J. Nonlinear Var. Anal. 1 (2017), No. 2, pp. 213-235
Available online at http://jnva.biemdas.com

ITERATION APPROACHES TO HIERARCHICAL VARIATIONAL INEQUALITIES FOR INFINITE NONEXPANSIVE MAPPINGS AND FINDING ZERO POINTS OF m-ACCRETIVE OPERATORS

LU-CHUAN CENG[1], CHING-FEN WEN[2], YONGHONG YAO[3,*]

[1]*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*
[2]*Center for Fundamental Science, and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 807, Taiwan*
[3]*Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China*

Abstract. In this paper, we introduce and analyze hybrid viscosity approximation methods for solving a hierarchical variational inequality over the common fixed points set of a countable family of nonexpansive mappings, and for finding zero points of m-accretive operators in a real reflexive Banach space. Under suitable assumptions, we establish several strong convergence theorems for the proposed iterative algorithms. The results presented in this paper improve and extend the corresponding results in the literature.

Keywords. Nonexpansive mapping; Fixed point; Accretive operator; Zero point.

2010 Mathematics Subject Classification. 49J30, 47H09, 47J20, 49M05.

1. Introduction

Let X be a real Banach space with its dual space X^*. Let C be a nonempty closed convex subset of X. Let the norms of X and X^* be denoted by the same notation $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the generalized duality pairing between X and X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{\phi \in X^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\}, \quad \forall x \in X.$$

It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers. A mapping $f : C \to C$ is called a contraction on C if there exists a constant $\rho \in (0,1)$ such that $\|f(x) - f(y)\| \le \rho\|x - y\|$ for all $x, y \in C$. Throughout, we use the notation Ξ_C to denote the collection of all contractions on C, i.e.,

$$\Xi_C = \{f : C \to C \quad \text{is} \quad \text{a} \quad \text{contraction}\}.$$

*Corresponding author.
E-mail addresses: zenglc@hotmail.com (L.-C. Ceng), cfwen@kmu.edu.tw (C.-F. Wen), yaoyonghong@aliyun.com (Y. Yao).

Received March 1, 2017; Accepted July 2, 2017.

Let $U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. Then the norm of X is said to be Gateaux differentiable if the limit

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y \in U$. In this case, X is said to be smooth. The norm of X is said to be uniformly Gateaux differentiable, if for each $y \in U$, the limit (1.1) is attained uniformly for $x \in U$. The norm of X is said to be Frechet differentiable, if for each $x \in U$, the limit (1.1) is attained uniformly for $y \in U$. The norm of X is said to be uniformly Frechet differentiable, if the limit (1.1) is attained uniformly for $x, y \in U$. It is well known that (uniform) Frechet differentiability of the norm of X implies (uniform) Gateaux differentiability of the norm of X.

A Banach space X is said to be strictly convex, if for $x, y \in U$ with $x \neq y$, one has $\|(1-\lambda)x + \lambda y\| < 1, \forall \lambda \in (0,1)$. X is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x-y\| \geq \varepsilon \Rightarrow \|\frac{x+y}{2}\| \leq 1-\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. Also, it is known that if a Banach space X is reflexive, then X is strictly convex if and only if X^* is smooth as well as X is smooth if and only if X^* is strictly convex. Moreover, if X is smooth, then the normalized duality mapping J is single-valued; if the norm of X is uniformly Gateaux differentiable, then J is norm-to-weak* uniformly continuous on every bounded subset of X; and if the norm of X is uniformly Frechet differentiable, then J is norm-to-norm uniformly continuous on every bounded subset of X.

Whenever X is a real Hilbert space H, we consider the following classical variational inequality (VI) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.2}$$

where $A : C \to H$ is a mapping. This problem is a fundamental problem in the variational analysis; in particular, in the optimization theory and mechanics; see e.g., [16, 21, 23] and the references therein. A large number of algorithms for solving this problem are essentially projection algorithms that employ projections onto the feasible set C of the VI, or onto some related set, so as to iteratively reach a solution. In particular, Korpelevich [24] proposed an algorithm for solving the VI in Euclidean space, known as the extragradient method. This method further has been improved by many researchers.

If X ia a real smooth Banach space, then the variational inequality (VI) is to find $x^* \in C$ such that

$$\langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

In [1], Aoyama, Iiduka and Takahashi proposed an iterative scheme to find approximate solutions of (1.3) and proved the weak convergence of the sequences generated by the proposed scheme. It is also well known (see [1, Lemma 2.7]) that this problem in a smooth Banach space is equivalent to a fixed-point equation. In [36], Yamada assumed that the feasible set is the set of common fixed points of a finite family of nonexpansive mappings and introduced a hybrid steepest-descent method. In this case, the variational inequality defined on such feasible set is also called a hierarchical variational inequality (HVI). Yamada's method is subsequently extended and modified to solve more complex problems, see, e.g., [38, 13, 37] and references therein.

On the other hand, let $\{T_k\}_{k=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on a nonempty closed convex subset C of a real Banach space X. Let $\{\rho_k\}_{k=1}^{\infty}$ be a sequence in $[0,1]$. Consider the nonexpansive mapping W_k defined by $U_{k,k+1} = I$ and

$$\begin{cases} U_{k,k} = \rho_k T_k U_{k,k+1} + (1-\rho_k)I, \\ \cdots \\ U_{k,i} = \rho_i T_i U_{k,i+1} + (1-\rho_i)I, \\ \cdots \\ U_{k,2} = \rho_2 T_2 U_{k,3} + (1-\rho_2)I, \\ W_k = U_{k,1} = \rho_1 T_1 U_{k,2} + (1-\rho_1)I, \quad \forall k \geq 1. \end{cases} \tag{1.4}$$

The mapping W_k is called a W-mapping, generated by $T_k, T_{k-1}, ..., T_1$ and $\rho_k, \rho_{k-1}, ..., \rho_1$. When $X = H$ a real Hilbert space, Takahashi [32] first introduced such a W-mapping, to find a common fixed point of $\{T_k\}_{k=1}^{\infty}$ (see also [29] for more details). Using the W-mapping and viscosity approximation method ([25]), Kikkawa and Takahashi [20, 19] studied an implicit iteration scheme that converges strongly to a solution of the stated problem. Recently, many authors proposed various efficient iteration methods for finding a common fixed point, which solves some variational inequality; see, e.g., [8, 11, 12].

In 2008, Ceng and Yao [9] introduced and analyzed relaxed implicit and explicit viscosity approximation methods for solving a hierarchical variational inequality over the common fixed point set of a countable family of nonexpansive mappings in a real strictly convex and reflexive Banach space with the uniformly Gateaux differentiable norm. Cai and Bu [7] introduced and analyzed an implicit iterative algorithm for solving a hierarchical variational inequality over the common fixed points set of a countable family of continuous pseudocontractions in a uniformly smooth Banach space. In [27], Qin, Cho and Wang established the strong convergence of the net $\{x_t\}_{t\in(0,1)}$ defined by the implicit viscosity approximation method with a continuous pseudocontraction, and proposed an iterative process with errors for finding a zero point of an m-accretive operator in a reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure. They proved the strong convergence of the proposed algorithm under some mild assumptions. Some related work, please refer to [10, 5, 14].

In this paper, we introduce and analyze hybrid viscosity approximation methods for solving a hierarchical variational inequality over the common fixed points set of a countable family of nonexpansive mappings, and for finding a zero point of an m-accretive operator in a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure. Under suitable assumptions, we establish several strong convergence theorems for the proposed iterative algorithms. The results presented in this paper improve, extend, supplement, and develop the corresponding results by Cai and Bu [7], Qin, Cho and Wang [27], and Ceng and Yao [9].

2. Preliminaries

Let C be a nonempty closed convex subset of a real Banach space X. We write $x_n \rightharpoonup x$ (respectively, $x_n \to x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x. An operator A

is said to be strongly positive if there exists a constants $\bar{\gamma} > 0$ such that

$$\langle Ax, J(x)\rangle \geq \bar{\gamma}\|x^2\|, \quad \|aI - bA\| = \sup_{\|x\|\leq 1} |\langle (aI - bA)x, J(x)\rangle|, a \in [0,1], b \in [-1,1], \tag{2.1}$$

where I is the identity mapping.

Let $T : C \to C$ be a mapping. Recall that T is said to be

(i) strongly pseudocontractive if there exists a constant $\alpha \in (0,1)$ and some $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \alpha\|x-y\|^2, \quad \forall x, y \in C;$$

(ii) pseudocontractive if there exists some $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2, \quad \forall x, y \in C.$$

Recall that a closed convex subset C of a Banach space X is said to have a normal structure if for each bounded convex subset K of C which contains at least two points, there exists an element x of K which is not a diametral point of K, i.e.,

$$\sup\{\|x-y\| : y \in K\} < d(K),$$

where $d(K)$ is the diameter of K. It is well known that a closed convex subset of a uniformly convex Banach space has the normal structure and a compact convex subset of a Banach space has the normal structure; see [33] for more details.

Proposition 2.1. [22] *Let C be a nonempty, bounded, closed and convex subset of a reflexive Banach space X which also has the normal structure. Let T be a nonexpansive self-mapping on C. Then,* $\mathrm{Fix}(T) \neq \emptyset$.

Proposition 2.2. [15] *Let C be a nonempty, closed and convex subset of a Banach space X, and $T : C \to C$ be a continuous and strong pseudocontraction. Then T has a unique fixed point in C.*

Lemma 2.1. [35] *Let $\{s_k\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{k+1} \leq (1-\alpha_k)s_k + \alpha_k\beta_k + \gamma_k, \quad \forall k \geq 1, \tag{2.2}$$

where $\{\alpha_k\}, \{\beta_k\}$ and $\{\gamma_k\}$ satisfy the following conditions:

(i) $\{\alpha_k\} \subset [0,1]$ *and* $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(ii) $\limsup_{k\to\infty} \beta_k \leq 0$;

(iii) $\gamma_k \geq 0$ *for all* $k \geq 1$, *and* $\sum_{k=1}^{\infty} \gamma_k < \infty$.

Then $\lim_{k\to\infty} s_k = 0$.

The following lemma is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

Lemma 2.2. *Let X be a real Banach space and J be the normalized duality map on X. Then, for all $x, y \in X$ one has*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y)$;

(ii) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, j(x)\rangle, \forall j(x) \in J(x)$.

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Proposition 2.3. [28] *Let C be a nonempty closed convex subset of a smooth Banach space X. Let D be a nonempty subset of C and Π be a retraction of C onto D. Then, the following are equivalent:*

(i) *Π is sunny and nonexpansive;*
(ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle, \forall x, y \in C$;
(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0, \forall x \in C, y \in D$.

It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Π_C coincides with the metric projection from X onto C.

