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HAHN-BANACH EXTENSION AND OPTIMIZATION RELATED TO FIXED POINT PROPERTIES AND AMENABILITY

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Dedicated to the memory of Jonathan Borwein for his tremendous contribution to optimization and fixed point theory

Abstract. The purpose of this paper is to present the relation of Hahn-Banach extension properties with optimization, convex analysis and fixed point properties. Some open problems are listed. A new extension of Caristi's fixed point theorem to families of mappings will be presented.

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1. Introduction

Throughout this paper, we assume that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. We first introduce some standard notation. Given $f\colon X\to\,]-\infty,+\infty]$, we set $\operatorname{dom} f := f^{-1}(\mathbb{R})$, and f^* is the *Fenchel conjugate* of f defined by $x^*\in X^*\mapsto \sup_{x\in X}\big(\langle x,x^*\rangle - f(x)\big)$. We say f is *proper* if $\operatorname{dom} f\neq\varnothing$. Let f be proper. Then $\partial f\colon X\rightrightarrows X^*\colon x\mapsto \big\{x^*\in X^* \mid (\forall y\in X)\ \langle y-x,x^*\rangle + f(x)\leq f(y)\big\}$ is the *subdifferential operator* of f. Set $\operatorname{argmin} f := \{x\in X \mid f(x)=\inf f\}$. Let $g\colon X\to\,]-\infty,+\infty]$. The *inf-convolution* $f\Box g$ is the function defined on X by

$$f\Box g\colon x\mapsto \inf_{z\in X}\Big(f(z)+g(x-z)\Big).$$

Given a subset C of X, $\operatorname{int} C$ is the *interior* of C, $\overline{C}$ is the *closure* of C, and $\operatorname{bdry} C$ is the *boundary* of C. The *indicator function* of C, written as ι_C, is defined at $x\in X$ by

$$\iota_C(x) := \begin{cases} 0, & \text{if } x\in C;\\ +\infty, & \text{otherwise.}\end{cases}$$

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The *normal cone* operator of the set C is $N_C := \partial\iota_C$. Let $x_0 \in C$. We say x_0 is a *support point* of C if there exists $x_0^* \in X^*\backslash\{0\}$ such that $\langle x_0, x_0^*\rangle = \sup\langle x_0^*, C\rangle$, i.e., $N_C(x_0) \neq \{0\}$. In this case, we say x_0^* is a *support functional* for C. Let $T : C \to C$. We say that T is *nonexpansive* if $\|Tx - Ty\| \leq \|x-y\|, \quad \forall x,y \in C$. The *distance function* to the set C, written as $\mathbf{dist}(\cdot, C)$, is defined by

$$x \longmapsto \inf\big\{\|x-c\| \mid c \in C\big\}.$$

We set $\mathbf{dist}^2(x,C) := [\mathbf{dist}(x,C)]^2, \forall x \in X$.

Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of A, and $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$ be the *domain* of A. Recall that A is *monotone* if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (x,x^*) \in \operatorname{gra} A \; \forall (y,y^*) \in \operatorname{gra} A.$$

We have subdifferential operators are always monotone. We say A is *maximally monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion).

The *open unit ball* in X is denoted by $U_X := \big\{x \in X \mid \|x\| < 1\big\}$, the *closed unit ball* in X is denoted by $B_X := \big\{x \in X \mid \|x\| \leq 1\big\}$, and $\mathbb{N} := \{1,2,3,\ldots\}$. We denote by $\longrightarrow$ and $\rightharpoonup$ the norm convergence and the weak convergence, respectively.

The rest of this paper are organized as follows. In Section 2, we collect some well known results in Optimization and Convex Analysis like Rockafellar's duality theorem, Bishop-Phelps Theorem, Ekeland's variational principle. We also present some new proof to some of these above results as well as a new generalization of the Caristi's fixed point theorem for a family of set-valued mappings. The recent progress for the famous sum problem in Monotone Operator Theory and the Chebyshev set problem are presented in Sections 3 and 4, respectively. In the Section 5, we focus on the fixed point properties with related to semigroups. We list some classical characterizations of the existence of left invariant means in the spaces: $LUC(S)$, $AP(S)$, and $WAP(S)$ according to the fixed point properties, and Hahn-Banach extension properties for semigroups. Some open problems in these areas are listed. It is our hope that this paper will be helpful to researchers and graduate students in the area.

2. Duality theory and Hahn-Banach Extension

We first introduce the two well known results by Rockafellar.

Theorem 2.1 (Rockafellar)**.** (See [36, Theorem 18.7] or [42, Theorem 3.2.8].) *Let $f : X \to \,]-\infty, +\infty]$ be proper, lower semicontinuous and convex. Then ∂f is maximally monotone.*

Theorem 2.2 (Rockafellar)**.** (See [34, Theorem 3] or [42, Theorem 2.8.7(iii)].) *Let $f, g : X \to \,]-\infty, +\infty]$ be proper convex. Assume that there exists $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that f is continuous at x_0. Then $(f+g)^* = f^* \Box g^*$ in X^* and the infimal convolution is exact everywhere. Furthermore, $\partial(f+g) = \partial f + \partial g$.*

The above Rockafellar's duality theorem is a very powerful tool in Convex Analysis and Optimization, which can deduce various current well known results like Bishop-Phelps theorem and Hahn-Banach

Extension. Applying Theorem 2.2, we can also obtain the following nice result, which was contributed by Marques Alves and Svaiter.

