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ON STRONG CONVERGENCE OF SOME MIDPOINT TYPE METHODS FOR NONEXPANSIVE MAPPINGS

GIUSEPPE MARINO[1,2,*], ROBERTA ZACCONE[1]

[1]*Dipartimento di Matematica e Informatica, Universitá della Calabria, 87036, Arcavacata di Rende (CS), Italy*
[2]*Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Abstract. Two variants of the implicit and explicit midpoint rules for nonexpansive mappings are introduced in the framework of Hilbert spaces. Strong convergence for these methods is proved under suitable assumptions on the parameters sequences.

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1. INTRODUCTION

Throughout this paper, the following notation will be used: H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$; $C \subset H$ is a nonempty, closed and convex set; $T : C \to C$ is a noexpansive mapping, that is, $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$, with fixed points set $Fix(T) = \{z \in C : Tz = z\}$. We always assume that $Fix(T)$ is nonempty in this article.

Iterative methods approximating fixed points for nonexpansive mappings can be applied to solve various problems in mathematics, for instance to find a solution for a variational inequality, zeros of accretive operators and a minimizer of a convex function. In light of this connection, in the recent years the convergence analysis for these methods has received a great deal of attention.

One of the most investigated methods in litterature dates back to 1953, due to Mann [11]. Let C be a nonempty, closed and convex subset of a Banach space X. Mann's scheme is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T(x_n), \quad n \geq 0, \end{cases} \tag{1.1}$$

where $(\alpha_n)_{n \in \mathbb{N}}$ is a real control sequence in $(0, 1)$.

A well known result (in [13]) states that if $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = +\infty$ and C is a subset of a uniformly convex Banach space X with Fréchet differentiable norm, then the sequence generated by (1.1) weakly converges to a fixed point of T. On the other hand, the convergence, in general, is not strong even in the

*Corresponding author.
E-mail addresses: giuseppe.marino@unical.it (G. Marino), zaccone@mat.unical.it (R. Zaccone).
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framework of Hilbert spaces, as was showed in a celebrated counterexample by Genel and Linderstrauss [4]. Ever since, several modifications have been proposed for the original method in order to get strong convergence (just to mention some of the most relevant, see [3], [6] and[12]).

In this paper, we continue to carry out the research along the same direction. In particular, we follow the investigation line introduced by Alghamdi *et al.* in [1]: the construction of iterative methods approximating fixed points of nonlinear mappings in an infinite-dimentional space, starting from numerical schemes approximating the solution of an initial value problem in a finite-dimensional space.

In detail, let us consider the initial value problem (IVP)

$$\begin{cases} x'(t) = \Phi(x(t)), \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where $\Phi : \mathbb{R}^s \to \mathbb{R}^s$ is a continuous function.

Given a time interval $[0,T]$ and fixed $N \in \mathbb{N}$, let

- $\{t_n\}_{n=0}^N$ the mesh of time nodes obtained subdividing $[0,T]$ in N subintervals of length $h = \frac{T}{N}$,
- y_n the approximed value of the soluton x to (1.2) at t_n,

the numerical Implicit Midpoint Rule (IMR) generates the set of values $\{y_n\}_{n=0}^N$ through the finite difference scheme

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = y_n + h\Phi(\frac{y_n+y_{n+1}}{2}), \quad n = 0, \cdots, N-1. \end{cases} \tag{1.3}$$

Connecting each pair of consecutive nodes $(t_n, y_n), (t_{n+1}, y_{n+1})$ by straight line, a polygonal function Y_N (called Euler polygonal) is obtained. If, in addition, Φ is a Lipschitz continuous and sufficiently smooth function, then sequence $\{Y_N\}_{N\in\mathbb{N}}$ converges to the exact solution of (1.2), as $N \to \infty$, uniformly on $t \in [t_0, T]$, for any fixed $T > 0$, (see [7, Theorem 7.3]).

If $\Phi(x) = x - g(x)$, then scheme (1.3) can be rewritten as

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = y_n + h[\frac{y_n+y_{n+1}}{2} - \Phi(\frac{y_n+y_{n+1}}{2})], \quad n = 0, \cdots, N-1 \end{cases}$$

and the problem of finding the critical points for Φ can be traited as the fixed point problem for g, $g(x) = x$. This fact motivated Alghamdi, Alghamdi and Shahzad [1] to employ the finite difference scheme (1.3) for the construction, by formal analogy, of an iterative method approximating fixed points for a nonexpansive mapping $T : H \to H$.

The proposed method is given by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = x_n - t_n[\frac{x_n+x_{n+1}}{2} - T(\frac{x_n+x_{n+1}}{2})], n \geq 0, \end{cases} \tag{1.4}$$

where $(t_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$.

It can be seen that this scheme is equivalent to the following one

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-t_n)x_n + t_n T(\frac{x_n+x_{n+1}}{2}), n \geq 0, \end{cases} \tag{1.5}$$

where $(t_n)_{n\in\mathbb{N}} \in (0,1)$.

This second formulation is a Mann type scheme and it is known as IMR for nonexpansive mappings.

For the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.5), the authors proved the following convergence result:

Theorem 1.1. [1, Theorem 2.6] *Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by (1.5), with $(t_n)_{n\in\mathbb{N}} \in (0,1)$ satisfying the conditions*

- $t_{n+1}^2 \leq at_n, \quad \forall n \geq 0$ *and some* $a > 0$,
- $\liminf_{n\to\infty} t_n > 0$.

The sequence $(x_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T.