Lemma 2.3. [31] *Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a continuous pseudocontractive mapping. We denote $B = (2I - T)^{-1}$. Then the following hold.*

(i) *The mapping B is a nonexpansive self-mapping on C;*
(ii) *If $\lim_{k\to\infty} \|x_k - Tx_k\| = 0$, then $\lim_{k\to\infty} \|x_k - Bx_k\| = 0$.*

Proposition 2.4. [1] *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and $A : C \to X$ be an accretive mapping. Then for all $\lambda > 0$,*

$$\mathrm{VI}(C, A) = \mathrm{Fix}(\Pi_C(I - \lambda A)).$$

Lemma 2.4. [7, Lemma 2.6] *Let C be a nonempty closed convex subset of a real Banach space X which has a uniformly Gateaux differentiable norm. Let $T : C \to C$ be a continuous pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and let $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\alpha \in (0,1)$ and Lipschitzian constant $L > 0$. Let $A : C \to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Assume that $C \pm C \subset C$ and that $\{x_t\}$ converges strongly to $x^* \in \mathrm{Fix}(T)$ as $t \to 0$, where x_t is defined by $x_t = tf(x_t) + (I - tA)Tx_t$. Suppose that $\{x_k\} \subset C$ is bounded and that $\lim_{k\to\infty} \|x_k - Tx_k\| = 0$. Then $\limsup_{k\to\infty} \langle (f - A)x^*, J(x_k - x^*)\rangle \leq 0$.*

To find a common fixed point of an infinite family $\{T_i\}_{i=1}^{\infty}$ of nonexpansive mappings on a nonempty, closed and convex subset C in a Hilbert space H, by using the W-mapping W_k defined by (1.4) and a contractive mapping f on C, Kikkawa and Takahashi [20] proved strong convergence of a sequence $\{x_k\}_{k=1}^{\infty}$, generated by the following implicit iterative scheme

$$x_k = \gamma_k f(x_k) + (1 - \gamma_k) W_k x_k, \tag{2.3}$$

where $0 < \alpha_1 \leq 1$ and $0 < \alpha_i \leq b < 1$, for $i = 2, 3, \ldots$. Subsequently, in [19], when C is a nonempty, closed and convex subset of a uniformly convex Banach space X with a uniformly Gateaux differentiable norm, they considered the following algorithm

$$x_k = (1 - \frac{1}{k})Ux_k + \frac{1}{k}f(x_k), \tag{2.4}$$

where

$$Ux = \lim_{k\to\infty} W_k x = \lim_{k\to\infty} U_{k,1}x. \tag{2.5}$$

Proposition 2.5. [6] *Assume that A is a strongly positive linear bounded operator on a smooth Banach space X with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

In 2011, Cai and Bu [7] introduced and analyzed an implicit iterative algorithm for solving a hierarchical variational inequality over the common fixed points set of a countable family of continuous pseudocontractions in a uniformly smooth Banach space.

Theorem 2.1. *(see* [7, Theorem 3.1]*). Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X such that $C \pm C \subset C$. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of continuous pseudocontractive mappings of C into itself such that $\mathscr{F} = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Let $f : C \to C$ be a fixed Lipschitz strongly pseudocontractive mapping with coefficient $\alpha \in (0,1)$ and Lipschitz constant $L > 0$. Let $A : C \to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} - \alpha < 1$. Let $\{x_k\}$ be a sequence generated by the following iterative process:*

$$\begin{cases} x_0 \in C, \\ x_k = \alpha_k f(x_k) + \beta_k x_{k-1} + ((1-\beta_k)I - \alpha_k A)T_k x_k, \quad \forall k \geq 1, \end{cases} \tag{2.6}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = 0$;

(ii) $\sum_{k=1}^{\infty} \frac{\alpha_k}{\alpha_k + \beta_k} = \infty$.

Assume that $\sum_{k=1}^{\infty} \sup_{x\in D} \|T_{k+1}x - T_k x\| < \infty$ for any bounded subset D of C, let T be a mapping of C into itself defined by $Tx = \lim_{k\to\infty} T_k x$ for all $x \in C$, and suppose that $F(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Then, $\{x_k\}$ converges strongly to a point $x^* \in \mathscr{F}$, which is the unique solution in $\mathscr{F}$ to the following VI:

$$\langle (f - A)x^*, J(p - x^*)\rangle \leq 0, \quad \forall p \in \mathscr{F}.$$

Recall that a possibly multivalued operator $\mathscr{A} \subset X \times X$ with domain $D(\mathscr{A})$ and range $R(\mathscr{A})$ in X is accretive if for each $x_i \in D(\mathscr{A})$ and $y_i \in \mathscr{A}x_i$ $(i = 1,2)$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$. An accretive operator $\mathscr{A}$ is said to satisfy the range condition if $\overline{D(\mathscr{A})} \subset R(I + r\mathscr{A})$ for all $r > 0$. An accretive operator $\mathscr{A}$ is m-accretive if $R(I + r\mathscr{A}) = X$ for each $r > 0$. If $\mathscr{A}$ is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$, a mapping $J_r : R(I + r\mathscr{A}) \to D(\mathscr{A})$ by $J_r = (I + r\mathscr{A})^{-1}$, which is called the resolvent of $\mathscr{A}$. It is well known that J_r is nonexpansive and $\mathrm{Fix}(J_r) = \mathscr{A}^{-1}0$ for all $r > 0$. Hence,

$$\mathrm{Fix}(J_r) = \mathscr{A}^{-1}0 = \{z \in D(\mathscr{A}) : 0 \in \mathscr{A}z\}.$$

If $\mathscr{A}^{-1}0 \neq \emptyset$, then the inclusion $0 \in \mathscr{A}z$ is solvable.

The following resolvent identity is well-known.

Lemma 2.5. [3] *For $\lambda > 0, \quad \mu > 0$, and $x \in X$,*

$$J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda x).$$

In [27], Qin, Cho and Wang established the strong convergence of the net $\{x_t\}_{t\in(0,1)}$ defined by the implicit viscosity approximation method with a continuous pseudocontraction, and proposed an iterative process with errors for finding a zero point of an m-accretive operator in a reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure.

Theorem 2.2. *(see* [27, Theorem 2.1]*). Let X be a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure, and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a continuous and bounded pseudocontraction with $\mathrm{Fix}(T) \neq \emptyset$, and let $f : C \to C$ be a fixed continuous and bounded strong pseudocontraction with coefficient $\alpha \in (0,1)$. Let $\{x_t\}$ be the net generated by the following*

$$x_t = tf(x_t) + (1-t)Tx_t, \quad \forall t \in (0,1), \tag{2.7}$$

Then $\{x_t\}$ converges strongly as $t \to 0$ to a point x^* in $\mathrm{Fix}(T)$, which is the unique solution in $\mathrm{Fix}(T)$ to the following VI:

$$\langle (I-f)x^*, J(x^*-p)\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T).$$

Theorem 2.3. [27, Theorem 2.5] *Let X be a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure, and let $\mathscr{A}$ be an m-accretive operator in X. Assume that $C := \overline{D(\mathscr{A})}$ is convex. Let $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ and $\{\delta_k\}$ be real number sequences in $(0,1)$. Let Π_C be a sunny nonexpansive retraction from X onto C. For an arbitrarily given $x_0 \in C$, let $\{x_k\}$ be the sequence generated by*

$$\begin{cases} y_k = \beta_k x_k + \gamma_k J_{r_k}(x_k + e_{k+1}) + \delta_k \Pi_C f_k, \\ x_{k+1} = \alpha_k u + (1-\alpha_k)y_k, \quad \forall k \geq 0, \end{cases} \tag{2.8}$$

where $\{e_k\}$ is a sequence in X, $\{f_k\}$ is a bounded sequence in X, $\{r_k\}$ is a sequence of positive real number, u is a fixed element in C and $J_{r_k} = (I + r_k\mathscr{A})^{-1}$. Assume that $\mathscr{A}^{-1}0 \neq \emptyset$ and the above control sequences satisfy the following restrictions:

(a) $\beta_k + \gamma_k + \delta_k = 1$ *for each* $k \geq 0$*;*
(b) $\lim_{k\to\infty} \alpha_k = 0$ *and* $\sum_{k=0}^{\infty} \alpha_k = \infty$*;*
(c) $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$*;*
(d) $\sum_{k=1}^{\infty} \|e_k\| < \infty$ *and* $\sum_{k=0}^{\infty} \delta_k < \infty$*;*
(e) $r_k \geq \varepsilon$ *for each* $k \geq 0$ *and* $\lim_{k\to\infty} |r_k - r_{k+1}| = 0$.

Then the sequence $\{x_k\}$ generated by (2.3) converges strongly to a zero $Q(u)$ of A, which is the unique solution of the following VI:

$$\langle u - Q(u), J(p - Q(u))\rangle \leq 0, \quad \forall p \in \mathscr{A}^{-1}0.$$

Let LIM be a continuous linear functional on l^∞ and $s = (a_1, a_2, \ldots) \in l^\infty$. We write $\mathrm{LIM}_k a_k$ instead of $\mathrm{LIM}(s)$. LIM is called a Banach limit if LIM satisfies $\|\mathrm{LIM}\| = \mathrm{LIM}_k 1 = 1$ and $\mathrm{LIM}_k a_{k+1} = \mathrm{LIM}_k a_k$ for all $(a_1, a_2, \ldots) \in l^\infty$. If LIM is a Banach limit, then there hold the following:

(i) for all $k \geq 1$, $a_k \leq c_k$ implies $\mathrm{LIM}_k a_k \leq \mathrm{LIM}_k c_k$;
(ii) $\mathrm{LIM}_k a_{k+m} = \mathrm{LIM}_k a_k$ for any fixed positive integer m;
(iii) $\liminf_{k\to\infty} a_k \leq \mathrm{LIM}_k a_k \leq \limsup_{k\to\infty} a_k$ for all $(a_1, a_2, \ldots) \in l^\infty$.

Lemma 2.6. [39] *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_k\} \in l^\infty$ satisfy the condition $\mathrm{LIM}_k a_k \leq a$ for all Banach limit* LIM. *If $\limsup_{k\to\infty}(a_{k+m} - a_k) \leq 0$, then $\limsup_{k\to\infty} a_k \leq a$.*

In particular, if $m = 1$ in Lemma 2.6, then we immediately obtain the following corollary.