Theorem 2.3 (Marques Alves and Svaiter)**.** (See [30, Theorem 3.4].) *Let $F : X \times X^* \to]-\infty, +\infty]$ be proper lower semicontinuous convex such that*

$$F \geq \langle \cdot, \cdot \rangle \quad \text{on } X \times X^* \quad \text{and} \quad F^* \geq \langle \cdot, \cdot \rangle \quad \text{on } X^* \times X^{**}.$$

Let $\varepsilon > 0$ and $(x_0, x_0^*) \in X \times X^*$. Assume that $F(x_0, x_0^*) < \langle x_0, x_0^* \rangle + \varepsilon$. Then there exists $(y_0, y_0^*) \in X \times X^*$ such that

$$F(y_0, y_0^*) = \langle y_0, y_0^* \rangle \quad \text{and} \quad \max\left\{\|x_0 - y_0\|, \|x_0^* - y_0^*\|\right\} < \sqrt{\varepsilon}.$$

For the following Bishop-Phelps Theorem, we directly apply Theorem 2.2 (along with Theorem 2.1) to establish its first part: (i). The part (ii) is proved by Theorem 2.3.

Theorem 2.4 (Bishop-Phelps)**.** (See [33, Theorem 3.18] or [42, Theorem 3.1.8].) *Let $C \subseteq X$ be a nonempty closed convex set with $C \neq X$. Then*

(i) *The set of support points of C is dense in* $\operatorname{bdry} C$.
(ii) $\operatorname{ran} N_C$ *is dense in* $\operatorname{dom} \iota_C^*$. *Therefore, the set of support functionals for the closed unit ball: B_X is dense in X^*.*

Proof. (i): Let $z \in \operatorname{bdry} C$, and $\delta > 0$. Let $f : X \to]-\infty, +\infty]$ be defined by

$$f(x) := \begin{cases} \dfrac{1}{\delta^2 - \|x - z\|^2}, & \text{if } \|x - z\| < \delta; \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $g(t) := \frac{1}{\delta^2 - t^2}$ is convex on the $[0, \delta[$ (by that $g''(t) = 2(\delta^2 - t^2)^{-2} + 8t^2(\delta^2 - t^2)^{-3} \geq 0, \forall t \in [0, \delta[$), f is convex. Thus we have f is proper lower semicontinuous and convex, so is $f + \iota_C$. Since f is continuous on its domain, we have

$$\operatorname{dom} \partial f = \operatorname{dom} f = z + \delta U_X. \tag{2.1}$$

By Theorem 2.2,

$$\partial(f + \iota_C) = \partial f + N_C. \tag{2.2}$$

Since $z \in \operatorname{bdry} C$,

$$C \cap \operatorname{dom} \partial f = C \cap [z + \delta U_X] \subsetneqq z + \delta U_X = \operatorname{dom} \partial f.$$

Then $\operatorname{gra} \partial f \cap [C \times X^*] \subsetneqq \operatorname{gra} \partial f$, and hence $\operatorname{gra} \partial f \cap [C \times X^*]$ is not a maximally monotone set. On the other hand, since $\partial(f + \iota_C)$ is maximally monotone (see Theorem 2.1), $\operatorname{gra} \partial(f + \iota_C) \neq \operatorname{gra} \partial f \cap [C \times X^*]$. Hence (2.2) implies that $\operatorname{gra}(\partial f + N_C) \neq \operatorname{gra} \partial f \cap [C \times X^*]$. Therefore, there exists $x \in \operatorname{dom} \partial f = z + \delta U_X$ with $x \in C$ such that $N_C(x) \neq \{0\}$. Then (i) holds.

(ii): Let $x^* \in \operatorname{dom} \iota_C^*$. Then there exist a sequence $(x_n)_{n \in \mathbb{N}}$ in C such that (for every $n \in \mathbb{N}$)

$$\iota_C^*(x^*) < \langle x_n, x^* \rangle + \frac{1}{n^2} \quad \text{and thus} \quad F(x_n, x^*) < \langle x_n, x^* \rangle + \frac{1}{n^2}, \tag{2.3}$$

where $F := \iota_C \oplus \iota_C^*$.

Clearly, we have $F \geq \langle\cdot,\cdot\rangle$ on $X \times X^*$ and $F^* = \iota_C^* \oplus \iota_C^{**} \geq \langle\cdot,\cdot\rangle$ on $X^* \times X^{**}$. Then applying Theorem 2.3 and (2.3), there exists $(y_n, y_n^*) \in X \times X^*$ such that

$$F(y_n, y_n^*) = (\iota_C \oplus \iota_C^*)(y_n, y_n^*) = \langle y_n, y_n^* \rangle \quad \text{and} \quad \|y_n^* - x^*\| < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Then $y_n^* \in N_C(y_n)$ and $y_n^* \longrightarrow x^*$. Thus $x^* \in \overline{\operatorname{ran} N_C}$ and hence $\operatorname{ran} N_C$ is dense in $\operatorname{dom} \iota_C^*$.

Now set $C := B_X$. Then the above result shows that

$$X^* = \operatorname{dom} \iota_C^* \subseteq \overline{\operatorname{ran} N_C} = \overline{[\operatorname{ran} N_C] \setminus \{0\}}.$$

$\square$

Remark 2.1. The proof of above Theorem 2.4 is new and is not in the literature. In particular, the proof of Theorem 2.4(i) comes from a new perspective.

Theorem 2.2 can be used to deduce the Hahn–Banach extension below when the sublinear function P is continuous (see the proof in [1, Theorem 11]).

Theorem 2.5 (Hahn–Banach Extension)**.** (See [36, Corollary 2.2].) *Let $P\colon X \to \mathbb{R}$ be a sublinear function. Suppose that F is a linear subspace of X and the function $M\colon F \to \mathbb{R}$ is linear such that $M \leq P$ on F. Then there exists a linear function $L : X \to \mathbb{R}$ such that $L \leq P$ on X and $L|_F = M$.*

We finish this section with the popular Ekeland's variational principle, which has numerous applications in Nonlinear Analysis.