As for the case of Mann's iterative process, with the aim to obtain strong convergence, a serie of variations provided to (1.5), generate a number of successive methods. We recall some of the most significant: the Viscosity Implicit Midpoint Rule (VIMR), with the introduction of a contraction mapping (Xu, Alghamdi and Shahzad [14]), the Generalized Viscosity Implicit Midpoint Rule (GVIMR), using a double convex combination (Ke and Ma [9]).

In this paper, following the same modification line of [8], we propose a modified IMR for nonexpansive mappings which differs from the previous ones available in literature. The proposed algorithm differs from scheme (1.5) for the introduction of a term $\alpha_n\mu_n(u-x_n)$ that can also be infinitesimal. The addition this term, that at first glance could appear non-significant, ensure the strong convergence of the method to a fixed point of the mapping.

In detail, let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be sequences in $(0,1]$, we introduce a modified implicit midpoint algorithm generating a sequence $(x_n)_{n\in\mathbb{N}}$ via the iterative procedure:

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}) + \alpha_n\mu_n(u-x_n), \quad n \geq 0, \end{cases} \tag{1.6}$$

where $T : H \to H$ is a nonexpansive mapping.

Assuming suitable conditions on the sequences of parameters $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$, we get strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.6) to the fixed point q_u of T nearest to u, that is, $\|u-q_u\| = \min_{x\in Fix(T)} \|u-x\|$.

In particular, if $u = 0 \in H$, then we get strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}) - \alpha_n\mu_n x_n, \quad n \geq 0 \end{cases}$$

to the fixed point q of T with minimum norm, namely $\|q\| = \min_{x\in Fix(T)} \|x\|$.

Moreover, as a consequence of the main convergence result, if $\mu_n = 1$ then algorithm (1.6) can be rewritten as

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \end{cases}$$

that is a Halpern type Method ([6]).

Going back to the idea born by Alghamdi *et al.* in [1], a different situation occurs if the starting point is the numerical Explicit Midpoint Rule (EMR), employed for numerically integrating an IVP of type (1.2). The finite difference scheme defining the EMR is given by:

$$\begin{cases} y_0 = x_0, \\ \bar{y}_{n+1} = y_n + h\Phi(y_n), \\ y_{n+1} = y_n + h\Phi\left(\frac{y_n + \bar{y}_{n+1}}{2}\right), \end{cases} \tag{1.7}$$

where y_n, t_n, h are defined as in the formulation of IMR (1.3).

Again, if function Φ is Lipschitz continuous and sufficiently smooth, then the polygonal functions sequence $(Y_N)_{N\in\mathbb{N}}$, identified by values $\{y_n\}_{n=0}^N$, converges to the exact solution of (1.2), as $N \to \infty$, uniformly on $t \in [t_0, T]$, for any fixed $T > 0$.

If function Φ is written as $\Phi(x) = x - g(x)$, then (1.7) becomes

$$y_{n+1} = y_n + h\Big(\frac{y_n + \bar{y}_{n+1}}{2} - g\Big(\frac{y_n + \bar{y}_{n+1}}{2}\Big)\Big) \tag{1.8}$$

and the critical points of IVP (1.2) are the fixed points of the function g, $x = g(x)$.

Therefore, as in the previous case, by formal analogy with (1.8), a recursive scheme for the Fixed Points Problem $x = Tx$, is obtained:

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = (1-t_n)x_n + t_n T(x_n) \\ x_{n+1} = x_n - t_n\Big(\frac{x_n + \bar{x}_{n+1}}{2} - T\Big(\frac{x_n + \bar{x}_{n+1}}{2}\Big)\Big), \end{cases} \tag{1.9}$$

where $(t_n)_{n\in\mathbb{N}} \in (0,1)$. It can bee seen that this algorithm is equivalent to the following one:

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = (1-t_n)x_n + t_n T(x_n) \\ x_{n+1} = (1-t_n)x_n + t_n T\left(\frac{x_n + \bar{x}_{n+1}}{2}\right), \quad n \geq 0, \end{cases} \tag{1.10}$$

with $(t_n)_{n\in\mathbb{N}} \in (0,1)$, known as Explicit Midpoint Rule (EMR) for nonexpansive mappings.

With the purpose of obtaining strong convergence, Marino, Scardamaglia and Zaccone [10] modified the EMR for nonexpansive mappings (1.10). They introduced a viscosity term and considered a double convex combination in the evaluation point of T.

As we have done for (1.6), we provide a modification for (1.10), on the same line of modification in [8], obtaining the following iterative scheme:

$$\begin{cases} x_0, u \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) + \alpha_n \mu_n (u - x_n), \quad n \geq 0, \end{cases} \tag{1.11}$$

where $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences in $(0,1]$. Under suitable assumptions on the sequences $(\alpha_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$, we prove that $(x_n)_{n\in\mathbb{N}}$ generated by (1.11) strongly converges to the fixed point ${x_u}^*$ of T nearest to u. If $u = 0$, then we obtain strong convergence of $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) - \alpha_n \mu_n x_n, \quad n \geq 0, \end{cases}$$

to the fixed point with minimum norm.

Moreover, as in the previous case, if $\mu_n = 1$, then the sequence generated by

$$\begin{cases} x_0, u \in H \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0, \end{cases}$$

strongly converges to the point ${x_u}^* \in Fix(T)$ nearest to u.

The paper is structured as follows. In Section 2, we collect some lemmas, which are essential to prove our main results. In Section 3, we claim the convergence results for the modified implicit midpoint method (1.6). In Section 4, we give the proofs of the convergence theorems holding for the modified explicit midpoint rule (1.11).

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $T : H \to H$ be a nonexpansive mapping with fixed points set $Fix(T) = \{z \in H : Tz = z\}$. In the remainder of the paper, we will always assume $Fix(T) \neq \emptyset$.

The weak and strong convergence will be denoted with $\rightharpoonup$ and $\to$, respectively.