Corollary 2.1. [30] *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_k\} \in l^\infty$ satisfy the condition $\mathrm{LIM}_k a_k \leq a$ for all Banach limit* LIM. *If $\limsup_{k\to\infty}(a_{k+1} - a_k) \leq 0$, then $\limsup_{k\to\infty} a_k \leq a$.*

Lemma 2.7. [33] *Let C be a nonempty, closed and convex subset of a Banach space X. Suppose that the norm of X is uniformly Gateaux differentiable. Let $\{x_k\}$ be a bounded sequence in X, and $x^* \in C$. Let* LIM *be a Banach limit. Then $\mathrm{LIM}_k \|x_k - x^*\|^2 = \min_{y\in C} \mathrm{LIM}_k \|x_k - y\|^2$ if and only if $\mathrm{LIM}_k \langle y - x^*, J(x_k - x^*)\rangle \leq 0$ for all $y \in C$.*

3. Iterative algorithms and convergence criteria

In this section, we introduce hybrid viscosity iterative algorithms for solving a hierarchical variational inequality (HVI) over the common fixed points set of a countable family of nonexpansive mappings, and for finding a zero point of an m-accretive operator. Under suitable assumptions, we establish some strong convergence theorems for the proposed iterative algorithms.

The following lemmas will be used to prove our main results in the sequel.

Lemma 3.1. *Let X be a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure, and let C be a nonempty closed convex subset of X such that $C \pm C \subset C$. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and let $f : C \to C$ be a fixed contraction with coefficient $\alpha \in (0,1)$. Let $A : C \to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} - \alpha < 1$ and $\bar{\gamma}\alpha < 1$. Let $\{x_t\}$ be the net generated in the implicit manner*

$$x_t = (I - \theta_t A)Tx_t + \theta_t[f(x_t) - t(Af(x_t) - Tx_t)], \quad \forall t \in (0,1), \tag{3.1}$$

where $\{\theta_t\}_{t\in(0,1)} \subset (0,\infty)$ and $\lim_{t\to 0}\theta_t = 0$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a point x^* in $\mathrm{Fix}(T)$, which is the unique solution in $\mathrm{Fix}(T)$ to the following VI:

$$\langle (A - f)x^*, J(x^* - p)\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{3.2}$$

Proof. First, let us show that the net $\{x_t\}$ is well defined. As a matter of fact, define the mapping $U_t : C \to C$ as follows:

$$U_t x = (I - \theta_t A)Tx + \theta_t[(I - tA)f(x) + tTx], \quad \forall x \in C.$$

Since $\lim_{t\to 0}\theta_t = 0$, we may assume, without loss of generality, that $\theta_t \in (0, \|A\|^{-1})$ for all $t \in (0, \bar{\delta})$, where $\bar{\delta} = \min\{\frac{\bar{\gamma}-\alpha}{2(1-\bar{\gamma}\alpha)}, \|A\|^{-1}\}$.

Utilizing Proposition 2.5, we obtain that for each $t \in (0, \bar{\delta})$

$$\begin{aligned}
\langle U_t x - U_t y, J(x-y)\rangle &= \langle (I-\theta_t A)Tx - (I-\theta_t A)Ty, J(x-y)\rangle + \theta_t\langle (I-tA)f(x) - (I-tA)f(y), J(x-y)\rangle \\
&\quad + \theta_t t\langle Tx - Ty, J(x-y)\rangle \\
&\le \|(I-\theta_t A)Tx - (I-\theta_t A)Ty\|\|x-y\| + \theta_t\|(I-tA)f(x) - (I-tA)f(y)\|\|x-y\| \\
&\quad + \theta_t t\|Tx-Ty\|\|x-y\| \\
&\le (1-\theta_t\bar{\gamma})\|Tx-Ty\|\|x-y\| + \theta_t(1-t\bar{\gamma})\|f(x)-f(y)\|\|x-y\| \\
&\quad + \theta_t t\|Tx-Ty\|\|x-y\| \\
&\le (1-\theta_t\bar{\gamma})\|x-y\|^2 + \theta_t(1-t\bar{\gamma})\alpha\|x-y\|^2 + \theta_t t\|x-y\|^2 \\
&= [1-\theta_t(\bar{\gamma}-\alpha-t(1-\bar{\gamma}\alpha))]\|x-y\|^2 \\
&\le [1-\theta_t(\bar{\gamma}-\alpha-\frac{\bar{\gamma}-\alpha}{2(1-\bar{\gamma}\alpha)}(1-\bar{\gamma}\alpha))]\|x-y\|^2 \\
&= (1-\frac{1}{2}\theta_t(\bar{\gamma}-\alpha))\|x-y\|^2.
\end{aligned}$$

It follows that for each $t \in (0, \bar{\delta})$, $\quad U_t : C \to C$ is a continuous and strongly pseudocontractive mapping with coefficient $1-\frac{1}{2}\theta_t(\bar{\gamma}-\alpha)$. Hence, by Proposition 2.2 we know that there exists a unique fixed point in C, denoted by x_t, which solves the fixed point equation

$$x_t = (I-\theta_t A)Tx_t + \theta_t[(I-tA)f(x_t) + tTx_t].$$

Let us show the uniqueness of solutions of the VI (3.2). Indeed, suppose that both $z_1 \in \mathrm{Fix}(T)$ and $z_2 \in \mathrm{Fix}(T)$ are solutions of the VI (3.2). Then we have

$$\langle (A-f)z_1, J(z_1-z_2)\rangle \le 0 \quad \text{and} \quad \langle (A-f)z_2, J(z_2-z_1)\rangle \le 0.$$

Adding up the above two inequalities, we obtain

$$\langle (A-f)z_1 - (A-f)z_2, J(z_1-z_2)\rangle \le 0.$$

Note that

$$\begin{aligned}
\langle (A-f)z_1 - (A-f)z_2, J(z_1-z_2)\rangle &= \langle A(z_1-z_2), J(z_1-z_2)\rangle - \langle f(z_1)-f(z_2), J(z_1-z_2)\rangle \\
&\ge \bar{\gamma}\|z_1-z_2\|^2 - \alpha\|z_1-z_2\|^2 \\
&= (\gamma-\alpha)\|z_1-z_2\|^2 \ge 0.
\end{aligned}$$

Consequently, we have $z_1 = z_2$, and the uniqueness is proved. We use x^* to denote the unique solution of (3.2).

Next, we prove that $\{x_t : t \in (0,\bar{\delta})\}$ is bounded. Indeed, we note that $0 < \theta_t < 1, \forall t \in (0,\bar{\delta})$. Take a fixed $p \in \mathrm{Fix}(T)$ arbitrarily. Again utilizing Proposition 2.5, we deduce that for all $t \in (0,\bar{\delta})$

$$\begin{aligned}
\|x_t - p\|^2 &= \langle (I-\theta_t A)Tx_t + \theta_t[(I-tA)f(x_t) + tTx_t] - p, J(x_t - p)\rangle \\
&= \langle (I-\theta_t A)Tx_t - (I-\theta_t A)Tp, J(x_t-p)\rangle + \theta_t\langle (I-tA)f(x_t) - (I-tA)f(p), J(x_t-p)\rangle \\
&\quad + \theta_t t\langle Tx_t - p, J(x_t-p)\rangle - \theta_t\langle (A-f)p, J(x_t-p)\rangle + \theta_t t\langle (I-Af)p, J(x_t-p)\rangle \\
&\le \|(I-\theta_t A)Tx_t - (I-\theta_t A)Tp\|\|x_t-p\| + \theta_t\|(I-tA)f(x_t) - (I-tA)f(p)\|\|x_t-p\| \\
&\quad + \theta_t t\langle Tx_t - p, J(x_t-p)\rangle - \theta_t\langle (A-f)p, J(x_t-p)\rangle + \theta_t t\langle (I-Af)p, J(x_t-p)\rangle \\
&\le (1-\theta_t\bar{\gamma})\|Tx_t - Tp\|\|x_t-p\| + \theta_t(1-t\bar{\gamma})\|f(x_t) - f(p)\|\|x_t-p\| \\
&\quad + \theta_t t\|Tx_t - p\|\|x_t-p\| - \theta_t\langle (A-f)p, J(x_t-p)\rangle + \theta_t t\|(I-Af)p\|\|x_t-p\| \\
&\le (1-\theta_t\bar{\gamma})\|x_t-p\|^2 + \theta_t(1-t\bar{\gamma})\alpha\|x_t-p\|^2 \\
&\quad + \theta_t t\|x_t-p\|^2 + \theta_t\langle (f-A)p, J(x_t-p)\rangle + \theta_t t\|(I-Af)p\|\|x_t-p\| \\
&= [1-\theta_t(\bar{\gamma}-\alpha-t(1-\bar{\gamma}\alpha))]\|x_t-p\|^2 + \theta_t\langle (f-A)p, J(x_t-p)\rangle + \theta_t t\|(I-Af)p\|\|x_t-p\| \\
&\le [1-\theta_t(\bar{\gamma}-\alpha-\frac{\bar{\gamma}-\alpha}{2(1-\bar{\gamma}\alpha)}(1-\bar{\gamma}\alpha))]\|x_t-p\|^2 + \theta_t\langle (f-A)p, J(x_t-p)\rangle \\
&\quad + \theta_t t\|(I-Af)p\|\|x_t-p\| \\
&= (1-\frac{1}{2}\theta_t(\bar{\gamma}-\alpha))\|x_t-p\|^2 + \theta_t\langle (f-A)p, J(x_t-p)\rangle + \theta_t t\|(I-Af)p\|\|x_t-p\|,
\end{aligned}$$

which immediately yields

$$\|x_t-p\|^2 \le \frac{2}{\bar{\gamma}-\alpha}(\langle (f-A)p, J(x_t-p)\rangle + t\|(I-Af)p\|\|x_t-p\|). \tag{3.3}$$

It follows that

$$\begin{aligned}
\|x_t-p\| &\le \frac{2}{\bar{\gamma}-\alpha}(\|(A-f)p\| + t\|(I-Af)p\|) \\
&\le \frac{2}{\bar{\gamma}-\alpha}(\|(A-f)p\| + \|A\|^{-1}\|(I-Af)p\|).
\end{aligned}$$

This shows that $\{x_t : t \in (0,\bar{\delta})\}$ is bounded.

Assume that $\{t_k\} \subset (0,\bar{\delta})$ and $t_k \to 0$ as $k \to \infty$. Set $\theta_k = \theta_{t_k}$ and $x_k := x_{t_k}$, and define $g : C \to \mathbf{R}$ by $g(x) = \mathrm{LIM}_k\|x_k - x\|^2, \forall x \in C$, where LIM is a Banach limit on l^∞. Then $g(x)$ is continuous, convex and $g(x) \to \infty$ as $\|x\| \to \infty$. We know that g attains its infinimum over C (see e.g., [33]). Set

$$D = \{x \in C : g(x) = \min_{y\in C} g(y)\}.$$

Then D is a nonempty bounded closed convex subset of C.

