) $\lim_{n\to\infty} \|T^n(x_n) - T^{n+1}(x_n)\| = 0$ *and* $\lim_{n\to\infty} \|T^n(y_n) - T^{n+1}(y_n)\| = 0$,

then $\{x_n\}$ converges strongly to x^* with the following error estimate:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \quad &\leq \quad (1 - \alpha_n\tau)\|x_n - x^*\|^2 + \mu^2\alpha_n^2\|\mathscr{F}(x^*)\|^2 + +2\alpha_n^2\mu^2 Lr\|\mathscr{F}(x^*)\| \\ &\quad +2\mu\alpha_n\langle\mathscr{F}(x^*), x^* - x_n\rangle + (\eta(T^n)^2 - 1)\|y_n - x^*\|^2 + \eta(T^n)^2(2\|y_n - x^*\| + a_n)a_n. \end{aligned}$$

Proof. **(a)** It follows from Remark 3.1.

(b) For every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|y_n - x^*\| &= \|x_n - \alpha_n\mu\mathscr{F}(x_n) - x^*\| \\ &= \|(I - \mu\alpha_n\mathscr{F})(x_n) - (I - \mu\alpha_n\mathscr{F})(x^*) - \mu\alpha_n\mathscr{F}(x^*)\| \\ &\leq \|(I - \mu\alpha_n\mathscr{F})(x_n) - (I - \mu\alpha_n\mathscr{F})(x^*)\| + \alpha_n\mu\|\mathscr{F}(x^*)\| \\ &\leq (1 - \alpha_n\tau)\|x_n - x^*\| + \alpha_n\mu\|\mathscr{F}(x^*)\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \lambda_n)y_n + \lambda_n T^n(y_n) - x^*\| \\ &\leq (1 - \lambda_n)\|y_n - x^*\| + \lambda_n\|T^n(y_n) - x^*\| \\ &\leq (1 - \lambda_n)\|y_n - x^*\| + \lambda_n\eta(T^n)(\|y_n - x^*\| + a_n) \\ &\leq \eta(T^n)[\|y_n - x^*\| + a_n] \\ &\leq \eta(T^n)[(1 - \alpha_n\tau)\|x_n - x^*\| + \alpha_n\mu\|\mathscr{F}(x^*)\| + a_n] \\ &\leq \eta(T^n)[(1 - \alpha_n\tau)\|x_n - x^*\| + \alpha_n\tau(\frac{\mu}{\tau}\|\mathscr{F}(x^*)\| + \frac{K}{\tau})] \\ &\leq \eta(T^n)\max\left\{\|x_n - x^*\|, \frac{1}{\tau}(\mu\|\mathscr{F}(x^*)\| + K)\right\}. \end{aligned}$$

Observe that

$$\begin{aligned} \|x_2 - x^*\| &\leq \eta(T)\max\left\{\|x_1 - x^*\|, \frac{1}{\tau}(\mu\|\mathscr{F}(x^*)\| + K)\right\} \\ &\leq \eta(T)r, \end{aligned}$$

and

$$\begin{aligned} \|x_3 - x^*\| &\leq \eta(T^2)\max\left\{\eta(T)r, \frac{\mu}{\tau}\|\mathscr{F}(x^*)\| + \frac{\eta_T}{\tau}K\right\} \\ &\leq \eta(T)\eta(T^2)\max\left\{r, \frac{\mu}{\tau}\|\mathscr{F}(x^*)\| + \frac{\eta_T}{\tau}K\right\} \\ &\leq \eta(T)\eta(T^2)r. \end{aligned}$$

Inductively, we have

$$\|x_n - x^*\| \leq \prod_{i=1}^{n-1}\eta(T^i)r, n \geq 2.$$

Since $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$, it follows that $\prod_{n=1}^{\infty}\eta(T^n) = 1$. Thus, $\{x_n\}$ is in $B_r[x^*] \cap C$. Note

$$\begin{aligned} \|y_n - x^*\| &\leq (1 - \alpha_n\tau)\|x_n - x^*\| + \alpha_n\mu\|\mathscr{F}(x^*)\| \\ &\leq r + \mu\|\mathscr{F}(x^*)\|. \end{aligned}$$