Recall that, for each $x, y \in H$ and $\lambda \in [0,1]$, the following relations hold:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.1}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.2}$$

Moreover, if C is a nonempty, closed and convex subset of H, for any $x \in H$ the nearest point projection $P_C(x)$ is defined as the unique point of C which is nearest to x; that is $y = P_C(x)$ if and only if $y \in C$ and $\|x-y\| = \inf\{\|x-z\| : z \in C\}$.

The metric projection P_C from the Hilbert space H to C is characterized by the following:

Lemma 2.1. [5] *$y = P_C(x)$ if and only if $\langle z-y, y-x\rangle \geq 0$ for each $z \in C$.*

In the proof techniques of the next section, we require the following lemma:

Lemma 2.2. [15] *Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers for which $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n$, $n \geq 0$, where (γ_n) is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that*

- $\sum_{n=1}^{\infty} \gamma_n = +\infty$
- $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 (Demiclosedness Priciple [2])**.** *Let C be a nonempty, closed and convex subset of a Hilbert space H, and let $T : C \to H$ be a nonexpansive mapping. Assume that $(x_n)_{n\in\mathbb{N}}$ is a sequence in C such that $x_n \to x$ weakly and $(I-T)x_n \to 0$ strongly. Then $(I-T)x = 0$ (i.e., $Tx = x$).*

3. Modified Implicit Midpoint Method

If $T : H \to H$ is a nonexpansive mapping, y, z, w are given points in H and $\alpha \in (0,1)$, then the mapping $\tilde{T} : H \to H$ defined

$$\tilde{T}x = \alpha y + (1-\alpha)T\left(\frac{z+x}{2}\right) + w$$

is a contraction with constant $\frac{1-\alpha}{2}$. Therefore it has a unique fixed point.

Then it is possible to define a sequence $(x_n)_{n\in\mathbb{N}}$ through the implicit algorithm:

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T\left(\frac{x_n+x_{n+1}}{2}\right) + \alpha_n\mu_n(u-x_n), \quad n \geq 0, \end{cases} \tag{3.1}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ are sequences in $(0,1]$.

First of all we prove the following

Lemma 3.1. *The sequence $(x_n)_{n\in\mathbb{N}}$ generated by (3.1) is bounded.*

Proof. Let $p \in Fix(T)$. For arbitrary $n \in \mathbb{N}$, we compute:

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(1-\mu_n)x_n + (1-\alpha_n)T\left(\frac{x_n+x_{n+1}}{2}\right) + \alpha_n\mu_n u - p\| \\
(\pm\alpha_n(1-\mu_n)p) &= \|\alpha_n(1-\mu_n)(x_n-p) + (1-\alpha_n)\left(T\left(\frac{x_n+x_{n+1}}{2}\right) - p\right) + \alpha_n\mu_n(u-p)\| \\
(T \text{ nonexpansive}) &\leq \alpha_n(1-\mu_n)\|x_n-p\| + (1-\alpha_n)\left\|\frac{x_n+x_{n+1}}{2} - p\right\| + \alpha_n\mu_n\|u-p\| \\
&\leq \alpha_n(1-\mu_n)\|x_n-p\| + \frac{1-\alpha_n}{2}\|x_n-p\| \\
&\quad + \frac{1-\alpha_n}{2}\|x_{n+1}-p\| + \alpha_n\mu_n\|u-p\|.
\end{aligned}$$

Thus, for $n \geq 0$, we have that

$$\left(\frac{1+\alpha_n}{2}\right)\|x_{n+1}-p\| \leq \left(\frac{1+\alpha_n}{2} - \alpha_n\mu_n\right)\|x_n-p\| + \alpha_n\mu_n\|u-p\|,$$

and hence

$$\begin{aligned}\|x_{n+1}-p\| &\leq \left(1-\frac{2\alpha_n\mu_n}{1+\alpha_n}\right)\|x_n-p\|+\frac{2\alpha_n\mu_n}{1+\alpha_n}\|u-p\| \\ &\leq \max\{\|x_n-p\|,\|u-p\|\}.\end{aligned}$$

Therefore, it results

$$\begin{aligned}\|x_{n+1}-p\| &\leq \max\{\|x_n-p\|,\|u-p\|\} \\ &\leq \max\{\|x_{n-1}-p\|,\|u-p\|\}.\end{aligned}$$

Hence, we have

$$\|x_{n+1}-p\| \leq \max\{\|x_0-p\|,\|u-p\|\}, \quad \forall n\in\mathbb{N}.$$

Then $(x_n)_{n\in\mathbb{N}}$ is bounded. □

For sequence $(x_n)_{n\in\mathbb{N}}$ generated by (3.1), the following convergence result holds.

Theorem 3.2. *Let H be a real Hilbert space and $T:H\to H$ a nonexpansive mapping with $Fix(T)\neq\emptyset$. Assume that the sequences $(\alpha_n)_{n\in\mathbb{N}},(\mu_n)_{n\in\mathbb{N}}\in(0,1]$ satisfy the conditions*

(1) $\lim_{n\to\infty}\alpha_n=0$,
(2) $\sum_{n=0}^{\infty}\alpha_n\mu_n=+\infty$,
(3) $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n\mu_n}=0$,
(4) $\lim_{n\to\infty}\frac{|\mu_n-\mu_{n-1}|}{\mu_n}=0$.

Then $(x_n)_{n\in\mathbb{N}}$, generated by (3.1), strongly converges to the point $q_u\in Fix(T)$ nearest to u, that is, $\|u-q_u\|=\min_{x\in Fix(T)}\|u-x\|$.

Proof. From the previous lemma, it is known that $(x_n)_{n\in\mathbb{N}}$ is bounded.