Now, we show that there exists a point $x^* \in D$ such that $Tx^* = x^*$. Note that $\|x_k - Tx_k\| = \theta_k\|f(x_k) - t_k(Af(x_k) - Tx_k) - ATx_k\| \to 0$ as $k \to \infty$. In terms of Lemma 2.3, we know that the mapping $B = (2I-T)^{-1} : C \to C$ is nonexpansive with $\mathrm{Fix}(T) = \mathrm{Fix}(B)$ and $\lim_{k\to\infty}\|x_k - Bx_k\| = 0$, where I denotes the identity operator. It follows that

$$g(Bx) = \mathrm{LIM}_k\|x_k - Bx\|^2 = \mathrm{LIM}_k\|Bx_k - Bx\|^2 \le \mathrm{LIM}_k\|x_k - x\|^2 = g(x),$$

which implies that $B(D) \subset D$; that is, D is invariant under B. Since D is a nonempty bounded closed convex subset of the reflexive Banach space X which also has the normal structure, by Proposition 2.1 we know that B has a fixed point, say $x^* \in D$. It follows from $\mathrm{Fix}(B) = \mathrm{Fix}(T)$ that $Tx^* = x^*$. From Lemma 2.7, we see that

$$\mathrm{LIM}_k \langle y - x^*, J(x_k - x^*)\rangle \le 0, \quad \forall y \in C.$$

In particular, we have

$$\mathrm{LIM}_k \langle f(x^*) - Ax^*, J(x_k - x^*)\rangle \le 0.$$

In view of (3.3), we arrive at $\mathrm{LIM}_k \|x_k - x^*\|^2 \le 0$. Hence there exists a subsequence which is still denoted by $\{x_k\}$ such that $x_k \to x^*$ as $k \to \infty$.

Next, we prove that x^* solves the VI (3.2). Taking into account

$$x_t = (I - \theta_t A)Tx_t + \theta_t[f(x_t) - t(Af(x_t) - Tx_t)],$$

we obtain that

$$x_t - Tx_t = \theta_t(f(x_t) - ATx_t) + \theta_t t(Tx_t - Af(x_t)).$$

Since T is nonexpansive, $I - T$ is accretive. So, from the accretiveness of $I - T$, it follows that, for any fixed $p \in \mathrm{Fix}(T)$,

$$\begin{aligned} 0 &\le \langle (I-T)x_t - (I-T)p, J(x_t - p)\rangle = \langle (I-T)x_t, J(x_t - p)\rangle \\ &= \theta_t \langle f(x_t) - ATx_t, J(x_t - p)\rangle + \theta_t t \langle Tx_t - Af(x_t), J(x_t - p)\rangle \\ &= \theta_t \langle (f - A)x_t, J(x_t - p)\rangle + \theta_t \langle Ax_t - ATx_t, J(x_t - p)\rangle \\ &\quad + \theta_t t \langle Tx_t - Af(x_t), J(x_t - p)\rangle. \end{aligned}$$

This implies that

$$\langle (A - f)x_t, J(x_t - p)\rangle \le \langle Ax_t - ATx_t, J(x_t - p)\rangle + t\langle Tx_t - Af(x_t), J(x_t - p)\rangle.$$

Now replacing t with t_k, letting $k \to \infty$, and noticing the boundedness of $\{Tx_{t_k} - Af(x_{t_k})\}$ and the fact that $Ax_{t_k} - ATx_{t_k} \to Ax^* - ATx^* = 0$ for $x^* \in \mathrm{Fix}(T)$, we conclude that

$$\langle (A - f)x^*, J(x^* - p)\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T).$$

That is, x^* is a solution of the VI (3.2) in $\mathrm{Fix}(T)$. In the same way, we infer that each cluster point of $\{x_t\}$ is a solution of the VI (3.2) in $\mathrm{Fix}(T)$, and hence it equals x^* by uniqueness. This completes the proof. □

Remark 3.1. Our Lemma 3.1 is very different from Cai and Bu [7, Lemma 2.5] and Qin, Cho and Wang [27, Theorem 2.1] in the following aspects:

(i) The uniform smoothness of the Banach space X in [7, Lemma 2.5] is replaced by the uniform Gateaux differentiability and the normal structure of the reflexive Banach space X in our Lemma 3.1. Our hybrid implicit viscosity scheme (3.1) is obviously different from the implicit viscosity iterative scheme $x_t = (I - tA)Tx_t + tf(x_t)$ in [7, Lemma 2.5] because we add the net $\{\theta_t\}$ satisfying $\lim_{t\to 0} \theta_t = 0$ in our iterative scheme (3.1).

(ii) The first and second terms $(1-t)Tx_t$ and $tf(x_t)$ in the implicit viscosity scheme $x_t = (1-t)Tx_t + tf(x_t)$ of [27, Lemma 2.1], are replaced by the first and second ones $(I-\theta_t A)Tx_t$ and $\theta_t[f(x_t) - t(Af(x_t) - Tx_t)]$ in our iterative scheme (3.1), where $\lim_{t\to 0}\theta_t = 0$. It is worth pointing out that we add a strongly positive bounded linear operator A and the net $\{\theta_t\}$ satisfying $\lim_{t\to 0}\theta_t = 0$ in our iterative scheme (3.1).

By using the same way, we can also derive the following lemma.

Lemma 3.2. *Let X be a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure, and let C be a nonempty closed convex subset of X such that $C \pm C \subset C$. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and let $f : C \to C$ be a fixed contraction with coefficient $\alpha \in (0,1)$. Let $A : C \to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $0 < \alpha < \bar{\gamma}$. Let $\{x_t\}$ be defined by*

$$x_t = tf(x_t) + (I - tA)Tx_t.$$

Then, as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point x^* of T such that x^* is the unique solution in $\mathrm{Fix}(T)$ to the VI

$$\langle (A-f)x^*, J(x^*-p)\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T).$$

Recall that a Banach space X is said to satisfy Opial's condition [26], if whenever $\{x_k\}$ is a sequence in X which converges weakly to x as $k \to \infty$, then

$$\limsup_{k\to\infty} \|x_k - x\| < \limsup_{k\to\infty} \|x_k - y\|, \quad \forall y \in X, \quad y \neq x.$$

Lemma 3.3. [18] *Let X be a reflexive Banach space satisfying Opial's condition, C be a nonempty closed convex subset of X and $T : C \to C$ be a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, if $\{x_k\}$ is a sequence of C such that $x_k \rightharpoonup x$ and $(I-T)x_k \to y$, then $(I-T)x = y$.*

Remark 3.2. Since if the duality mapping $J : x \mapsto \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ from X into X^* is single-valued and weakly sequentially continuous, then X satisfies Opial's condition; see [17, Theorem 1]. Each Hilbert space and the sequence spaces l^p with $1 < p < \infty$ satisfy Opial's condition; see [17]. Though an L^p-space with $p \neq 2$ does not usually satisfy Opial's condition, each separable Banach space can be equivalently renormed so that it satisfies Opial's condition; see [34].

Lemma 3.4. [2] *Let C be a nonempty closed convex subset of a Banach space X. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of mappings of C into itself. Suppose that $\sum_{k=1}^{\infty} \sup\{\|T_{k+1}x - T_k x\| : x \in C\} < \infty$. Then, for each $y \in C$, $\{T_k y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{k\to\infty} T_k y$, for all $y \in C$. Then $\lim_{k\to\infty} \sup\{\|Tx - T_k x\| : x \in C\} = 0$.*

We are now in a position to state and prove our main results in this paper.

Theorem 3.1. *Let X be a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure, and let C be a nonempty closed convex subset of X such that $C \pm C \subset C$. Let*

$\{T_i\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $\mathscr{F} = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$, and let $f : C \to C$ be a fixed contraction with coefficient $\alpha \in (0,1)$. Let $A : C \to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} \in (\alpha, 1+\alpha)$. For an arbitrarily given $x_0 \in C$, let the sequence $\{x_k\}$ be generated iteratively by

$$\begin{cases} y_k = \alpha_k f(y_k) + \beta_k x_k + ((1-\beta_k)I - \alpha_k A)T_k y_k, \\ x_{k+1} = \gamma_k f(y_k) + (I - \gamma_k A)T_k y_k, \quad \forall k \geq 0, \end{cases} \tag{3.4}$$

where $\{\alpha_k\}, \{\beta_k\}$ and $\{\gamma_k\}$ are four sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = 0$;

(ii) $\limsup_{k\to\infty} \gamma_k/\alpha_k < \infty$ *and* $\sum_{k=0}^{\infty} \frac{\alpha_k}{\alpha_k+\beta_k} = \infty$.

Assume that $\sum_{k=0}^{\infty} \sup_{x\in D} \|T_{k+1}x - T_k x\| < \infty$ for any bounded subset D of C, let T be a mapping of C into itself defined by $Tx = \lim_{k\to\infty} T_k x$ for all $x \in C$, and suppose that $\mathrm{Fix}(T) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i)$. Then, $\{x_k\}$ converges strongly to a point $x^* \in \mathscr{F}$ which is the unique solution in $\mathscr{F}$ to the following VI:

$$\langle (f-A)x^*, J(p-x^*)\rangle \leq 0, \quad \forall p \in \mathscr{F}. \tag{3.5}$$

Proof. By condition (i), we may assume, without loss of generality, that $\alpha_k \leq (1-\beta_k)\|A\|^{-1}$ for all $k \geq 0$. Since A is a $\bar{\gamma}$-strongly positive linear bounded operator on C, from (2.6) we have

$$\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in C, \quad \|u\| = 1\}.$$

Observe that

$$\begin{aligned} \langle ((1-\beta_k)I - \alpha_k A)u, J(u)\rangle &= 1 - \beta_k - \alpha_k \langle Au, J(u)\rangle \\ &\geq 1 - \beta_k - \alpha_k \|A\| \\ &\geq 0. \end{aligned}$$

It follows that

$$\begin{aligned} \|(1-\beta_k)I - \alpha_k A\| &= \sup\{\langle ((1-\beta_k)I - \alpha_k A)u, J(u)\rangle : u \in C, \quad \|u\| = 1\} \\ &= \sup\{1 - \beta_k - \alpha_k \langle Au, J(u)\rangle : u \in C, \quad \|u\| = 1\} \\ &\leq 1 - \beta_k - \alpha_k \bar{\gamma}. \end{aligned}$$

Next, we show that $\{y_k\}$ is well defined. For each $k \geq 0$, define a mapping $S_k : C \to C$ by

$$S_k x = \alpha_k f(x) + \beta_k x_k + ((1-\beta_k)I - \alpha_k A)T_k x, \quad \forall x \in C.$$