(**c**) Note

$$\begin{aligned} x_{n+1} &= (1 - \lambda_n)y_n + \lambda_n T^n(y_n) \\ &= (1 - \lambda_n)(x_n - \alpha_n\mu\mathscr{F}(x_n)) + \lambda_n T^n(y_n) \\ &= (1 - \lambda_n)x_n + \lambda_n T^n(y_n) - \alpha_n(1 - \lambda_n)\mu\mathscr{F}(x_n) \\ &= (1 - \lambda_n)x_n + \lambda_n z_n, \end{aligned}$$

where

$$z_n = \frac{1}{\lambda_n}\left[\lambda_n T^n(y_n) - \alpha_n(1 - \lambda_n)\mu\mathscr{F}(x_n)\right].$$

Note

$$\begin{aligned} z_{n+1} - z_n &= T^{n+1}(y_{n+1}) - \frac{\mu\alpha_{n+1}(1 - \lambda_{n+1})}{\lambda_{n+1}}\mathscr{F}(x_{n+1}) \ - \left[T^n(y_n) - \frac{\mu\alpha_n(1 - \lambda_n)}{\lambda_n}\mathscr{F}(x_n)\right] \\ &= T^{n+1}(y_{n+1}) - T^n(y_n) + \frac{\mu\alpha_n(1 - \lambda_n)}{\lambda_n}\mathscr{F}(x_n) - \frac{\mu\alpha_{n+1}(1 - \lambda_{n+1})}{\lambda_{n+1}}\mathscr{F}(x_{n+1}) \\ &= T^{n+1}(y_{n+1}) - T^{n+1}(y_n) + T^{n+1}(y_n) - T^n(y_n) \\ &\quad + \frac{\mu\alpha_n(1 - \lambda_n)}{\lambda_n}\mathscr{F}(x_n) - \frac{\mu\alpha_{n+1}(1 - \lambda_{n+1})}{\lambda_{n+1}}\mathscr{F}(x_{n+1}). \end{aligned}$$

Set $\eta_T = \sup_{n\in\mathbb{N}} \eta(T^n)$. Hence

$$\begin{aligned}
& ||z_{n+1}-z_n|| \\
\leq\ & \eta(T^{n+1})(\|y_{n+1}-y_n\|+a_{n+1})+\left\|T^{n+1}(y_n)-T^n(y_n)\right\| \\
& +\frac{\mu\alpha_n(1-\lambda_n)}{\lambda_n}\|\mathscr{F}(x_n)\|+\frac{\mu\alpha_{n+1}(1-\lambda_{n+1})}{\lambda_{n+1}}\|\mathscr{F}(x_{n+1})\| \\
\leq\ & \|y_{n+1}-y_n\|+\left\|T^{n+1}(y_n)-T^n(y_n)\right\|+(\eta(T^{n+1})-1)\,\|y_{n+1}-y_n\|+a_{n+1}\eta_T \\
& +\frac{\mu\alpha_n(1-\lambda_n)}{\lambda_n}\|\mathscr{F}(x_n)\|+\frac{\mu\alpha_{n+1}(1-\lambda_{n+1})}{\lambda_{n+1}}\|\mathscr{F}(x_{n+1})\| \\
=\ & \|x_{n+1}-\mu\alpha_{n+1}\mathscr{F}(x_{n+1})-(x_n-\mu\alpha_n\mathscr{F}(x_n))\|+\left\|T^{n+1}(y_n)-T^n(y_n)\right\| \\
& +(\eta(T^{n+1})-1)K_1+a_{n+1}\eta_T+\frac{\mu\alpha_n(1-\lambda_n)}{\lambda_n}\|\mathscr{F}(x_n)\|+\frac{\mu\alpha_{n+1}(1-\lambda_{n+1})}{\lambda_{n+1}}\|\mathscr{F}(x_{n+1})\| \\
\leq\ & \|x_{n+1}-x_n\|+\left\|T^{n+1}(y_n)-T^n(y_n)\right\|+\mu\alpha_n\|\mathscr{F}(x_n)\|+\mu\alpha_{n+1}\|\mathscr{F}(x_{n+1})\| \\
& +(\eta(T^{n+1})-1)K_1+a_{n+1}\eta_T+\frac{\mu\alpha_n(1-\lambda_n)}{\lambda_n}\|\mathscr{F}(x_n)\|+\frac{\mu\alpha_{n+1}(1-\lambda_{n+1})}{\lambda_{n+1}}\|\mathscr{F}(x_{n+1})\|
\end{aligned}$$