We are going to prove that $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular, that is, $(x_n)_{n\in\mathbb{N}}$ verifies

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Indeed, it results that

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n(1-\mu_n)x_n+(1-\alpha_n)T(\frac{x_n+x_{n+1}}{2})+\alpha_n\mu_n u \\
&\quad -\alpha_{n-1}(1-\mu_{n-1})x_{n-1}-(1-\alpha_{n-1})T(\frac{x_{n-1}+x_n}{2})-\alpha_{n-1}\mu_{n-1}u\| \\
(\pm\alpha_n(1-\mu_n)x_{n-1}) &= \|\alpha_n(1-\mu_n)(x_n-x_{n-1})+\alpha_n(\mu_{n-1}-\mu_n)x_{n-1} \\
(\pm\alpha_n(1-\mu_{n-1})x_{n-1}) &\quad +(\alpha_n-\alpha_{n-1})((1-\mu_{n-1})x_{n-1}-T(\frac{x_{n-1}+x_n}{2})) \\
(\pm\alpha_n T(\tfrac{x_{n-1}+x_n}{2})) &\quad +(1-\alpha_n)(T(\frac{x_n+x_{n+1}}{2})-T(\frac{x_{n-1}+x_n}{2})) \\
(\pm\alpha_n\mu_{n-1}u) &\quad +\alpha_n(\mu_n-\mu_{n-1})u+\mu_{n-1}(\alpha_n-\alpha_{n-1})u\| \\
\text{(T nonexpansive)} &\leq \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(\|x_{n-1}\|+\|u\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|(\|(1-\mu_{n-1})x_{n-1}-T(\frac{x_{n-1}+x_n}{2})\|+\mu_{n-1}\|u\|) \\
&\quad +(1-\alpha_n)\|\frac{x_n+x_{n+1}}{2}-\frac{x_{n-1}+x_n}{2}\| \\
&\leq \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(\|x_{n-1}\|+\|u\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|(\|(1-\mu_{n-1})x_{n-1}-T(\frac{x_{n-1}+x_n}{2})\|+\mu_{n-1}\|u\|) \\
&\quad +\frac{(1-\alpha_n)}{2}\|x_n-x_{n-1}\|+\frac{(1-\alpha_n)}{2}\|x_{n+1}-x_n\|.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
(\frac{1+\alpha_n}{2})\|x_{n+1}-x_n\| &\leq (\frac{1+\alpha_n}{2}-\alpha_n\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(\|x_{n-1}\|+\|u\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|(\|(1-\mu_{n-1})x_{n-1}-T(\frac{x_{n-1}+x_n}{2})\|+\mu_{n-1}\|u\|),
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\leq (1-\frac{2\alpha_n\mu_n}{1+\alpha_n})\|x_n-x_{n-1}\|+\frac{2\alpha_n}{1+\alpha_n}|\mu_n-\mu_{n-1}|(\|x_{n-1}\|+\|u\|) \\
&\quad +\frac{2}{1+\alpha_n}|\alpha_n-\alpha_{n-1}|(\|(1-\mu_{n-1})x_{n-1}-T(\frac{x_{n-1}+x_n}{2})\|+\mu_{n-1}\|u\|).
\end{aligned} \tag{3.2}
$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exist two positive constants M and L such that

$$\|x_n\| \leq M,$$

$$\|(1-\mu_{n-1})x_{n-1}-T(\frac{x_{n-1}+x_n}{2})\| \leq L,$$

for $n \geq 0$. Therefore, inequality (3.2) becomes

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-\frac{2\alpha_n\mu_n}{1+\alpha_n})\|x_n-x_{n-1}\|+\frac{2\alpha_n}{1+\alpha_n}|\mu_n-\mu_{n-1}|(M+\|u\|)\\ &\quad+\frac{2}{1+\alpha_n}|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|).\end{aligned}$$

From hypotheses $(1),(2)$ and since $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ are sequences in $(0,1]$, it follows that

- $\lim_{n\to\infty}\frac{\alpha_n\mu_n}{1+\alpha_n}=0$,
- $\sum_{n=0}^{\infty}\frac{\alpha_n\mu_n}{1+\alpha_n}=+\infty$,

These last, added to hypotheses (3) and (4), allows us (Lemma 2.2) to conclude that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3}$$

By definition of (3.1), it can be noticed that

$$\|x_{n+1}-T(\frac{x_n+x_{n+1}}{2})\|=\alpha_n\|(1-\mu_n)x_n+\mu_n u-T(\frac{x_n+x_{n+1}}{2})\|.$$

Since $\lim_{n\to\infty}\alpha_n=0$, we have that

$$\lim_{n\to\infty}\|x_{n+1}-T(\frac{x_n+x_{n+1}}{2})\|=0. \tag{3.4}$$

On the other hand, the following estimation holds

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &\leq \|x_{n+1}-T(\frac{x_n+x_{n+1}}{2})\|+\|T(\frac{x_n+x_{n+1}}{2})-Tx_{n+1}\|\\ &\leq \|x_{n+1}-T(\frac{x_n+x_{n+1}}{2})\|+\frac{\|x_{n+1}-x_n\|}{2}.\end{aligned}$$

Therefore, from (4.2) and (3.4), we get that

$$\lim_{n\to\infty}\|x_{n+1}-Tx_{n+1}\|=0, \tag{3.5}$$

that is, $(x_n)_{n\in\mathbb{N}}$ is an approximating fixed point sequence for T. From the Demiclosedness Principle and (3.5), follows that that every weak cluster point of $(x_n)_{n\in\mathbb{N}}$ is a fixed point for T, i.e. $\omega_w(x_n)\subset Fix(T)$.