For every $x, y \in C$, we have

$$\begin{aligned} \langle S_k x - S_k y, J(x-y)\rangle &= \alpha_k \langle f(x) - f(y), J(x-y)\rangle + \langle ((1-\beta_k)I - \alpha_k A)(T_k x - T_k y), J(x-y)\rangle \\ &\leq \alpha_k \alpha \|x-y\|^2 + (1 - \beta_k - \alpha_k \bar{\gamma})\|T_k x - T_k y\| \|x-y\| \\ &\leq \alpha_k \alpha \|x-y\|^2 + (1 - \beta_k - \alpha_k \bar{\gamma})\|x-y\|^2 \\ &= [1 - \beta_k - \alpha_k(\bar{\gamma} - \alpha)]\|x-y\|^2, \end{aligned}$$

Therefore, S_k is a continuous strong pseudocontraction for each $k \geq 0$. By Proposition 2.2, we see that there exists a unique fixed point y_k for each $k \geq 0$ such that

$$y_k = \alpha_k f(y_k) + \beta_k x_k + ((1-\beta_k)I - \alpha_k A)T_k y_k.$$

That is, the sequence $\{y_k\}$ is well defined. Next, we prove that $\{x_k\}$ is bounded. Take a fixed $p \in \mathscr{F}$ arbitrarily. Then we have

$$\begin{aligned}
\|y_k - p\|^2 &= \alpha_k \langle f(y_k) - Ap, J(y_k - p)\rangle + \beta_k \langle x_k - p, J(y_k - p)\rangle \\
&\quad + \langle ((1-\beta_k)I - \alpha_k A)(T_k y_k - p), J(y_k - p)\rangle \\
&\leq \alpha_k \langle f(y_k) - f(p), J(y_k - p)\rangle + \alpha_k \langle f(p) - Ap, J(y_k - p)\rangle + \beta_k \|x_k - p\| \|y_k - p\| \\
&\quad + (1 - \beta_k - \alpha_k \bar{\gamma}) \|T_k y_k - p\| \|y_k - p\| \\
&\leq \alpha_k \alpha \|y_k - p\|^2 + \alpha_k \langle f(p) - Ap, J(y_k - p)\rangle + \beta_k \|x_k - p\| \|y_k - p\| \\
&\quad + (1 - \beta_k - \alpha_k \bar{\gamma}) \|y_k - p\|^2 \\
&= (1 - \beta_k - \alpha_k(\bar{\gamma} - \alpha)) \|y_k - p\|^2 + \beta_k \|x_k - p\| \|y_k - p\| + \alpha_k \langle f(p) - Ap, J(y_k - p)\rangle \\
&\leq (1 - \beta_k - \alpha_k(\bar{\gamma} - \alpha)) \|y_k - p\|^2 + \beta_k \|x_k - p\| \|y_k - p\| + \alpha_k \|f(p) - Ap\| \|y_k - p\|,
\end{aligned}$$

which implies that

$$\|y_k - p\| \leq \frac{\beta_k}{\beta_k + \alpha_k(\bar{\gamma} - \alpha)} \|x_k - p\| + \frac{\alpha_k(\bar{\gamma} - \alpha)}{\beta_k + \alpha_k(\bar{\gamma} - \alpha)} \frac{\|f(p) - Ap\|}{\bar{\gamma} - \alpha}.$$

Therefore, we have

$$\begin{aligned}
\|x_{k+1} - p\| &= \|\gamma_k f(y_k) + (I - \gamma_k A)T_k y_k - p\| \\
&= \|\gamma_k (f(y_k) - f(p)) + (I - \gamma_k A)T_k y_k - (I - \gamma_k A)T_k p + \gamma_k (f(p) - Ap)\| \\
&\leq \gamma_k \|f(y_k) - f(p)\| + \|(I - \gamma_k A)(T_k y_k - T_k p)\| + \gamma_k \|f(p) - Ap\| \\
&\leq \gamma_k \alpha \|y_k - p\| + (1 - \gamma_k \bar{\gamma}) \|y_k - p\| + \gamma_k \|f(p) - Ap\| \\
&= (1 - \gamma_k(\bar{\gamma} - \alpha)) \|y_k - p\| + \gamma_k \|f(p) - Ap\| \\
&= (1 - \gamma_k(\bar{\gamma} - \alpha)) [\frac{\beta_k}{\beta_k + \alpha_k(\bar{\gamma} - \alpha)} \|x_k - p\| \\
&\quad + \frac{\alpha_k(\bar{\gamma} - \alpha)}{\beta_k + \alpha_k(\bar{\gamma} - \alpha)} \cdot \frac{\|f(p) - Ap\|}{\bar{\gamma} - \alpha}] + \gamma_k \|f(p) - Ap\| \\
&\leq (1 - \gamma_k(\bar{\gamma} - \alpha)) \max\{\|x_k - p\|, \frac{\|f(p) - Ap\|}{\bar{\gamma} - \alpha}\} + \gamma_k \|f(p) - Ap\| \\
&= (1 - \gamma_k(\bar{\gamma} - \alpha)) \max\{\|x_k - p\|, \frac{\|f(p) - Ap\|}{\bar{\gamma} - \alpha}\} + \gamma_k(\bar{\gamma} - \alpha) \frac{\|f(p) - Ap\|}{\bar{\gamma} - \alpha} \\
&\leq \max\{\|x_k - p\|, \frac{\|f(p) - Ap\|}{\bar{\gamma} - \alpha}\}.
\end{aligned}$$

By induction, we get

$$\|x_k - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - Ap\|}{\bar{\gamma} - \alpha}\}, \quad \forall k \geq 0.$$

Therefore, $\{x_k\}$ is bounded and so are the sequences $\{y_k\}, \{T_k y_k\}$. We observe that

$$\begin{aligned}\|y_k - T_k y_k\| &= \|\alpha_k(f(y_k) - AT_k y_k) + \beta_k(x_k - T_k y_k)\| \\ &\leq \alpha_k\|f(y_k) - AT_k y_k\| + \beta_k\|x_k - T_k y_k\|,\end{aligned}$$

which together with condition (i), implies that

$$\lim_{k\to\infty}\|y_k - T_k y_k\| = 0.$$

On the other hand, we have

$$\|y_k - Ty_k\| \leq \|y_k - T_k y_k\| + \|T_k y_k - Ty_k\|.$$

Utilizing Lemma 3.4, we immediately derive

$$\lim_{k\to\infty}\|y_k - Ty_k\| = 0.$$

Let $x_t = tf(x_t) + (I - tA)Tx_t$. Utilizing Lemmas 2.4 and 3.2, we conclude that $\{x_t\}$ converges strongly to $x^* \in \mathrm{Fix}(T) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(T_i) = \mathscr{F}$ and

$$\limsup_{k\to\infty}\langle (f - A)x^*, J(y_k - x^*)\rangle \leq 0. \tag{3.6}$$

Finally, we show that $x_k \to x^*$ as $k \to \infty$. We observe that

$$\begin{aligned}\|y_k - x^*\| &= \alpha_k\langle f(y_k) - Ax^*, J(y_k - x^*)\rangle + \beta_k\langle x_k - x^*, J(y_k - x^*)\rangle \\ &\quad + \langle ((1-\beta_k)I - \alpha_k A)(T_k y_k - x^*), J(y_k - x^*)\rangle \\ &\leq (1 - \beta_k - \alpha_k\bar{\gamma})\|y_k - x^*\|^2 + \beta_k\|x_k - x^*\|\|y_k - x^*\| \\ &\quad + \alpha_k\langle f(y_k) - f(x^*), J(y_k - x^*)\rangle + \alpha_k\langle f(x^*) - Ax^*, J(y_k - x^*)\rangle \\ &\leq (1 - \beta_k - \alpha_k\bar{\gamma})\|y_k - x^*\|^2 + \beta_k\|x_k - x^*\|\|y_k - x^*\| + \alpha_k\alpha\|y_k - x^*\|^2 \\ &\quad + \alpha_k\langle f(x^*) - Ax^*, J(y_k - x^*)\rangle \\ &\leq (1 - \beta_k - \alpha_k\bar{\gamma})\|y_k - x^*\|^2 + \frac{\beta_k}{2}\|x_k - x^*\|^2 + \frac{\beta_k}{2}\|y_k - x^*\|^2 \\ &\quad + \alpha_k\alpha\|y_k - x^*\|^2 + \alpha_k\langle f(x^*) - Ax^*, J(y_k - x^*)\rangle \\ &= (1 - \frac{\beta_k}{2} - \alpha_k(\bar{\gamma} - \alpha))\|y_k - x^*\|^2 + \frac{\beta_k}{2}\|x_k - x^*\|^2 + \alpha_k\langle f(x^*) - Ax^*, J(y_k - x^*)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|y_k - x^*\|^2 &\leq \frac{\beta_k}{\beta_k + 2\alpha_k(\bar{\gamma} - \alpha)}\|x_k - x^*\|^2 + \frac{2\alpha_k}{\beta_k + 2\alpha_k(\bar{\gamma} - \alpha)}\langle f(x^*) - Ax^*, J(y_k - x^*)\rangle \\ &= (1 - \frac{2\alpha_k(\bar{\gamma} - \alpha)}{\beta_k + 2\alpha_k(\bar{\gamma} - \alpha)})\|x_k - x^*\|^2 + \frac{2\alpha_k(\bar{\gamma} - \alpha)}{\beta_k + 2\alpha_k(\bar{\gamma} - \alpha)}\frac{\langle f(x^*) - Ax^*, J(y_k - x^*)\rangle}{\bar{\gamma} - \alpha}.\end{aligned}$$

Furthermore, utilizing Lemma 2.2 (i), from the last relation we have

$$\begin{aligned}
\|x_{k+1}-x^*\|^2 &= \|\gamma_k(f(y_k)-f(x^*))+(I-\gamma_k A)T_k y_k-(I-\gamma_k A)T_k x^*+\gamma_k(f(x^*)-Ax^*)\|^2 \\
&\le \|\gamma_k(f(y_k)-f(x^*))+(I-\gamma_k A)T_k y_k-(I-\gamma_k A)T_k x^*\|^2 \\
&\quad +2\gamma_k\langle f(x^*)-Ax^*, J(x_{k+1}-x^*)\rangle \\
&\le [\gamma_k\alpha\|y_k-x^*\|^2+(1-\gamma_k\bar{\gamma})\|T_k y_k-T_k x^*\|]^2+2\gamma_k\|f(x^*)-Ax^*\|\|x_{k+1}-x^*\| \\
&\le [\gamma_k\alpha\|y_k-x^*\|^2+(1-\gamma_k\bar{\gamma})\|y_k-x^*\|]^2+2\gamma_k\|f(x^*)-Ax^*\|\|x_{k+1}-x^*\| \\
&= (1-\gamma_k(\bar{\gamma}-\alpha))^2\|y_k-x^*\|^2+2\gamma_k\|f(x^*)-Ax^*\|\|x_{k+1}-x^*\| \\
&\le \|y_k-x^*\|^2+2\gamma_k\|f(x^*)-Ax^*\|\|x_{k+1}-x^*\| \\
&\le (1-\frac{2\alpha_k(\bar{\gamma}-\alpha)}{\beta_k+2\alpha_k(\bar{\gamma}-\alpha)})\|x_k-x^*\|^2+\frac{2\alpha_k(\bar{\gamma}-\alpha)}{\beta_k+2\alpha_k(\bar{\gamma}-\alpha)}\frac{\langle f(x^*)-Ax^*, J(y_k-x^*)\rangle}{\bar{\gamma}-\alpha} \\
&\quad +2\gamma_k\|f(x^*)-Ax^*\|\|x_{k+1}-x^*\| \\
&= (1-\frac{2\alpha_k(\bar{\gamma}-\alpha)}{\beta_k+2\alpha_k(\bar{\gamma}-\alpha)})\|x_k-x^*\|^2+\frac{2\alpha_k(\bar{\gamma}-\alpha)}{\beta_k+2\alpha_k(\bar{\gamma}-\alpha)}\{\frac{\langle f(x^*)-Ax^*, J(y_k-x^*)\rangle}{\bar{\gamma}-\alpha} \\
&\quad +\frac{\beta_k+2\alpha_k(\bar{\gamma}-\alpha)}{\bar{\gamma}-\alpha}\cdot\frac{\gamma_k}{\alpha_k}\cdot\|f(x^*)-Ax^*\|\|x_{k+1}-x^*\|\}.
\end{aligned} \tag{3.7}$$