Theorem 2.6 (Ekeland)**.** (See [33, Theorem 3.13] or [42, Corollary 1.4.2].) *Let $f : X \to\,]-\infty, +\infty]$ be a proper lower semicontinuous and lower bounded function. Assume that there exist $x_0 \in X$ and $\varepsilon > 0$ such that*

$$f(x_0) \leq \inf f + \varepsilon.$$

Then for every $\lambda > 0$, there exists $\widetilde{x} \in \operatorname{dom} f$ such that

(i) $\|\widetilde{x} - x_0\| \leq \dfrac{\varepsilon}{\lambda}$. (ii) $\lambda\|\widetilde{x} - x_0\| \leq f(x_0) - f(\widetilde{x})$.

(iii) $f(x) + \lambda\|x - \widetilde{x}\| > f(\widetilde{x}), \quad \forall x \neq \widetilde{x}$.

Applying Ekeland's variational principle, we may deduce the following extension of Caristi's fixed point theorem for a family of mappings (See [11, Theorem (2.1)′] or [17].). The proof of Theorem 2.7 was inspired by [42, Exercise 1.14, page 303].

Let $C \subseteq X$ be a nonempty set and $T : C \rightrightarrows X$ be a set-valued mapping. The set of the fixed points of T is $\operatorname{Fix} T := \{x \in C \mid x \in Tx\}$. Let S be a nonempty set, and $\mathscr{S} := \{T_s : C \rightrightarrows X \mid s \in S\}$. We say C has *a common fixed point* for S if

$$\bigcap_{s \in S} \operatorname{Fix} T_s \neq \varnothing, \quad \text{i.e.,} \quad \{x \in C \mid x \in T_s x,\ \forall s \in S\} \neq \varnothing.$$

In this case, $\bigcap_{s \in S} \operatorname{Fix} T_s$ is the set of all common fixed points of C for S.

Theorem 2.7. *Let $\lambda_0 > 0$ and $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous and lower bounded function with $C := \operatorname{dom} f$. Let S be a nonempty set, and $\mathscr{S} := \{T_s : C \rightrightarrows X \mid s \in S\}$. Assume that for every $x \in \operatorname{dom} f$ and every $s \in S$, there exists $z \in T_s x$ such that $f(z) + \lambda_0\|z - x\| \leq f(x)$. Then*

(i) $\operatorname{argmin} f \subseteq \bigcap_{s\in S} \operatorname{Fix} T_s \neq \varnothing$. *Therefore, C has a common fixed point for S. Moreover, if* $\operatorname{argmin} f = \varnothing$, *then C has infinite common fixed points for S.*

(ii) *C has the unique common fixed point for S* $\iff$ $\operatorname{argmin} f = \bigcap_{s\in S} \operatorname{Fix} T_s$ *is a singleton.*

Proof. (i): We consider two cases.

Case 1: $\operatorname{argmin} f \neq \varnothing$.

Let $x \in \operatorname{argmin} f$. Fix $s \in S$. By the assumption, there exists $z_s \in T_s x$ such that

$$f(z_s) + \lambda_0\|z_s - x\| \leq f(x) \leq f(z_s).$$

Thus $\|z_s - x\| = 0$ and then $z_s = x$. Thus $x \in T_s x$ and hence $x \in \operatorname{Fix} T_s$. Then we have

$$x \in \operatorname{Fix} T_s, \qquad \forall s \in S.$$

Then we have $\operatorname{argmin} f \subseteq \bigcap_{s\in S} \operatorname{Fix} T_s$ and then $\bigcap_{s\in S} \operatorname{Fix} T_s \neq \varnothing$. Hence C has a common fixed point for S.

Case 2: $\operatorname{argmin} f = \varnothing$.

Clearly, $\varnothing = \operatorname{argmin} f \subseteq \operatorname{Fix} T$. Now we show that $\bigcap_{s\in S} \operatorname{Fix} T_s$ has infinite elements.

By the assumption, there exists $x_n \in X$ such that

$$f(x_n) \leq \inf f + \frac{1}{n}, \qquad \forall n \in \mathbb{N}. \tag{2.4}$$

Fix $n \in \mathbb{N}$. Applying Theorem 2.6 that $\lambda = \lambda_0$, there exists $y_n \in \operatorname{dom} f$ such that

$$\lambda_0\|y_n - x_n\| \leq f(x_n) - f(y_n) \quad \text{and} \quad f(x) + \lambda_0\|x - y_n\| > f(y_n), \quad \forall x \neq y_n. \tag{2.5}$$

Let $s \in S$. By the assumption again, there exists $z_{s,n} \in T_s y_n$ such that

$$f(z_{s,n}) + \lambda_0\|z_{s,n} - y_n\| \leq f(y_n).$$

Then combining with the second part of (2.5), we have $z_{s,n} = y_n$ and then $y_n \in T_s y_n$. Hence

$$y_n \in T_s y_n, \quad \forall s \in S, \quad \forall n \in \mathbb{N}.$$

This gives us that

$$y_n \in \bigcap_{s\in S} \operatorname{Fix} T_s, \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Now suppose to the contrary that $\bigcap_{s\in S} \operatorname{Fix} T_s$ only has finite elements. By (2.6), there exists a subsequence of $(y_n)_{n\in\mathbb{N}}$, for convenience, still written as $(y_n)_{n\in\mathbb{N}}$ such that

$$y_n = y_0, \quad \forall n \in \mathbb{N}.$$

Then combining the first part of (2.5) with (2.4),

$$f(y_0) = f(y_n) \le f(x_n) \le \inf f + \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Thus $f(y_0) \le \inf f$ and hence $y_0 \in \operatorname{argmin} f$, which is a contradiction with that $\operatorname{argmin} f = \varnothing$.