for some $K_1 > 0$. It follows that

$$\limsup_{n\to\infty}(||z_{n+1}-z_n||-||x_{n+1}-x_n||) \leq 0.$$

From Lemma 2.2, we obtain that

$$\lim_{n\to\infty}||x_n-z_n|| = 0. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned}
||x_{n+1}-U_n(x_n)|| &= ||U_n(y_n)-U_n(x_n)|| \\
&\leq \eta(T^n)(\|y_n-x_n\|+a_n) \\
&= \eta(T^n)(\alpha_n\mu||\mathscr{F}(x_n)||+a_n) \to 0 \text{ as } n\to\infty
\end{aligned}$$

and

$$||x_{n+1}-x_n|| = \lambda_n||z_n-x_n|| \leq ||z_n-x_n|| \to 0 \text{ as } n\to\infty.$$

It follows that $\lim_{n\to\infty}||x_n-U_n(x_n)|| = 0$. By the definition of U_n, we have that

$$\begin{aligned}
||x_n-T^n(x_n)|| &\leq ||x_n-U_n(x_n)||+||(1-\lambda_n)x_n+\lambda_nT^nx_n-T^n(x_n)|| \\
&\leq ||x_n-U_n(x_n)||+(1-\lambda_n)||x_n-T^n(x_n)||.
\end{aligned}$$

Thus,

$$a||x_n-T^n(x_n)|| \leq \|x_n-U_n(x_n)|| \to 0 \text{ as } n\to\infty.$$

Note

$$\begin{aligned}
||x_n-T(x_n)|| &\leq ||x_n-T^n(x_n)||+||T^nx_n-T^{n+1}(x_n)||+||T^{n+1}x_n-T(x_n)|| \\
&\leq ||x_n-T^n(x_n)||+||T^nx_n-T^{n+1}(x_n)||+\omega_T(||T^nx_n-x_n||),
\end{aligned}$$

which gives us that $||x_n-T(x_n)|| \to 0$ as $n\to\infty$. Since T is uniformly continuous, we have $||x_n - T^m(x_n)|| \to 0$ as $n\to\infty$ for all $m\in\mathbb{N}$.

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. Note $x^* \in \mathrm{Fix}(T)$ is the unique solution of (1.1) with $D = Fix(T)$. Assume that there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty}\langle \mathscr{F}(x^*), x^* - x_n\rangle = \limsup_{k\to\infty}\langle \mathscr{F}(x^*), x^* - x_{n_k}\rangle.$$

Since $\{x_n\}_{n=1}^{\infty}$ is in $B_r[x^*] \cap C$, without loss of generality, we can assume that $\{x_{n_k}\}_{k=1}^{\infty}$ converges weakly to $z \in B_r[x^*] \cap C$. Since $x_{n_k} \rightharpoonup z$ and $\limsup_{m\to\infty}\limsup_{n\to\infty}||x_n - T^m(x_n)|| = 0$, then from Theorem 3.1, we have $z \in \mathrm{Fix}(T)$. Thus,

$$\limsup_{n\to\infty}\langle \mathscr{F}(x^*), x^* - y_n\rangle = \langle \mathscr{F}(x^*), x^* - z\rangle \le 0.$$