We note that

$$\frac{2\alpha_k(\bar{\gamma}-\alpha)}{\beta_k+2\alpha_k(\bar{\gamma}-\alpha)} > \frac{2\alpha_k(\bar{\gamma}-\alpha)}{2\beta_k+2\alpha_k} = (\bar{\gamma}-\alpha)\frac{\alpha_k}{\alpha_k+\beta_k}.$$

Therefore, condition (ii) leads to $\sum_{k=0}^{\infty}\frac{2\alpha_k(\bar{\gamma}-\alpha)}{\beta_k+2\alpha_k(\bar{\gamma}-\alpha)}=\infty$. In addition, since $\alpha_k\to 0$, $\beta_k\to 0$ and $\limsup_{k\to\infty}\frac{\gamma_k}{\alpha_k}<\infty$, we get from (3.6) that

$$\limsup_{k\to\infty}\{\frac{\langle f(x^*)-Ax^*, J(y_k-x^*)\rangle}{\bar{\gamma}-\alpha}+\frac{\beta_k+2\alpha_k(\bar{\gamma}-\alpha)}{\bar{\gamma}-\alpha}\cdot\frac{\gamma_k}{\alpha_k}\cdot\|f(x^*)-Ax^*\|\|x_{k+1}-x^*\|\}\le 0.$$

Applying Lemma 2.1 to (3.7), we have that $x_k\to x^*$ as $k\to\infty$. This completes the proof. □

Next, we prove strong convergence of the iterative process with errors for an m-accretive operator.

Theorem 3.2. *Let X be a real reflexive Banach space which has the uniformly Gateaux differentiable norm, the normal structure and the weakly sequentially continuous duality mapping J. Let $\mathscr{A}$ be an m-accretive operator in X such that $\mathscr{A}^{-1}0\neq\emptyset$. Assume that $C:=\overline{D(\mathscr{A})}$ is convex such that $C\pm C\subset C$. Let $f:C\to C$ be a fixed contraction with coefficient $\alpha\in(0,1)$ and let $A:C\to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma}>0$ such that $0<\bar{\gamma}-\alpha<1$ and $\bar{\gamma}\alpha<1$. For any given $x_0\in C$, let $\{x_k\}_{k=1}^{\infty}$ be the sequence defined by*

$$\begin{cases} x_{k+1}=(I-\beta_k A)J_{r_k}(x_k+u_k)+\beta_k y_k, \\ y_k=(I-\alpha_k A)f(x_k)+\alpha_k J_{r_k}(x_k+v_k), \quad \forall k\ge 0, \end{cases} \tag{3.8}$$

where $\{u_k\}$ and $\{v_k\}$ are two bounded sequences in C, $\{r_k\}$ is a sequence in $[\tilde{r},\infty)$ for some $\tilde{r}>0$, $J_{r_k}=(I+r_k\mathscr{A})^{-1}$, and $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences in $(0,1)$. Suppose that:

(i) $\sum_{k=0}^{\infty}\alpha_k<\infty$, $\lim_{k\to\infty}\beta_k=0$, $\sum_{k=0}^{\infty}\beta_k=\infty$;

(ii) $\sum_{k=1}^{\infty}|\beta_k-\beta_{k-1}|<\infty$ *or* $\lim_{k\to\infty}\beta_{k-1}/\beta_k=1$;

(iii) $\sum_{k=1}^{\infty}\|u_k\|<\infty$ *and* $\sum_{k=1}^{\infty}|r_k-r_{k-1}|<\infty$.

Then $\{x_k\}$ *converges strongly to a point* $x^*\in\mathscr{A}^{-1}0$ *which is the unique solution in* $\mathscr{A}^{-1}0$ *to the following VI:*

$$\langle (A-f)x^*, J(x^*-p)\rangle \le 0, \quad \forall p\in\mathscr{A}^{-1}0.$$

Proof. First, since A is a $\bar\gamma$-strongly positive linear bounded operator on C, from (2.6) we have

$$\|A\| = \sup\{|\langle Au, J(u)\rangle| : u\in C, \quad \|u\|=1\}.$$

Let us show that $\{x_k\}$ is bounded. Indeed, since $\lim_{k\to\infty}\alpha_k=\lim_{k\to\infty}\beta_k=0$, without loss of generality, we may assume that $0<\alpha_k\le\min\{\frac{\bar\gamma-\alpha}{2(1-\bar\gamma\alpha)},\|A\|^{-1}\}$ and $0<\beta_k\le\|A\|^{-1}$ for all $k\ge 0$. Take a fixed $p\in\mathscr{A}^{-1}0$ arbitrarily. Then it follows that $p=J_{r_k}p, \forall k\ge 0$, and

$$\begin{aligned} x_{k+1}-p &= (I-\beta_kA)J_{r_k}(x_k+u_k)-(I-\beta_kA)J_{r_k}p+\beta_k[(I-\alpha_kA)f(x_k)-(I-\alpha_kA)f(p)\\ &\quad+\alpha_k(J_{r_k}(x_k+v_k)-p)]+\beta_k(f-A)p+\beta_k\alpha_k(I-Af)p. \end{aligned}$$

Hence we deduce from $0<\alpha_k\le\min\{\frac{\bar\gamma-\alpha}{2(1-\bar\gamma\alpha)},\|A\|^{-1}\}$ that

$$\begin{aligned} \|x_{k+1}-p\| &= \|(I-\beta_kA)J_{r_k}(x_k+u_k)-(I-\beta_kA)J_{r_k}p+\beta_k[(I-\alpha_kA)f(x_k)-(I-\alpha_kA)f(p)\\ &\quad+\alpha_k(J_{r_k}(x_k+v_k)-p)]+\beta_k(f-A)p+\beta_k\alpha_k(I-Af)p\|\\ &\le \|(I-\beta_kA)J_{r_k}(x_k+u_k)-(I-\beta_kA)J_{r_k}p\|+\beta_k[\|(I-\alpha_kA)f(x_k)-(I-\alpha_kA)f(p)\|\\ &\quad+\alpha_k\|J_{r_k}(x_k+v_k)-p\|]+\beta_k\|(f-A)p\|+\beta_k\alpha_k\|(I-Af)p\|\\ &\le (1-\beta_k\bar\gamma)\|J_{r_k}(x_k+u_k)-J_{r_k}p\|+\beta_k[(1-\alpha_k\bar\gamma)\|f(x_k)-f(p)\|\\ &\quad+\alpha_k\|J_{r_k}(x_k+v_k)-J_{r_k}p\|]+\beta_k\|(f-A)p\|+\beta_k\alpha_k\|(I-Af)p\|\\ &\le (1-\beta_k\bar\gamma)\|(x_k+u_k)-p\|+\beta_k[(1-\alpha_k\bar\gamma)\alpha\|x_k-p\|\\ &\quad+\alpha_k\|(x_k+v_k)-p\|]+\beta_k\|(f-A)p\|+\beta_k\alpha_k\|(I-Af)p\|\\ &\le (1-\beta_k\bar\gamma)(\|x_k-p\|+\|u_k\|)+\beta_k[(1-\alpha_k\bar\gamma)\alpha\|x_k-p\|\\ &\quad+\alpha_k(\|x_k-p\|+\|v_k\|)]+\beta_k\|(f-A)p\|+\beta_k\alpha_k\|(I-Af)p\|\\ &\le [1-\beta_k(\bar\gamma-\alpha-\alpha_k(1-\bar\gamma\alpha))]\|x_k-p\|+\beta_k\|(f-A)p\|+\beta_k\alpha_k\|(I-Af)p\|\\ &\quad+\|u_k\|+\alpha_k\beta_k\|v_k\|\\ &\le [1-\beta_k(\bar\gamma-\alpha-\frac{\bar\gamma-\alpha}{2(1-\bar\gamma\alpha)}(1-\bar\gamma\alpha))]\|x_k-p\|+\beta_k(\|(f-A)p\|+\|(I-Af)p\|)\\ &\quad+\|u_k\|+\alpha_k\beta_k\|v_k\|\\ &\le (1-\frac{1}{2}\beta_k(\bar\gamma-\alpha))\|x_k-p\|+\beta_k(\|(f-A)p\|+\|(I-Af)p\|)+\|u_k\|+\alpha_k\beta_k\|v_k\|\\ &\le \max\{\|x_k-p\|,\frac{2(\|(f-A)p\|+\|(I-Af)p\|)}{\bar\gamma-\alpha}\}+\|u_k\|+\alpha_k\beta_k\|v_k\|. \end{aligned}$$

By induction

$$\|x_k - p\| \le \max\{\|x_0 - p\|, \frac{2(\|(f-A)p\| + \|(I-Af)p\|)}{\bar{\gamma} - \alpha}\} + \sum_{i=0}^{k}(\|u_i\| + \alpha_i\beta_i\|v_i\|).$$

This implies that $\{x_k\}$ is bounded and so are $\{J_{r_k}(x_k+u_k)\}, \{J_{r_k}(x_k+v_k)\}$ and $\{f(x_k)\}$.