Hence $\bigcap_{s\in S} \operatorname{Fix} T_s$ has infinite number of elements. Therefore, C has infinite number common fixed points for S. Combining all the above results, we have (i) holds.

(ii): $\Longleftarrow$: Clear.

$\Longrightarrow$: Since $\bigcap_{s\in S} \operatorname{Fix} T_s$ is a singleton, (i) gives us that $\operatorname{argmin} f \neq \varnothing$. Then by (i) again,

$$\varnothing \neq \operatorname{argmin} f \subseteq \bigcap_{s\in S} \operatorname{Fix} T_s.$$

Hence we have $\operatorname{argmin} f = \bigcap_{s\in S} \operatorname{Fix} T_s$ is a singleton. $\square$

By Theorem 2.7, we can directly deduce the following result by Feng and Liu.

Corollary 2.1 (Feng and Liu)**.** (See [19, Theorem 3.1].) *Let $C \subseteq X$ be a nonempty set and $0 < \gamma < \kappa$. Let $T : C \rightrightarrows X$ bet a set-valued mapping. Assume the following two conditions hold.*

(a) *For every $x \in C$, there exists $y \in Tx$ with $y \in C$ such that*

$$\kappa\|x-y\| \le \mathbf{dist}(x, Tx) \quad \textit{and} \quad \mathbf{dist}(y, Ty) \le \gamma\|x-y\|.$$

(b) *The function*

$$f(x) := \begin{cases} \mathbf{dist}(x, Tx), & \textit{if} \quad x \in C; \\ +\infty, & \textit{otherwise} \end{cases}$$

is lower semicontinuous.

Then $\operatorname{Fix} T \neq \varnothing$.

Proof. By the assumption (b), we have $f(x)$ is proper, lower semicontinuous and lower bounded. Let $x \in \operatorname{dom} f$ and hence $x \in C$, and let $\lambda_0 := \kappa - \gamma > 0$. Then by the assumption (a), there exists $y \in Tx$ with $y \in C$ such that

$$\begin{aligned} f(x) = \mathbf{dist}(x, Tx) \ge &\kappa\|x-y\| \\ &= \gamma\|x-y\| + (\kappa-\gamma)\|x-y\| \\ &\ge \mathbf{dist}(y, Ty) + \lambda_0\|x-y\| = f(y) + \lambda_0\|x-y\|. \end{aligned}$$

Thus, applying Theorem 2.7, we have $\operatorname{Fix} T \neq \varnothing$. $\square$

3. Monotone Operator Theory

Monotone operators are important in modern Optimization and Analysis [3, 7, 33, 36, 42, 43, 44].

Let A and B be monotone operators from X to X^* with $\operatorname{dom} A \cap \operatorname{dom} B \neq \varnothing$. Clearly, the *sum operator* $A+B\colon X \rightrightarrows X^*\colon x \mapsto Ax+Bx := \big\{a^*+b^* \mid a^* \in Ax \text{ and } b^* \in Bx\big\}$ is monotone.

The following *sum problem* is the most famous open problem in Monotone Operator Theory, which has attracted most interest since 1970s.

> **The sum problem:** Let $A, B : X \rightrightarrows X^*$ be maximally monotone. Assume that A and B satisfy the classical *Rockafellar's constraint qualification*:
>
> $$\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing.$$
>
> Is $A+B$ necessarily maximally monotone?

The sum problem has an affirmative answer in the setting of reflexive spaces, which was established by Rockafellar in 1970 (see [35]). The recent progress for the sum problem can be found in [36, 10, 9, 39, 40]. In particular, it shows in [39] that the sum problem is equivalent to its special case: Problem 3.1 (see Figure 1).

The sum problem

$\Updownarrow$

Open Problem 3.1. *Let $A : X \rightrightarrows X^*$ be maximally monotone, and $C \subseteq X$ be a nonempty closed and convex set. Assume that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. *Is $A+N_C$ necessarily maximally monotone?*

FIGURE 1. The equivalences

Another crucial problem in current Monotone Operator Theory is the following Simons' *domain problem* (see [36, Problem 31.3]):

Open Problem 3.2. *Let $A : X \rightrightarrows X^*$ be a maximally monotone operator. Is $\overline{\operatorname{dom} A}$ necessarily convex?*

If there was an affirmative answer to the sum problem, then we would answer affirmatively the above domain problem [36, 7]. The recent development for Problem 3.2 can be referred to [36, 8].

4. Chebyshev set problem

Let H be a real Hilbert space throughout Section 4. Assume that C is a nonempty subset of X. We define the *projector* to the set C, written as P_C, by

$$P_C(x) := \Big\{v \in C \mid \|x-v\| = \mathbf{dist}(x, C)\Big\}, \quad \forall x \in X.$$

We say C is a *Chebyshev set* if $P_C(x)$ is a singleton for every $x \in X$.

The following problem was posed by V. Klee in 1961 [23], which is so called *Chebyshev set problem*.

Open Problem 4.1. Let $C \subseteq H$ be a nonempty Chebyshev set. Is C necessarily convex?

The Chebyshev set problem might be the most famous open problem in Best Approximation Theory. Historical notes on the development of Problem 4.1 was stated in [15, Chapter 12, pp. 307–309].

The convexity of Chebyshev sets has been proved under various additional assumptions. Now we collect some classical results on the Chebyshev set problem as follows. For readers' convenience, we present the direct and self-contained proofs here (see Theorem 4.2).

Let $C \subseteq H$ be a nonempty Chebyshev set. The $\varphi_C : H \to \mathbb{R}$ is the *Asplund function* associated with C [2], defined by

$$x \mapsto \frac{1}{2}\|x\|^2 - \frac{1}{2}\,\mathbf{dist}^2(x,C)) = \left(\frac{1}{2}\|\cdot\|^2 + \iota_C\right)^*(x).$$

Then φ_C is continuous and convex, and

$$\operatorname{gra} P_C \subseteq \operatorname{gra} \partial\varphi_C \quad \text{(by [2])}. \tag{4.1}$$

Theorem 4.2. *Let $C \subseteq H$ be a nonempty Chebyshev set. Then C is convex if one of the following conditions hold.*

(i) $P_C = \partial\varphi_C$.