The next goal is to prove that

$$\lim_{n\to+\infty}\|x_n-q_u\|=0,$$

where q_u is the point of $Fix(T)$ nearest to u. In this direction, we compute:

$$\begin{aligned}
\|x_{n+1}-q_u\|^2 &= \|\alpha_n(1-\mu_n)x_n+(1-\alpha_n)T(\frac{x_n+x_{n+1}}{2})+\alpha_n\mu_n u-q_u\|^2 \\
(\pm\alpha_n(1-\mu_n)q_u) &= \|\alpha_n(1-\mu_n)(x_n-q_u)+(1-\alpha_n)\big(T(\frac{x_n+x_{n+1}}{2})-q_u\big) \\
&\quad +\alpha_n\mu_n(u-q_u)\|^2 \\
(2.2) &\leq \|\alpha_n(1-\mu_n)(x_n-q_u)+(1-\alpha_n)\big(T(\frac{x_n+x_{n+1}}{2})-q_u\big)\|^2 \\
&\quad +2\alpha_n\mu_n\langle u-q_u,x_{n+1}-q_u\rangle \\
(2.1) &\leq \alpha_n\|(1-\mu_n)(x_n-q_u)\|^2+(1-\alpha_n)\|T(\frac{x_n+x_{n+1}}{2})-q_u\|^2 \\
&\quad +2\alpha_n\mu_n\langle u-q_u,x_{n+1}-q_u\rangle \\
(\text{T nonexpansive}) &\leq \alpha_n(1-\mu_n)^2\|x_n-q_u\|^2+(1-\alpha_n)\|\frac{x_n+x_{n+1}}{2}-q_u\|^2 \\
&\quad +2\alpha_n\mu_n\langle u-q_u,x_{n+1}-q_u\rangle \\
(\mu_n\leq 1) &\leq \alpha_n(1-\mu_n)\|x_n-q_u\|^2+(\frac{1-\alpha_n}{2})\|x_n-q_u\|^2 \\
&\quad +(\frac{1-\alpha_n}{2})\|x_{n+1}-q_u\|^2+2\alpha_n\mu_n\langle u-q_u,x_{n+1}-q_u\rangle \\
&\leq (\frac{1+\alpha_n}{2}-\alpha_n\mu_n)\|x_n-q_u\|^2 \\
&\quad +(\frac{1-\alpha_n}{2})\|x_{n+1}-q_u\|^2+2\alpha_n\mu_n\langle u-q_u,x_{n+1}-q_u\rangle,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\|x_{n+1}-q_u\|^2 &\leq (1-\frac{2\alpha_n\mu_n}{1+\alpha_n})\|x_n-q_u\|^2 \\
&\quad \frac{4\alpha_n\mu_n}{1+\alpha_n}\langle u-q_u,x_{n+1}-q_u\rangle. \qquad (3.6)
\end{aligned}$$

In order to apply Lemma 2.2 to the real sequence $(\|x_{n+1}-q_u\|^2)_{n\in\mathbb{N}}$, we claim that

$$\limsup_{n\to+\infty}\langle u-q_u,x_{n+1}-q_u\rangle\leq 0.$$

First of all, we designate with $(x_{n_k})_{k\in\mathbb{N}}$ the subsequence of $(x_n)_{n\in\mathbb{N}}$, such that

$$\limsup_{n\to+\infty}\langle u-q_u,x_{n+1}-q_u\rangle=\lim_{k\to+\infty}\langle u-q_u,x_{n_k+1}-q_u\rangle.$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exists a subsequence, for semplicity of notation denoted with $(x_{n_k})_{k\in\mathbb{N}}$, such that $x_{n_k}\rightharpoonup\bar{x}\in H$, for $k\to\infty$. In particular, cosidering that $\omega_w(x_n)\subset Fix(T)$, $\bar{x}\in Fix(T)$.

Therefore, we have that

$$\begin{aligned}
\limsup_{n\to+\infty}\langle u-q_u,x_{n+1}-q_u\rangle &= \lim_{k\to+\infty}\langle u-q_u,x_{n_k+1}-q_u\rangle \\
&= \langle u-q_u,\bar{x}-q_u\rangle \\
(\text{Lemma 2.1}) &\leq 0.
\end{aligned}$$

Finally, we get from Lemma 2.2 that $\lim_{n\to\infty}\|x_{n+1}-q_u\|=0$. □

Remark 3.3. If $u=0\in H$, under the same hypotheses of Theorem 3.2, we get that $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0\in H, \\ x_{n+1}=\alpha_n x_n+\alpha_n T(\frac{x_n+x_{n+1}}{2})-\alpha_n\mu_n x_n \quad n\geq 0 \end{cases}$$

strongly converges to the point $q\in Fix(T)$ nearest to $0\in H$, that is, the fixed point of T with minimum norm $\|q\|=\min_{x\in Fix(T)}\|x\|$.

A particular case of Theorem 3.2 is obtained when $\mu_n=1$, and it is given by the following:

Corollary 3.4. *Let H be a real Hilbert space and $T:H\to H$ a nonexpansive maping with $Fix(T)\neq\emptyset$. If the sequence $(\alpha_n)_{n\in\mathbb{N}},\in(0,1]$ satisfies the conditions*

(1) $\lim_{n\to\infty}\alpha_n=0$,
(2) $\sum_{n=0}^{\infty}\alpha_n=+\infty$,
(3) $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}=0$.