Now we claim that

$$\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0. \tag{3.9}$$

Indeed, first of all, we observe that

$$\begin{aligned}
\|y_k - y_{k-1}\| &= \|(I-\alpha_k A)f(x_k) + \alpha_k J_{r_k}(x_k+v_k) - (I-\alpha_{k-1}A)f(x_{k-1}) - \alpha_{k-1}J_{r_{k-1}}(x_{k-1}+v_{k-1})\| \\
&= \|\alpha_k(J_{r_k}(x_k+v_k) - J_{r_{k-1}}(x_{k-1}+v_{k-1}) + (\alpha_k - \alpha_{k-1})(J_{r_{k-1}}(x_{k-1}+v_{k-1}) - Af(x_{k-1})) \\
&\quad + (I-\alpha_k A)f(x_k) - (I-\alpha_k A)f(x_{k-1})\| \\
&\le \alpha_k\|J_{r_k}(x_k+v_k) - J_{r_{k-1}}(x_{k-1}+v_{k-1})\| + |\alpha_k - \alpha_{k-1}|\|J_{r_{k-1}}(x_{k-1}+v_{k-1}) - Af(x_{k-1})\| \\
&\quad + \|I-\alpha_k A\|\|f(x_k) - f(x_{k-1})\| \\
&\le \alpha_k(\|J_{r_k}(x_k+v_k) - J_{r_k}(x_{k-1}+v_{k-1})\| + \|J_{r_k}(x_{k-1}+v_{k-1}) - J_{r_{k-1}}(x_{k-1}+v_{k-1})\|) \\
&\quad + |\alpha_k - \alpha_{k-1}|\|J_{r_{k-1}}(x_{k-1}+v_{k-1}) - Af(x_{k-1})\| + (1-\alpha_k\bar{\gamma})\alpha\|x_k - x_{k-1}\| \\
&\le \alpha_k(\|x_k - x_{k-1}\| + \|v_k - v_{k-1}\| + \|J_{r_k}(x_{k-1}+v_{k-1}) - J_{r_{k-1}}(x_{k-1}+v_{k-1})\| \\
&\quad + |\alpha_k - \alpha_{k-1}|\|J_{r_{k-1}}(x_{k-1}+v_{k-1}) - Af(x_{k-1})\| + (1-\alpha_k\bar{\gamma})\alpha\|x_k - x_{k-1}\| \\
&= (\alpha - \alpha_k(\bar{\gamma}\alpha - 1))\|x_k - x_{k-1}\| + |\alpha_k - \alpha_{k-1}|\|J_{r_{k-1}}(x_{k-1}+v_{k-1}) - Af(x_{k-1})\| \\
&\quad + \alpha_k(\|v_k - v_{k-1}\| + \|J_{r_k}(x_{k-1}+v_{k-1}) - J_{r_{k-1}}(x_{k-1}+v_{k-1})\|.
\end{aligned} \tag{3.10}$$

On the other hand, if $r_{k-1} \le r_k$, using the resolvent identity in Lemma 2.5 we have

$$J_{r_k}(x_{k-1}+v_{k-1}) = J_{r_{k-1}}(\frac{r_{k-1}}{r_k}(x_{k-1}+v_{k-1}) + (1-\frac{r_{k-1}}{r_k})J_{r_k}(x_{k-1}+v_{k-1})),$$

which hence implies that

$$\begin{aligned}
&\|J_{r_k}(x_{k-1}+v_{k-1}) - J_{r_{k-1}}(x_{k-1}+v_{k-1})\| \\
&= \|J_{r_{k-1}}(\frac{r_{k-1}}{r_k}(x_{k-1}+v_{k-1}) + (1-\frac{r_{k-1}}{r_k})J_{r_k}(x_{k-1}+v_{k-1})) - J_{r_{k-1}}(x_{k-1}+v_{k-1})\| \\
&\le \frac{r_k - r_{k-1}}{r_k}\|J_{r_k}(x_{k-1}+v_{k-1}) - (x_{k-1}+v_{k-1})\| \\
&\le \frac{1}{\tilde{r}}|r_k - r_{k-1}|\|J_{r_k}(x_{k-1}+v_{k-1}) - (x_{k-1}+v_{k-1})\|.
\end{aligned}$$

If $r_k \le r_{k-1}$, we derive in a similar way

$$\|J_{r_k}(x_{k-1}+v_{k-1}) - J_{r_{k-1}}(x_{k-1}+v_{k-1})\| \le \frac{1}{\tilde{r}}|r_{k-1} - r_k|\|J_{r_{k-1}}(x_{k-1}+v_{k-1})) - (x_{k-1}+v_{k-1})\|.$$

Combining the last two inequalities we obtain

$$\|J_{r_k}(x_{k-1}+v_{k-1}) - J_{r_{k-1}}(x_{k-1}+v_{k-1})\| \le M_0|r_{k-1} - r_k|, \quad \forall k \ge 1, \tag{3.11}$$

where $\sup_{k\geq 1}\{\frac{1}{\bar{r}}(\|J_{r_k}(x_{k-1}+v_{k-1})-(x_{k-1}+v_{k-1})\|+\|J_{r_{k-1}}(x_{k-1}+v_{k-1})-(x_{k-1}+v_{k-1})\|)\}\leq M_0$ for some $M_0>0$.

Similarly, it is not hard to find that there exists a constant $M_1>0$ such that

$$\|J_{r_k}(x_{k-1}+u_{k-1})-J_{r_{k-1}}(x_{k-1}+u_{k-1})\|\leq M_1|r_k-r_{k-1}|,\quad \forall k\geq 1. \tag{3.12}$$

So, it follows from (3.8) and (3.10)-(3.12) that

$$\begin{aligned}
\|x_{k+1}-x_k\| &= \|(I-\beta_kA)J_{r_k}(x_k+u_k)+\beta_ky_k-(I-\beta_{k-1}A)J_{r_{k-1}}(x_{k-1}+u_{k-1})-\beta_{k-1}y_{k-1}\| \\
&= \|\beta_k(y_k-y_{k-1})+(\beta_k-\beta_{k-1})(y_{k-1}-AJ_{r_{k-1}}(x_{k-1}+u_{k-1})) \\
&\quad +(I-\beta_kA)J_{r_k}(x_k+u_k)-(I-\beta_kA)J_{r_{k-1}}(x_{k-1}+u_{k-1})\| \\
&\leq \beta_k\|y_k-y_{k-1}\|+|\beta_k-\beta_{k-1}|\|y_{k-1}-AJ_{r_{k-1}}(x_{k-1}+u_{k-1})\| \\
&\quad +(1-\beta_k\bar{\gamma})\|J_{r_k}(x_k+u_k)-J_{r_{k-1}}(x_{k-1}+u_{k-1})\| \\
&\leq \beta_k\|y_k-y_{k-1}\|+|\beta_k-\beta_{k-1}|\|y_{k-1}-AJ_{r_{k-1}}(x_{k-1}+u_{k-1})\| \\
&\quad +(1-\beta_k\bar{\gamma})(\|J_{r_k}(x_k+u_k)-J_{r_k}(x_{k-1}+u_{k-1})\| \\
&\quad +\|J_{r_k}(x_{k-1}+u_{k-1})-J_{r_{k-1}}(x_{k-1}+u_{k-1})\|) \\
&\leq \beta_k\|y_k-y_{k-1}\|+|\beta_k-\beta_{k-1}|\|y_{k-1}-AJ_{r_{k-1}}(x_{k-1}+u_{k-1})\| \\
&\quad +(1-\beta_k\bar{\gamma})(\|(x_k+u_k)-(x_{k-1}+u_{k-1})\|+\|J_{r_k}(x_{k-1}+u_{k-1})-J_{r_{k-1}}(x_{k-1}+u_{k-1})\|) \\
&\leq \beta_k[(\alpha-\alpha_k(\bar{\gamma}\alpha-1))\|x_k-x_{k-1}\|+|\alpha_k-\alpha_{k-1}|\|J_{r_{k-1}}(x_{k-1}+v_{k-1})-Af(x_{k-1})\| \\
&\quad +\alpha_k(\|v_k-v_{k-1}\|+\|J_{r_k}(x_{k-1}+v_{k-1})-J_{r_{k-1}}(x_{k-1}+v_{k-1})\|)] \\
&\quad +|\beta_k-\beta_{k-1}|\|y_{k-1}-AJ_{r_{k-1}}(x_{k-1}+u_{k-1})\|+(1-\beta_k\bar{\gamma})(\|x_k-x_{k-1}\| \\
&\quad +\|u_k-u_{k-1}\|+\|J_{r_k}(x_{k-1}+u_{k-1})-J_{r_{k-1}}(x_{k-1}+u_{k-1})\|) \\
&\leq \beta_k[(\alpha-\alpha_k(\bar{\gamma}\alpha-1))\|x_k-x_{k-1}\|+|\alpha_k-\alpha_{k-1}|\|J_{r_{k-1}}(x_{k-1}+v_{k-1})-Af(x_{k-1})\| \\
&\quad +\alpha_k(\|v_k-v_{k-1}\|+M_0|r_{k-1}-r_k|)]+|\beta_k-\beta_{k-1}|\|y_{k-1}-AJ_{r_{k-1}}(x_{k-1}+u_{k-1})\| \\
&\quad +(1-\beta_k\bar{\gamma})\|x_k-x_{k-1}\|+\|u_k-u_{k-1}\|+M_1|r_k-r_{k-1}| \\
&\leq [1-\beta_k(\bar{\gamma}-\alpha-\alpha_k(1-\bar{\gamma}\alpha))]\|x_k-x_{k-1}\|+\beta_k[|\alpha_k-\alpha_{k-1}|M \\
&\quad +\alpha_k(\|v_k-v_{k-1}\|+M_0|r_{k-1}-r_k|)]+\|u_k-u_{k-1}\|+M_1|r_k-r_{k-1}|+|\beta_k-\beta_{k-1}|M \\
&\leq [1-\beta_k(\bar{\gamma}-\alpha-\frac{\bar{\gamma}-\alpha}{2(1-\bar{\gamma}\alpha)}(1-\bar{\gamma}\alpha))]\|x_k-x_{k-1}\|+\beta_k[|\alpha_k-\alpha_{k-1}|M \\
&\quad +\alpha_k(\|v_k-v_{k-1}\|+M_0|r_{k-1}-r_k|)]+\|u_k-u_{k-1}\|+M_1|r_k-r_{k-1}|+|\beta_k-\beta_{k-1}|M \\
&= [1-\frac{1}{2}\beta_k(\bar{\gamma}-\alpha)]\|x_k-x_{k-1}\|+\beta_k[|\alpha_k-\alpha_{k-1}|M+\alpha_k(\|v_k-v_{k-1}\|+M_0|r_{k-1}-r_k|)] \\
&\quad +\|u_k-u_{k-1}\|+M_1|r_k-r_{k-1}|+|\beta_k-\beta_{k-1}|M,
\end{aligned}$$

where $\sup_{k\geq 0}\{\|J_{r_k}(x_k+v_k)-Af(x_k)\|+\|y_k-AJ_{r_k}(x_k+u_k)\|\}\leq M$ for some $M>0$ (it is easy to see that $\{y_k\}$ is bounded due to the boundedness of $\{x_k\}$). Utilizing Lemma 2.1, we conclude from conditions (i)-(iii) that (3.9) holds.