By Lemma 2.3, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|(I-\alpha_n\mu\mathscr{F})(x_n) - (I-\alpha_n\mathscr{F})(x^*) - \alpha_n\mu\mathscr{F}(x^*)\|^2 \\
&= \|(I-\alpha_n\mu\mathscr{F})(x_n) - (I-\alpha_n\mu\mathscr{F})(x^*)\|^2 + \mu^2\alpha_n^2\|\mathscr{F}(x^*)\|^2 \\
&\quad +2\alpha_n\mu\langle \mathscr{F}(x^*), (I-\alpha_n\mu\mathscr{F})(x^*) - (I-\alpha_n\mu\mathscr{F})(x_n)\rangle \\
&\le (1-\alpha_n\tau)\|x_n - x^*\|^2 + \mu^2\alpha_n^2\|\mathscr{F}(x^*)\|^2 \\
&\quad +2\alpha_n\mu\langle \mathscr{F}(x^*), x^* - x_n + \alpha_n\mu(\mathscr{F}(x_n) - \mathscr{F}(x^*))\rangle \\
&\le (1-\alpha_n\tau)\|x_n - x^*\|^2 + \mu^2\alpha_n^2\|\mathscr{F}(x^*)\|^2 \\
&\quad +2\alpha_n^2\mu^2 Lr\|\mathscr{F}(x^*)\| + 2\mu\alpha_n\langle \mathscr{F}(x^*), x^* - x_n\rangle.
\end{aligned}$$

By virtue of (3.2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\lambda_n)y_n + \lambda_n T^n(y_n) - x^*\|^2 \\
&= (1-\lambda_n)\|y_n - x^*\|^2 + \lambda_n\|T^n(y_n) - x^*\|^2 - (1-\lambda_n)\lambda_n\|y_n - T^n(y_n)\|^2 \\
&\le (1-\lambda_n)\|y_n - x^*\|^2 + \lambda_n(\eta(T^n)^2(\|y_n - x^*\| + a_n)^2) \\
&= (1-\lambda_n)\|y_n - x^*\|^2 + \lambda_n(\eta(T^n)^2(\|y_n - x^*\|^2 + (2\|y_n - x^*\| + a_n)a_n) \\
&\le \eta(T^n)^2\|y_n - x^*\|^2 + \eta(T^n)^2(2\|y_n - x^*\| + a_n)a_n \\
&= \|y_n - x^*\|^2 + (\eta(T^n)^2 - 1)\|y_n - x^*\|^2 + (\eta(T^n)^2(2\|y_n - x^*\| + a_n)a_n) \\
&\le (1-\alpha_n\tau)\|x_n - x^*\|^2 + \mu^2\alpha_n^2\|\mathscr{F}(x^*)\|^2 + 2\alpha_n^2\mu^2 Lr\|\mathscr{F}(x^*)\| \\
&\quad +2\mu\alpha_n\langle \mathscr{F}(x^*), x^* - x_n\rangle + (\eta(T^n)^2 - 1)\|y_n - x^*\|^2 + \eta(T^n)^2(2\|y_n - x^*\| + a_n)a_n.
\end{aligned}$$

Note $\lim_{n\to\infty}\frac{a_n}{\alpha_n} = 0$. Hence, by Lemma 2.1, we conclude that $x_n \to x^*$ as $n \to \infty$. □

For the class of nonexpansive mappings, assumption (A4) is not required in Theorem 3.2. In fact, we have the following.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space X and $T : C \to C$ a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let $\mathscr{F} : C \to X$ be η-strongly monotone and L-Lipschitz* *continuous for some positive constants η and L and let $\mu \in (0, 2\eta/L^2)$ such that $(I - \alpha\mu\mathscr{F})(C) \subseteq C$ for all $\alpha \in (0,1]$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} := (1-\lambda_n)y_n + \lambda_n T(y_n), \\ y_n = x_n - \alpha_n \mu \mathscr{F}(x_n) \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that assumptions (A1)-(A2) hold. Then we have the following:

- **(a)** $\Omega[\mathrm{VIP}_{\mathrm{Fix}(T)}(\mathscr{F}, C)] = \{x^*\}$.
- **(b)** *The orbit $\{x_n\}$ of the Algorithm 3.1 is well defined in the closed convex set $B_r[x^*] \cap C$, where r is a positive constant such that*

$$\max\left\{\|x_1 - x^*\|, \frac{\mu}{\tau}||\mathscr{F}(x^*)\|\right\} \leq r.$$

- **(c)** *$\{x_n\}$ converges strongly to x^* with the following error estimate:*

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \quad \leq \quad & (1-\alpha_n\tau)\|x_n - x^*\|^2 + \mu^2\alpha_n^2\|\mathscr{F}(x^*)\|^2 + 2\alpha_n^2\mu^2 Lr\|\mathscr{F}(x^*)\| \\ & +2\mu\alpha_n\langle \mathscr{F}(x^*), x^* - x_n\rangle \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Theorem 3.3 is more general in nature. It is an improvement upon corresponding results of Buong and Duong [6] and Zhou and Wang [28].