Then the sequence $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0,u\in H, \\ x_{n+1}=\alpha_n u+(1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}), \quad n\geq 0 \end{cases} \tag{3.7}$$

strongly converges to the point $q_u\in Fix(T)$ nearest to u, that is, $\|u-q_u\|=\min_{x\in Fix(T)}\|u-x\|$

Remark 3.5. Also in this case, it is considered the eventuality $u=0\in H$. Therefore, we get, under the same assumptions of Corollary 3.4, that sequence $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0\in H, \\ x_{n+1}=(1-\alpha_n)T(\frac{x_n+x_{n+1}}{2}), \quad n\geq 0 \end{cases}$$

strongly converges to the point $q\in Fix(T)$ nearest to $0\in H$, that is, the fixed point of T with minimum norm $\|q\|=\min_{x\in Fix(T)}\|x\|$.

4. Modified Explicit Midpoint Method

Let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by

$$\begin{cases} x_0,u\in H, \\ \bar{x}_{n+1}=\beta_n x_n+(1-\beta_n)Tx_n, \quad n\geq 0, \\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)T(s_n x_n+(1-s_n)\bar{x}_{n+1})+\alpha_n\mu_n(u-x_n), \quad n\geq 0, \end{cases} \tag{4.1}$$

with $(\alpha_n)_{n\in\mathbb{N}},(\mu_n)_{n\in\mathbb{N}},(\beta_n)_{n\in\mathbb{N}},(s_n)_{n\in\mathbb{N}}\in(0,1]$.

First of all we prove the following lemma.

Lemma 4.1. *The sequence $(x_n)_{n\in\mathbb{N}}$ generated by (4.5) is bounded.*

Proof. Letting $p \in Fix(T)$ and $n \in \mathbb{N}$, let us compute:

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n(1-\mu_n)x_n+(1-\alpha_n)T(s_nx_n+(1-s_n)\bar{x}_{n+1})+\alpha_n\mu_n u-p\| \\
(\pm\alpha_n(1-\mu_n)p) &= \|\alpha_n(1-\mu_n)(x_n-p)+(1-\alpha_n)(T(s_nx_n+(1-s_n)\bar{x}_{n+1})-p) \\
&\quad +\alpha_n\mu_n(u-p)\| \\
&\leq \alpha_n(1-\mu_n)\|x_n-p\|+(1-\alpha_n)\|T(s_nx_n+(1-s_n)\bar{x}_{n+1})-p\| \\
&\quad +\alpha_n\mu_n\|u-p\| \\
\text{(T nonexpansive)} &\leq \alpha_n(1-\mu_n)\|x_n-p\|+(1-\alpha_n)\|s_nx_n+(1-s_n)\bar{x}_{n+1}-p\| \\
&\quad +\alpha_n\mu_n\|u-p\| \\
&\leq \alpha_n(1-\mu_n)\|x_n-p\|+(1-\alpha_n)s_n\|x_n-p\|+(1-\alpha_n)(1-s_n)\|\bar{x}_{n+1}-p\| \\
&\quad +\alpha_n\mu_n\|u-p\| \\
&\leq \alpha_n(1-\mu_n)\|x_n-p\|+(1-\alpha_n)s_n\|x_n-p\| \\
&\quad +(1-\alpha_n)(1-s_n)\beta_n\|x_n-p\| \\
&\quad +(1-\alpha_n)(1-s_n)(1-\beta_n)\|Tx_n-p\| \\
&\leq (1-\alpha_n\mu_n)\|x_n-p\|+\alpha_n\mu_n\|u-p\| \\
&\leq \max\{\|x_n-p\|,\|u-p\|\}.
\end{aligned}$$

Consequently, it results that

$$\|x_{n+1}-p\| \leq \max\{\|x_0-p\|,\|u-p\|\}.$$

Therefore the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. □

The following is the main convergence result for the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (4.5):

Theorem 4.2. *Let H be a real Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Under the assumptions $(1),(2),(3),(4)$ of Theorem 3.2, if the sequences $(\alpha_n)_{n\in\mathbb{N}},(\mu_n)_{n\in\mathbb{N}},(\beta_n)_{n\in\mathbb{N}},(s_n)_{n\in\mathbb{N}} \in (0,1]$ satisfy also the hypotheses*

(5) $\lim_{n\to\infty}\frac{|s_n-s_{n-1}|}{\alpha_n\mu_n}=0$
(6) $\lim_{n\to\infty}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n\mu_n}=0$
(7) $\limsup_{n\to\infty}\beta_n(1-s_n)+s_n>0$,

then $(x_n)_{n\in\mathbb{N}}$ generated by (4.5) strongly converges to the point $x_u{}^* \in Fix(T)$ nearest to u, that is

$$\|u-x_u{}^*\| = \min_{x\in Fix(T)}\|u-x\|.$$

Proof. An example of parameters sequences satisfing conditions $(1)-(7)$ is given by $\alpha_n = s_n = \mu_n = \frac{1}{\sqrt{n}}$ and $\beta_n = \dfrac{n}{n+1}$.

The first goal is to prove that sequence $(x_n)_{n\in\mathbb{N}}$, defined by (4.5), is asymptotically regular. Indeed, let us compute