(ii) P_C *is continuous* (Vlasov/Asplund, 1967/1969 [2, 41]).

(iii) P_C is maximally monotone (Frerking–Westphal, 1978 [4]).

(iv) $\mathbf{dist}^2(\cdot,C)$ is Gâteaux differentiable (Vlasov-Efimov-Stechkin, [7, 42]).

(v) C is weakly closed (Klee, 1961 [23]).

Proof. (i): Since $P_C = \partial\varphi_C$, the Brøndsted-Rockafellar Theorem (see [42, Theorem 3.1.2, page 161]) shows that

$$C = \overline{C} = \overline{\operatorname{ran} P_C} = \overline{\operatorname{ran} \partial\varphi_C} = \overline{\operatorname{dom} \varphi_C^*} \quad \text{is convex.}$$

(ii)&(iii): Then we have P_C is maximally monotone, and thus $P_C = \partial\varphi_C$ by (4.1) since $\partial\varphi_C$ is monotone. Then apply (i) directly.

(iv): Since $\mathbf{dist}^2(\cdot,C)$ is Gâteaux differentiable, $\varphi_C = \frac{1}{2}\|\cdot\|^2 - \frac{1}{2}\mathbf{dist}^2(\cdot,C)$ is Gâteaux differentiable. Thus $\partial\varphi_C(x) = \nabla\varphi_C(x)$ is a singleton for every $x \in H$. Then (4.1) shows that $P_C = \partial\varphi_C$. Then (i) shows that C is convex.

(v): We show that P_C is continuous. Let $x_n \longrightarrow x$. Since $\mathbf{dist}(\cdot,C)$ is continuous (1-Liptschitz), we have

$$\|x_n - P_C(x_n)\| \longrightarrow \|x - P_C(x)\| = \mathbf{dist}(x,C). \tag{4.2}$$

Since $(P_C(x_n))_{n\in\mathbb{N}}$ is bounded, there exists a subsequence $(P_C(x_{n_k}))_{k\in\mathbb{N}}$ of $(P_C(x_n))_{n\in\mathbb{N}}$ such that $P_C(x_{n_k}) \rightharpoonup x^* \in H$. By the assumption that C is weakly closed, $x^* \in C$. Thus (4.2) shows that

$$\|x - x^*\| \leq \liminf_{k\to\infty} \|x_{n_k} - P_C(x_{n_k})\| = \mathbf{dist}(x,C). \tag{4.3}$$

Since $x^* \in C$ and C is a Chebyshev set, the above inequality shows that $x^* = P_C(x)$. Since $P_C(x)$ is the unique weak cluster point of $(P_C(x_{n_k}))_{k\in\mathbb{N}}$, we have $P_C(x_n) \rightharpoonup P_C(x)$.

Since $\|x_n - P_C(x_n)\| \longrightarrow \|x - P_C(x)\|$ (see (4.2)) and $x_n - P_C(x_n) \rightharpoonup x - P_C(x)$, we have $x_n - P_C(x_n) \longrightarrow x - P_C(x)$ and hence $P_C(x_n) \longrightarrow P_C(x)$. Hence $P_C(x)$ is continuous. Now apply (ii) to have C is convex. □

Remark 4.1. Clearly, when the set C is closed convex, the statements (i)-(v) in Theorem 4.2 all hold [42, 15, 3]. The proof of Theorem 4.2(v) follows closely the lines of the proof of [7, Proposition 4.5.8].

The following is the Chebyshev set problem in the setting of a more general space: a strictly convex and reflexive real Banach space.

Open Problem 4.3. Assume that X is a strictly convex and reflexive real Banach space. Let $C \subseteq X$ be a nonempty Chebyshev set. Is C necessarily convex?

M. M. Day first proved that every closed convex set in a strictly convex and reflexive Banach space is a Chebyshev set in 1941 (see [12] or [42, Theorem 3.8.1, page 238]), i.e., the converse of Problem 4.3 has an affirmative answer.

In this section, we also recall the following open problem posed by Borwein and Fitzpatrick in 1989 [6]:

Open Problem 4.4. Assume that X is a reflexive real Banach space. Let $C \subseteq X$ be a nonempty closed set with $C \neq X$. Is $P_C(X \backslash C)$ necessarily dense on bdryC?

5. Fixed point properties and Hahn-Banach extension properties

In this section, we will focus on the fixed point properties related to semingroups. We first present some preliminaries and notations.

Let S be a *semitopological semigroup*, i.e., S is a semigroup with Hausdorff topology such that for every $a \in S$, the mappings $s \mapsto sa$ and $s \mapsto as$ from S into S are continuous. We say $\mathbf{S} := \{T_s \mid s \in S\}$ is a representation of S on the subset C of a locally convex space if for each $s \in S$, $T_s : C \to C$ and $T_{st}(x) = T_s(T_t x), \quad \forall s,t \in S,\ x \in C$. Recall that C has a common fixed point for S if

$$\bigcap_{s \in S} \operatorname{Fix} T_s \neq \varnothing, \quad \text{i.e.,} \quad \big\{x \in C \mid T_s(x) = x,\ \forall s \in S\big\} \neq \varnothing.$$

Let $\ell^\infty(S)$ denote the C^*-algebra of bounded complexed-valued functions on S with the supremum norm and pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$, let $\ell_a f$ and $r_a f$ denote the *left and right translate* of f by a respectively, i.e., $(\ell_a f)(s) := f(as)$ and $(r_a f)(s) := f(sa), \quad \forall s \in S$. Let E be a closed subspace of $\ell^\infty(S)$ containing constants and invariant under translations. Then a linear functional $m \in E^*$ is called a *mean* if $\|m\| = m(1) = 1$. We say a mean m is a *left invariant mean*, denoted by LIM, if $m(\ell_a f) = m(f)$ for all $a \in S,\ f \in E$.