4. Applications

In this section, we consider the convex minimization problem (1.2) of finding a minimizer of a differentiable convex functional in a Hilbert space.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space X. Let $\mathscr{F} : C \to X$ be η-strongly monotone and L-Lipschitz continuous for some positive constants η and L and let $\mu \in (0, 2\eta/L^2)$ such that $(I - \alpha\mu\mathscr{F})(C) \subseteq C$ for all $\alpha \in (0,1]$. Let $\psi : X \to \mathbb{R}$ be a function such that*

- (A5) *$T := I - \nabla\psi$, $T(C) \subseteq C$ and T is a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(\eta(T^n), a_n)\}$*
- (A6) $\mathbb{S} = \arg\min_{x\in C} \psi(x) = \{z \in C : \psi(z) = \min_{z\in C} \psi(z)\} \neq \emptyset$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C generated by

$$\begin{cases} x_{n+1} := (1-\lambda_n)y_n + \lambda_n (I - \nabla\psi)^n(y_n), \\ y_n = x_n - \alpha_n \mu \mathscr{F}(x_n) \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that assumptions (A1)-(A3) hold with $T = I - \nabla\psi$. Then we have the following:

- **(a)** $\Omega[\mathrm{VIP}_{\mathbb{S}}(\mathscr{F}, C)] = \{x^*\}$.
- **(b)** *The orbit $\{x_n\}$ of the Algorithm 3.1 is well defined in the closed convex $B_r[x^*] \cap C$, where r is a positive constant such that*

$$\max\left\{\|x_1 - x^*\|, \frac{1}{\tau}(\mu||\mathscr{F}(x^*)\| + K)\right\} \leq r.$$

(c) *If the assumption (A4) holds for $T = I - \nabla\psi$, then $\{x_n\}$ converges strongly to x^*.*

Theorem 4.1 is applicable when $\nabla\psi$ is not necessarily Lipschitzian. Therefore, Theorem 4.1 provides an affirmative answer of Problem 1.1.

In particular, we have the following.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space X. Let $b \in X$ and $Q\colon X \to X$ a self-adjoint bounded linear operator and strongly positive; that is, there exists $\eta > 0$ such that $\langle Q(x), x\rangle \geq \eta||x||^2$ for all $x \in X$. Define a quadratic functional $f\colon X \to \mathbb{R}$ by*

$$f(x) = \frac{1}{2}\langle Q(x), x\rangle + \langle x, b\rangle \text{ for all } x \in X$$

and let $\mu \in (0, 2\eta/||Q||^2)$ such that $(I - \alpha\mu\mathscr{F})(C) \subseteq C$ for all $\alpha \in (0,1]$. Let $\psi : X \to \mathbb{R}$ be a functional such that (A5) and (A6) hold.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C generated by

$$\begin{cases} x_{n+1} := (1-\lambda_n)y_n + \lambda_n(I - \nabla\psi)^n(y_n), \\ y_n = x_n - \alpha_n\mu\nabla f(x_n) \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that assumptions (A1)-(A3) hold with $T = I - \nabla\psi$. Then we have the following:

(a) $\Omega[\mathrm{VIP}_{\mathbb{S}}(\nabla\psi, C)] = \{x^*\}$.

(b) *The orbit $\{x_n\}$ of the Algorithm 3.1 is well defined in the closed convex $B_r[x^*] \cap C$, where r is a positive constant such that*

$$\max\left\{\|x_1 - x^*\|, \frac{1}{\tau}(\mu||\mathscr{F}(x^*)\| + K)\right\} \leq r.$$

(c) *If the assumption (A4) holds for $T = I - \nabla\psi$, then $\{x_n\}$ converges strongly to x^*.*

5. Conclusion

For finding a solution of a variational inequality problem with a strongly monotone mapping over the set of fixed points of a nearly asymptotically nonexpansive mapping on Hilbert spaces, we have introduced a steepest descent method. Its strong convergence has been proved without the Lipschitz continuity of the mapping in the framework of Hilbert spaces.

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