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n(1-\mu_n)x_n+(1-\alpha_n)T(s_nx_n+(1-s_n)\bar{x}_{n+1}) \\
&\quad +\alpha_n\mu_nu-\alpha_{n-1}(1-\mu_{n-1})x_{n-1} \\
&\quad -(1-\alpha_{n-1})T(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)-\alpha_{n-1}\mu_{n-1}u\| \\
\pm\alpha_n(1-\mu_n)x_{n-1} &= \|\alpha_n(1-\mu_n)(x_n-x_{n-1})+\alpha_n(\mu_{n-1}-\mu_n)x_{n-1} \\
\pm\alpha_n(1-\mu_{n-1})x_{n-1} &\quad +(\alpha_n-\alpha_{n-1})((1-\mu_{n-1})x_{n-1}-T(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)) \\
\pm\alpha_nT(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n) &\quad +(1-\alpha_n)(T(s_nx_n+(1-s_n)\bar{x}_{n+1})-T(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)) \\
\pm\alpha_n\mu_{n-1}u &\quad +\mu_{n-1}(\alpha_n-\alpha_{n-1})u+\alpha_n(\mu_n-\mu_{n-1})u\| \\
&\le \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&\quad +|\alpha_n-\alpha_{n-1}|\|(1-\mu_{n-1})x_{n-1} \\
&\quad -T(s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)\|+(1-\alpha_n)\|s_nx_n+(1-s_n)\bar{x}_{n+1} \\
&\quad -s_{n-1}x_{n-1}+(1-s_{n-1})\bar{x}_n)\|+\mu_{n-1}|\alpha_n-\alpha_{n-1}|\|u\| \\
&\quad +\alpha_n|\mu_n-\mu_{n-1}|\|u\| \\
&= \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(M+\|u\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|)+(1-\alpha_n)\|s_nx_n+(1-s_n)\beta_nx_n \\
&\quad +(1-s_n)(1-\beta_n)Tx_n-s_{n-1}x_{n-1}-(1-s_{n-1})\beta_{n-1}x_{n-1} \\
&\quad -(1-s_{n-1})(1-\beta_{n-1})Tx_{n-1}\| \\
(\pm s_nx_{n-1}) &= \alpha_n(1-\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(M+\|u\|) \\
(\pm(1-s_n)\beta_nx_{n-1}) &\quad +|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|)+(1-\alpha_n)\|s_n(x_n-x_{n-1}) \\
(\pm(1-s_n)(1-\beta_n)Tx_{n-1}) &\quad +(1-s_n)\beta_n(x_n-x_{n-1})+(1-s_n)(1-\beta_n)(Tx_n-Tx_{n-1}) \\
(\pm\beta_ns_{n-1}x_{n-1}) &\quad +(s_n-s_{n-1}+\beta_n-\beta_{n-1}+\beta_n(s_{n-1}-s_n) \\
(\pm\beta_ns_{n-1}Tx_{n-1}) &\quad +s_{n-1}(\beta_{n-1}-\beta_n))x_{n-1}+(s_{n-1}-s_n+\beta_{n-1}-\beta_n \\
&\quad +\beta_n(s_n-s_{n-1})+s_{n-1}(\beta_n-\beta_{n-1}))Tx_{n-1}\|.
\end{aligned}
$$

Hence, we have

$$\begin{aligned}
& \|x_{n+1}-x_n\| \\
\leq\ & (\alpha_n(1-\mu_n)+1-\alpha_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(M+\|u\|) \\
& +|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|)+((1-s_{n-1})|\beta_n-\beta_{n-1}|+(1-\beta_n)|s_n-s_{n-1}|)M \\
& +((1-s_{n-1})|\beta_n-\beta_{n-1}|+(1-\beta_n)|s_n-s_{n-1}|)N \\
=\ & (1-\alpha_n\mu_n)\|x_n-x_{n-1}\|+\alpha_n|\mu_n-\mu_{n-1}|(M+\|u\|) \\
& +|\alpha_n-\alpha_{n-1}|(L+\mu_{n-1}\|u\|)+((1-s_{n-1})|\beta_n-\beta_{n-1}|+(1-\beta_n)|s_n-s_{n-1}|)(M+N),
\end{aligned}$$

where L and M are positive constants defined as in the previous section, and $N>0$ is such that $\|Tx_n\|\leq N$ for all $n\geq 0$. From conditions $(1)-(6)$, thanks to Lemma 2.1, we finally get that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.2}$$

Moreover, it results that

$$\begin{aligned}
\|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T(s_nx_n+(1-s_n)\bar{x}_{n+1})\| \\
&\quad +\|T(s_nx_n+(1-s_n)\bar{x}_{n+1})-Tx_n\| \\
&\leq \|x_n-x_{n+1}\|+\alpha_n\|(1-\mu_n)x_n+\mu_nu-T(s_nx_n+(1-s_n)\bar{x}_{n+1})\| \\
&\quad +\|s_nx_n+(1-s_n)\bar{x}_{n+1}-x_n\| \\
&\leq \|x_n-x_{n+1}\|+\alpha_n\|(1-\mu_n)x_n+\mu_nu-T(s_nx_n+(1-s_n)\bar{x}_{n+1})\| \\
&\quad +(1-s_n)\|\bar{x}_{n+1}-x_n\| \\
&\leq \|x_n-x_{n+1}\|+\alpha_n\|(1-\mu_n)x_n+\mu_nu-T(s_nx_n+(1-s_n)\bar{x}_{n+1})\| \\
&\quad +(1-s_n)(1-\beta_n)\|x_n-Tx_n\|.
\end{aligned}$$

It follows that

$$(\beta_n(1-s_n)+s_n)\|x_n-Tx_n\| \leq \|x_n-x_{n+1}\|+\alpha_n\|(1-\mu_n)x_n+\mu_nu-T(s_nx_n+(1-s_n)\bar{x}_{n+1})\|.$$

Since condition (1) holds, and from (4.2), we deduce that

$$\limsup_{n\to\infty}(\beta_n(1-s_n)+s_n)\|x_n-Tx_n\|\leq 0.$$

By hypothesis (7), it follows that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. Consequently, in light of the Demiclosedness Principle, we have that $\omega_w(x_n)\subset Fix(T)$.