Let $CB(S)$ denote the space of all bounded continuous real-valued functions on S with the supremum norm. Let $LUC(S)$ be the space of all $f \in CB(S)$ such that the mappings $a \to \ell_a f$ from S into $CB(S)$ are continuous. Set $\mathscr{LO}(f) := \{\ell_s f \mid s \in S\}$, where $f \in CB(S)$.

The spaces $AP(S)$ and $WAP(S)$ are respectively denoted by the followings:

$$\begin{aligned} AP(S) = & \text{ the space of all } f \in CB(S) \text{ such that } \mathscr{L}O(f) \text{ is relatively compact in the} \\ & \text{norm topology of } CB(S); \\ WAP(S) = & \text{ the space of all } f \in CB(S) \text{ such that } \mathscr{L}O(f) \text{ is relatively compact in the} \\ & \text{weak topology of } CB(S). \end{aligned}$$

In general, we have the following inclusions.

$$AP(S) \subseteq LUC(S) \subseteq CB(S) \quad \text{and} \quad AP(S) \subseteq WAP(S) \subseteq CB(S).$$

The following important common fixed point theorem of Markov-Kakutani for a commutative semigroup is well known:

Theorem 5.1 (Markov-Kakutani Fixed Point Theorem)**.** *Let S be a commutative semigroup. Let C be a nonempty compact convex subset of a separated locally convex space, and $\mathbf{S} := \{T_s \mid s \in S\}$ be a representation of S as continuous affine mappings from C to C. Then C has a common fixed point for S.*

A semitopological semigroup S is called *left amenable* if $LUC(S)$ has a left invariant mean (LIM). It follows from Theorem 5.1 that any commutative semigroup S is left amenable. However, the free group (or free semigroup) on two generators is not amenable. In the following, we present some well known characterizations on the existences of left invariant means in the spaces of $LUC(S)$, $AP(S)$, and $WAP(S)$.

Theorem 5.2 (Mitchell)**.** (See [31, Theorem 2].) *Let S be a semitopological semigroup. Then $LUC(S)$ has a left invariant mean (LIM) $\Longleftrightarrow$ S has the following fixed point property:*

(F_*): *Whenever $\mathbf{S} := \{T_s \mid s \in S\}$ is a jointly continuous representation of S as continuous affine mappings on a nonempty compact convex subset C of a separated locally convex space, then C has a common fixed point for S.*

Day proved the above result when S is a discrete semigroup in [13].

Theorem 5.3. (See [26] and [24, Theorem 4.1].) *Let S be a semitopological semigroup. Then the following are equivalent.*

(i) *$AP(S)$ has a left invariant mean (LIM).*

(ii) *S has the fixed point property (F_a).*

(F_a): *Whenever $\mathbf{S} := \{T_s \mid s \in S\}$ is a continuous representation of S as equicontinuous continuous affine mappings on a compact convex subset C of a separated locally convex space, then C has a common fixed point for S.*

(iii) *S has the fixed point property (F_b).*

(F_b): *Whenever $\mathbf{S} := \{T_s \mid s \in S\}$ is a continuous representation of S as nonexpansive mappings on a nonempty compact convex subset C of a Banach space, then C has a common fixed point for S.*

Theorem 5.4. (See [26] and [29, Theorem 3.4].) *Let S be a semitopological semigroup. Then the following are equivalent.*

(i) *$WAP(S)$ has a left invariant mean (LIM).*

(ii) *S has the fixed point property (G_1).*

(G_1): *Whenever $\mathbf{S} := \{T_s \mid s \in S\}$ is a continuous representation of S as affine continuous mappings from a weakly compact convex subset C of a Banach space such that $\overline{\mathbf{S}} \subseteq C^C$ (product of C with the weak topology) consists entirely of continuous functions, then C has a common fixed point for S.*

(iii) *S has the fixed point property (G_2).*

(G_2): *Whenever $\mathbf{S} := \{T_s \mid s \in S\}$ is a continuous representation of S as weakly continuous norm nonexpansive mappings on a nonempty weakly compact convex subset C of a Banach space such that $\overline{\mathbf{S}} \subseteq C^C$ consists entirely of weakly continuous functions, then C has a common fixed point for S.*

A semitopological semigroup S is *left reversible* if $\overline{aS} \cap \overline{bS} \neq \varnothing, \quad \forall\, a, b \in S$.

$$\begin{array}{ccl} & & \nearrow\!\!\!\nearrow \text{ left reversible} \\ \text{Commutative} & & \\ & & \searrow\!\!\!\searrow \text{ left amenable} \end{array}$$

If S is left amenable as a discrete semigroup, then S is left reversible.

Open Problem 5.5. If S is a left reversible semitopological semigroup, does S have fixed point property (G_1)?

If S is a discrete semigroup, then the following holds:

$$\begin{array}{ccc} S \text{ left reversible} & \Longrightarrow & WAP(S) \text{ has } LIM \\ \upuparrows & \searrow\!\!\!\searrow & \downdownarrows \\ S \text{ left amenable} & & AP(S) \text{ has } LIM \end{array}$$

The following result of R. Hsu [22] answers Open Problem 5.5 when S is discrete.

Theorem 5.6. *Let S be a left reversible discrete semigroup. Then S has the following fixed point property:*

$(G_{2'})$: *Whenever $\mathbf{S} := \{T_s \mid s \in S\}$ is a continuous representation of S as weakly continuous norm nonexpansive mappings on a nonempty weakly compact convex subset C of a Banach space, then C contains a common fixed point for S.*

In particular, S has fixed point property (G_2).