Next purpose is to claim that $\lim_{n\to\infty}\|x_n-x_u{}^*\|=0$, where $x_u{}^*$ is the unique point of $Fix(T)$ with the property

$$\|u-x_u{}^*\|=\min_{x\in Fix(T)}\|u-x\|.$$

Indeed, it holds that

$$\begin{aligned}
\|x_{n+1}-x_u{}^*\|^2 &= \|\alpha_n(1-\mu_n)x_n+(1-\alpha_n)T(s_nx_n+(1-s_n)\bar{x}_{n+1})+\alpha_n\mu_n u-x_u{}^*\|^2\\
(\pm\alpha_n(1-\mu_n)x_u{}^*) &= \|\alpha_n(1-\mu_n)(x_n-x_u{}^*)+(1-\alpha_n)(T(s_nx_n+(1-s_n)\bar{x}_{n+1})-x_u{}^*)\\
&\quad +\alpha_n\mu_n(u-x_u{}^*)\|^2\\
&\leq \|\alpha_n(1-\mu_n)(x_n-x_u{}^*)+(1-\alpha_n)(T(s_nx_n+(1-s_n)\bar{x}_{n+1})-x_u{}^*)\|^2\\
&\quad +2\alpha_n\mu_n\langle u-x_u{}^*,x_{n+1}-x_u{}^*\rangle\\
&\leq \alpha_n(1-\mu_n)^2\|x_n-x_u{}^*\|^2+(1-\alpha_n)\|T(s_nx_n+(1-s_n)\bar{x}_{n+1})-x_u{}^*\|^2\\
&\quad +2\alpha_n\mu_n\langle u-x_u{}^*,x_{n+1}-x_u{}^*\rangle\\
&\leq \alpha_n(1-\mu_n)\|x_n-x_u{}^*\|^2+(1-\alpha_n)\|s_nx_n+(1-s_n)\bar{x}_{n+1}-x_u{}^*\|^2\\
&\quad +2\alpha_n\mu_n\langle u-x_u{}^*,x_{n+1}-x_u{}^*\rangle\\
&\leq (1-\alpha_n\mu_n)\|x_n-x_u{}^*\|^2+2\alpha_n\mu_n\langle u-x_u{}^*,x_{n+1}-x_u{}^*\rangle.
\end{aligned}$$

Argumenting as in the proof of Theorem 3.2, we are going to claim that

$$\limsup_{n\to\infty}\langle u-x_u{}^*,x_{n+1}-x_u{}^*\rangle \leq 0.$$

First of all, there exists a sunsequence $(x_{n_k})_{k\in\mathbb{N}} \in (x_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty}\langle u-x_u{}^*,x_{n+1}-x_u{}^*\rangle = \lim_{k\to\infty}\langle u-x_u{}^*,x_{n_k}-x_u{}^*\rangle.$$

On the other hand, from Lemma 4.1, it is known that $(x_n)_{n\in\mathbb{N}}$ is bounded. From reflexivity of H it follows that there exists a subsequence, that we still denote with $(x_{n_k})_{k\in\mathbb{N}}$, for semplicity of notation, such that $x_{n_k} \rightharpoonup x \in H$, as $k\to\infty$. Since every weak cluster point of $(x_n)_{n\in\mathbb{N}}$ is a fixed point, it follows that $x \in Fix(T)$. Therefore, we have that

$$\begin{aligned}
\limsup_{n\to\infty}\langle u-x_u{}^*,x_{n+1}-x_u{}^*\rangle &= \lim_{k\to\infty}\langle u-x_u{}^*,x_{n_k}-x_u{}^*\rangle\\
&= \langle u-x_u{}^*,x-x_u{}^*\rangle\\
\text{(Lemma 2.1)} &\leq 0.
\end{aligned}$$

From Lemma 2.2, we finally conclude that $\lim_{n\to\infty}\|x_{n+1}-x_u{}^*\| = 0$. □

Remark 4.3. In case $u=0$, under the same assumptions of Theorem 4.2, we obtain strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$, generated by

$$\begin{cases}
x_0 \in H,\\
\bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n\geq 0,\\
x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(s_nx_n+(1-s_n)\bar{x}_{n+1}) - \alpha_n\mu_n x_n, \quad n\geq 0,
\end{cases}$$

to the point $x* \in Fix(T)$ nearest to $0 \in H$, that is the fixed point of T with minimum norm $\|x^*\| = \min_{x \in Fix(T)} \|x\|$.

As in the previous section, in the particular case $\mu_n = 1$, for all $n \in \mathbb{N}$, the following result holds.

Corollary 4.4. *Let H be a real Hilbert space and $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that conditions $(1),(2),(3)$ of Corollary 3.4 hold and that the sequences $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ satisfy also the hypotheses:*

(4) $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{\alpha_n} = 0$,
(5) $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0$,
(6) $\limsup_{n\to\infty} \beta_n(1-s_n) + s_n > 0$,

then the sequence $(x_n)_{n\in\mathbb{N}}$ generated by

$$\begin{cases} x_0, u \in H \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) T x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0, \end{cases} \tag{4.3}$$

strongly converges to the point ${x_u}^* \in Fix(T)$ nearest to u.

Remark 4.5. For $u = 0$, under the same assumptions of Corollary 4.4, we obtain strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$, generated by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) T x_n, \quad n \geq 0, \\ x_{n+1} = (1-\alpha_n) T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0, \end{cases}$$

to the point $x* \in Fix(T)$ nearest to $0 \in H$, that is, the fixed point of T with minimum norm $\|x^*\| = \min_{x \in Fix(T)} \|x\|$.

5. Open problems

(1) Do the results hold in the setting of a more general Banach space?
(2) Do the proposed algorithm converge also for a family of mapping?
(3) Is it possible to obtain convergence results of the proposed schemes for the more general class of quasi-nonexpansive mappings?

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