Open Problem 5.7. Let S be a semitopological semigroup. Does the left reversibility of S imply $(G_{2'})$?

Let

$S_1 =$ the semigroup generated by a unit e and two other elements p, q subject to the relation:

$$pq = e.$$

$S_2 =$ the semigroup generated by a unit element e and three more elements a, b, c subject to the condition:

$$ab = ac = e.$$

$S_3 =$ the semigroup generated by a unit e and four more elements a, b, c, d subject to the relation:

$$ac = bd = e.$$

Theorem 5.8 (Duncan and Namioka)**.** (See [16].) *S_1 is amenable. In particular, S_1 is left (and right) reversible.*

Remark 5.1. (a) The bicyclic semigroup S_2 and S_3 are not left amenable.
(b) The bicyclic semigroup S_2 is right amenable.
(c) The bicyclic semigroup S_3 is not right amenable.

Theorem 5.9 (Mitchell)**.** (See [32].) *$AP(S_2)$ and $AP(S_3)$ have a left invariant mean.*

Proof. $S_2 =$ semigroup generated by $\{a, b, c, 1\}$, where 1 is the identity with the relation $ab = 1$, and $ac = 1$. Then Δ the spectrum of $AP(S_2)$ is a compact topological group. Hence if $m(f) = \int_\Delta \widehat{f}(t)d\mu(t)$, where μ is the normalized Haar mean on $\Delta f \in AP(S_2)$. Then m is a left invariant mean on $AP(S)$. □

Remark 5.2. S_2 (S_3) is not left reversible since $cS \cap bS = \varnothing$.

Theorem 5.10 (Lau and Zhang)**.** (See [29, Theorem 4.13].) *$WAP(S_3)$ does not have a fixed point property (G_1).*

Remark 5.3. The above theorem answers negatively a problem posed in Dalhousie 1976:

Is there a semigroup S such that $AP(S)$ has a LIM but $WAP(S)$ does not have a LIM?

Theorem 5.11 (Mitchell)**.** (See [32].) *There is a semigroup S such that $AP(S)$ has a* LIM *but S is not left reversible.*

Theorem 5.12 (Lau and Zhang)**.** (See [29, Theorem 4.11].) *The bicyclic semigroup S_2 is not left reversible but $WAP(S_2)$ has a left invariant mean. In particular, S_2 has fixed point property (G_1).*

Remark 5.4. Theorem 5.12 above answered an open problem of Mitchell in [32].

In a summary, we have

$$S \quad \text{left amenable}$$
$$\Downarrow \quad \not\Uparrow$$
$$S \quad \text{left reversible}$$
$$\Downarrow \quad \not\Uparrow$$
$$\text{WAP(S)} \quad \text{has LIM}$$
$$\Downarrow \quad \not\Uparrow$$
$$\text{AP(S) has LIM.}$$

Theorem 5.13 (Lau, Nishiura and Takahashi)**.** (See [28].) *Let S be a semitopological semigroup. If S is left reversible or left amenable, then the following fixed point property holds.*

(G_3) : *Whenever $\mathbf{S} := \{T_s \mid s \in S\}$ is a norm nonexpansive representation of S on a nonempty norm separable weak* compact convex set C of the dual space of a Banach space E and the mapping $(s,x) \mapsto T_s(x)$ from $S \times C$ to C is jointly continuous when C is endowed with the weak* topology of E^*, then there is a common fixed point for S in C.*

Open Problem 5.14. Does the semigroup S_3 have property (G_3)?

Let S be a semitopological semigroup S and E be a topological vector space. Then *a right linear action* of S on E is a separately continuous map from $S \times E \to E$, denoted by $(s,x) \to s \cdot x$ satisfying

(i) $(ab) \cdot x = b \cdot (a \cdot x)$ for all, $a, b \in E$;
(ii) for each $s \in S$, the map $x \to s \cdot x$ is a linear from E into E.

Theorem 5.15 (a)⇒(b) is due to Silverman [37] for the case when S has the discrete topology, (see also [14, Page 4] and [38, Page 576]). Let P be a real-valued function on a vector space E. We say that P is *sublinear* if $P(x+y) \leq P(x) + P(y)$ and $P(\lambda x) = \lambda P(x)$ for all $x, y \in E, \lambda \geq 0$.

Theorem 5.15. (See [25, Theorem 1].) *Let S be a semitopological semigroup. The following conditions on S are equivalent:*

(a) *S is left amenable.*
(b) *For any right linear action of S on a real topological vector space E, if P is a continuous sublinear map on E such that $P(s \cdot x) \leq P(x)$ for all $s \in S, x \in E$, and if ϕ is an invariant linear functional on an invariant subspace F of E such that $\phi \leq P$, then there exists an invariant extension $\widetilde{\phi}$ of ϕ to E such that $\widetilde{\phi} \leq P$.*
(c) *For any right linear action of S on a topological vector space E, if U is an invariant open convex subset of E containing an invariant element, and M is an invariant subspace of E which does not meet U, then there exists a closed invariant hyperplane H of E such that H contains M and H does not meet U.*
(d) *For any right linear action of S on a Haudsroff topological vector space E with a base of the neighbourhoods of the origin consisting of invariant open convex sets, then any two points*

in $E_f := \{x \in E \mid sx = x \quad \text{for all } s \in S\}$ *can be separated by a continuous invariant linear functional on* E.

Corollary 5.1. (See [25, Corollary].) *Let* S *be a semitopological semigroup. If* S *is abelian,a solvable group, or a compact semigroup with finite intersection property for right ideals, then* S *has properties* $(b),(c)$ *and* (d) *of Theorem 5.15.*

Let E be a partially ordered topological vector space over the real field. An element $e \in E$ is a topological order unit if e is an *order unit* (i.e., for each $x \in E$, there exists $\lambda > 0$ such that $-\lambda e \leq x \leq \lambda e$) and the absolutely convex set $[-e,e]$ is a neighbourhood of E, where $[a,b] := \{x \in E \mid a \leq x \leq b\}$ for any $a,b \in E$. A subspace I of E is a *proper ideal* if $I \neq E$ and $x \in E$ implies $[0,x] \subseteq I$. An action of S on E is *positive* if $s \cdot x \geq 0$, 0 for all $s \in S$ and $x \geq 0$. The action is *normalized* (with respect a topological order unit e) if $s \cdot e = e$, for all $s \in S$. Note that if E is a partially order vector space (no topology) and e is an order unit of E. Then P, the *Minkowski functional* of $[-e,e]$ on E, i.e.,

$$P(x) := \inf\left\{\lambda > 0 \mid -\lambda e \leq x \leq \lambda e\right\}$$

is a *semi-norm* on E, and e will become a topological order unit of the locally convex space E equipped with the topology determined by P.

An important example of partially ordered topological vector space that we shall be concerned with is $LUC(S)$ with the natural ordering $f \leq g$ if and only if $f(s) \leq g(s)$ for all $s \in S$. In this case, 1, the constant one function on S, is a topological order unit of $LUC(S)$.

It is known [5, Page 124] that if $\mathscr{F}$ is a commuting family of positive normalised linear endomorphisms of a partially ordered vector space E with a unit, then E contains a proper maximal ideal which is invariant under each map in $\mathscr{F}$. Our next result shows that a much stronger result also holds.

Theorem 5.16. (See [25, Theorem 2].) *Let* S *be a semitopological semigroup. The following conditions on* S *are equivalent:*

(a) *S is left amenable.*
(b) *For any positive normalised right linear action of S on a partially ordered topological vector space E with a topological order unit e, if F is an invariant subspace of E containing e, and ϕ is an invariant monotonic linear functional on F, then there exists an invariant monotonic linear function $\widetilde{\phi}$ on E extending ϕ.*
(c) *For any positive normalised right linear action on a partially ordered topological vector space E with a topological order unit e, E contains a maximal proper ideal which is invariant under S.*

Corollary 5.2. (See [25, Corollary].) *Let* S *be a semitopological semigroup. If* S *is abelian, a solvable group, or a compact semigroup with finite intersection property for right ideals, then* S *has properties* (b) *and* (c) *of Theorem 5.16.*

Let S be a semitopological semigroup. We say that the action of S on a Hausdorff real topological vector space E is *almost periodic* (resp. *weakly almost periodic*) if for each $x \in E$, the orbit: $\{s \cdot x \mid s \in S\}$ is relatively compact in the topology of E (resp. weak topology).

Theorem 5.17. (See [27, Theorem 1].) *Let S be a semitopological semigroup. The following conditions on S are equivalent:*

(a) *$AP(S)$ has a left invariant mean (LIM).*

(b) *For any almost periodic continuous right action of S on a Hausdorff real topological vector space E, if F is an invariant subspace of E and K is a convex subset of E such that $K - x_0$ is invariant for some $x_0 \in F \cap \operatorname{int} K$, then for each invariant linear functional ϕ on F such that $\phi(x) \leq \alpha$ for all $x \in F \cap K$ and some fixed real number α, then there exists an invariant linear extension $\widetilde{\phi}$ of ϕ to E such that $\widetilde{\phi}(x) \leq \alpha$ for all $x \in K$.*

Theorem 5.18. (See [27, Theorem 2].) *Let S be a semitopological semigroup. The following conditions on S are equivalent:*

(a) *$WAP(S)$ has a left invariant mean (LIM).*

(b) *For any weakly almost periodic continuous right action of S on a Hausdorff real topological vector space E, if F is an invariant subspace of E and K is a convex subset of E such that $K - x_0$ is invariant for some $x_0 \in F \cap \operatorname{int} K$, then for each invariant linear functional ϕ on F such that $\phi(x) \leq \alpha$ for all $x \in F \cap K$ and some fixed real number α, then there exists an invariant linear extension $\widetilde{\phi}$ of ϕ to E such that $\widetilde{\phi}(x) \leq \alpha$ for all $x \in K$.*

Remark 5.5. (i) Since $AP(S)$ and $WAP(S)$ always have a left-invariant mean for any group S (see [20, Page 38]) our Theorem 5.17 implies [18, Theorem 1] and our Theorem 5.18 implies [18, Theorem 3] without completeness hypothesis.

(ii) If $aS \cap bS \neq \varnothing$, $\forall a, b \in E$, then S has property (b) of Theorem 5.17, since in this case $AP(S)$ has a left-invariant mean (see [21]).

(iii) We have actually proved implicitly the following analogue of [37, Theorem 15.A] by Silverman:

Theorem 5.19. (See [27, Theorem 3].) *Let S be a semitopological semigroup. The following conditions on S are equivalent:*

(a) *$AP(S)$ has a left invariant mean (LIM).*

(b) *For any almost periodic continuous right action of S on a Hausdorff real topological vector space E, if P is a continuous sublinear functional on E such that $P(s \cdot x) \leq P(x)$, $\forall s \in S, x \in E$, and if ϕ is an invariant linear functional defined on a subspace F of E such that $\phi \leq P$, then there exists an invariant linear extension $\widetilde{\phi}$ of ϕ to E such that $\widetilde{\phi} \leq P$.*

A similar statement for $WAP(S)$ with "almost periodic continuous right action" replaced by "weakly almost periodic weakly continuous right action" also holds.

Remark 5.6. See also Fan [18].

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