Next let us show that

$$\lim_{k\to\infty}\|x_k - J_{\tilde{r}}x_k\| = 0. \tag{3.13}$$

Indeed, from (3.8), (3.9) and $\lim_{k\to\infty}\beta_k = 0$, it follows that as $k\to\infty$,

$$\begin{aligned}\|x_k - J_{r_k}(x_k+u_k)\| &\le \|x_k - x_{k+1}\| + \|x_{k+1} - J_{r_k}(x_k+u_k)\| \\ &\le \|x_k - x_{k+1}\| + \beta_k\|y_k - AJ_{r_k}(x_k+u_k)\| \to 0,\end{aligned}$$

which together with the relation

$$\begin{aligned}\|x_k - J_{r_k}x_k\| &\le \|x_k - J_{r_k}(x_k+u_k)\| + \|J_{r_k}(x_k+u_k) - J_{r_k}x_k\| \\ &\le \|x_k - J_{r_k}(x_k+u_k)\| + \|(x_k+u_k) - x_k\|\|x_k - J_{r_k}(x_k+u_k)\| + \|u_k\|,\end{aligned}$$

implies that

$$\lim_{k\to\infty}\|x_k - J_{r_k}x_k\| = 0. \tag{3.14}$$

Since $r_k \ge \tilde{r}$ for all $k\ge 1$, utilizing Lemma 2.5 we have

$$\begin{aligned}\|J_{r_k}x_k - J_{\tilde{r}}x_k\| &= \|J_{\tilde{r}}(\frac{\tilde{r}}{r_k}x_k + (1-\frac{\tilde{r}}{r_k})J_{r_k}x_k) - J_{\tilde{r}}x_k\| \\ &\le \|x_k - J_{r_k}x_k\|,\end{aligned}$$

and hence

$$\|x_k - J_{\tilde{r}}x_k\| \le \|x_k - J_{r_k}x_k\| + \|J_{r_k}x_k - J_{\tilde{r}}x_k\| \le 2\|x_k - J_{r_k}x_k\|.$$

Thus, from (3.14) it is easy to see that (3.13) holds.

Let $x_t = (I-\theta_t A)J_{\tilde{r}}x_t + \theta_t[f(x_t) - t(Af(x_t) - J_{\tilde{r}}x_t)]$. According to Lemma 3.1, we know that $\{x_t\}$ converges strongly to $x^*\in \mathrm{Fix}(J_{\tilde{r}}) = \mathscr{A}^{-1}0$, which is the unique solution in $\mathscr{A}^{-1}0$ to the VI:

$$\langle (A-f)x^*, J(x^*-p)\rangle \le 0, \quad \forall p\in \mathscr{A}^{-1}0. \tag{3.15}$$

Further, let us show that

$$\limsup_{k\to\infty}\langle (f-A)x^*, J(x_k - x^*)\rangle \le 0. \tag{3.16}$$

Indeed, take a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

$$\limsup_{k\to\infty}\langle (f-A)x^*, J(x_k - x^*)\rangle = \lim_{i\to\infty}\langle (f-A)x^*, J(x_{k_i} - x^*)\rangle. \tag{3.17}$$

Without loss of generality, we may assume that $x_{k_i} \rightharpoonup \tilde{x}$. Utilizing Lemma 3.3 we obtain from (3.13) that $\tilde{x}\in \mathrm{Fix}(J_{\tilde{r}}) = \mathscr{A}^{-1}0$. Hence from (3.15) and (3.17) we get

$$\limsup_{k\to\infty}\langle (f-A)x^*, J(x_k - x^*)\rangle = \langle (f-A)x^*, J(\tilde{x} - x^*)\rangle \le 0. \tag{3.18}$$

As required, finally let us show that $x_k\to x^*$ as $k\to\infty$.

As a matter of fact, observe that

$$\begin{aligned}
\|x_{k+1}-x^*\|^2 &= \|(I-\beta_k A)J_{r_k}(x_k+u_k)-(I-\beta_k A)J_{r_k}x^*+\beta_k[(I-\alpha_k A)f(x_k)-(I-\alpha_k A)f(x^*)\\
&\quad +\alpha_k(J_{r_k}(x_k+v_k)-x^*)]+\beta_k(f-A)x^*+\beta_k\alpha_k(I-Af)x^*\|^2\\
&\le \|(I-\beta_k A)J_{r_k}(x_k+u_k)-(I-\beta_k A)J_{r_k}x^*+\beta_k[(I-\alpha_k A)f(x_k)-(I-\alpha_k A)f(x^*)\\
&\quad +\alpha_k(J_{r_k}(x_k+v_k)-x^*)]\|^2+2\beta_k\langle(f-A)x^*,J(x_{k+1}-x^*)\rangle\\
&\quad +2\beta_k\alpha_k\langle(I-Af)x^*,J(x_{k+1}-x^*)\rangle\\
&\le [\|(I-\beta_k A)J_{r_k}(x_k+u_k)-(I-\beta_k A)J_{r_k}x^*\|+\beta_k(\|I-\alpha_k A\|\|f(x_k)-f(x^*)\|\\
&\quad +\alpha_k\|J_{r_k}(x_k+v_k)-x^*\|)]^2+2\beta_k\langle(f-A)x^*,J(x_{k+1}-x^*)\rangle\\
&\quad +2\beta_k\alpha_k\|(I-Af)x^*\|\|x_{k+1}-x^*\|\\
&\le \{(1-\beta_k\bar{\gamma})\|J_{r_k}(x_k+u_k)-J_{r_k}x^*\|+\beta_k[(1-\alpha_k\bar{\gamma})\alpha\|x_k-x^*\|+\alpha_k\|(x_k+v_k)-x^*\|]\}^2\\
&\quad +2\beta_k\langle(f-A)x^*,J(x_{k+1}-x^*)\rangle+2\beta_k\alpha_k\|(I-Af)x^*\|\|x_{k+1}-x^*\|\\
&= [1-\beta_k(\bar{\gamma}-\alpha-\alpha_k(1-\bar{\gamma}\alpha)]^2\|x_k-x^*\|^2\\
&\quad +2\beta_k\langle(f-A)x^*,J(x_{k+1}-x^*)\rangle+2\beta_k\alpha_k\|(I-Af)x^*\|\|x_{k+1}-x^*\|\\
&\le [1-\beta_k(\bar{\gamma}-\alpha-\alpha_k(1-\bar{\gamma}\alpha)]\|x_k-x^*\|^2\\
&\quad +2\beta_k\langle(f-A)x^*,J(x_{k+1}-x^*)\rangle+2\beta_k\alpha_k\|(I-Af)x^*\|\|x_{k+1}-x^*\|\\
&\le [1-\beta_k(\bar{\gamma}-\alpha-\frac{\bar{\gamma}-\alpha}{2(1-\bar{\gamma}\alpha)}(1-\bar{\gamma}\alpha)]\|x_k-x^*\|^2\\
&\quad +2\beta_k\langle(f-A)x^*,J(x_{k+1}-x^*)\rangle+2\beta_k\alpha_k\|(I-Af)x^*\|\|x_{k+1}-x^*\|\\
&= [1-\frac{1}{2}\beta_k(\bar{\gamma}-\alpha)]\|x_k-x^*\|^2+2\beta_k\langle(f-A)x^*,J(x_{k+1}-x^*)\rangle\\
&\quad +2\beta_k\alpha_k\|(I-Af)x^*\|\|x_{k+1}-x^*\|\\
&= (1-\mu_k)\|x_k-x^*\|^2+\mu_k\nu_k,
\end{aligned}$$

where $\mu_k=\frac{1}{2}\beta_k(\bar{\gamma}-\alpha)$ and

$$\nu_k=\frac{4(\langle(f-A)x^*,J(x_{k+1}-x^*)\rangle+\alpha_k\|(I-Af)x^*\|\|x_{k+1}-x^*\|)}{\bar{\gamma}-\alpha}.$$

It can be easily seen from (3.16) and condition (i) that

$$\sum_{k=0}^{\infty}\mu_k=\infty \quad \text{and} \quad \limsup_{k\to\infty}\nu_k\le 0.$$

In terms of Lemma 2.1, we infer that $x_k\to x^*$ as $k\to\infty$. □

Acknowledgements

This research was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002)

and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100). This research was partially supported by the grant NSC 99-2115-M-037-002-MY3.

REFERENCES

[1] K. Aoyama, H. Iiduka, W. Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, Fixed Point Theory Appl. 2006 (2006), Article ID 35390.

[2] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007), 2350-2360.

[3] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leiden, 1976.

[4] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967), 197-228.

[5] G. Cai, S.Q. Bu, Convergence analysis for variational inequality problems and fixed point problems in 2-uniformly smooth and uniformly convex Banach spaces, Math. Comput. Modelling, 55 (2012), 538-546.

[6] G. Cai, C.S. Hu, Strong convergence theorems of a general iterative process for a finite family of λ_i pseudocontractions in q-uniformly smooth Banach spaces, Comput. Math. Appl. 59 (2010), 149-160.

[7] G. Cai, S.Q. Bu, Approximation of common fixed points of a countable family of continuous pseudocontractions in a uniformly smooth Banach space, Appl. Math. Lett. 24 (2011), 1998-2004.

[8] L.C. Ceng, J.C. Yao, Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinite many nonexpansive mappings, Appl. Math. Comput. 198 (2008), 729-741.

[9] L.C. Ceng, J.C. Yao, Relaxed viscosity approximation methods for fixed point problems and variational inequality problems, Nonlinear Anal. 69 (2008), 3299-3309.

[10] L.C. Ceng, C.F. Wen, Hybrid extragradient methods for finding zeros of accretive operators and solving variational inequality and fixed point problems in Banach spaces, Abstr. Appl. Anal. 2013 (2013), Article ID 894926.

[11] L.C. Ceng, N.C. Wong, J.C. Yao, Strong and weak convergence theorems for an infinite family of nonexpansive mappings and applications, Fixed Point Theory Appl. 2012 (2012), Article ID 117.

[12] L.C. Ceng, Z.R. Kong, C.F. Wen, On general systems of variational inequalities, Comput. Math. Appl. 66 (2013), 1514-1532.

[13] L.C. Ceng, S.M. Guu, J.C. Yao, Hybrid iterative method for finding common solutions of generalized mixed equilibrium and fixed point problems, Fixed Point Theory Appl. 2012 (2012), Article ID 92.

[14] L.C. Ceng, A. Al-Otaibi, Q.H. Ansari, A. Latif, Relaxed and composite viscosity methods for variational inequalities, fixed points of nonexpansive mappings and zeros of accretive operators, Fixed Point Theory Appl., 2014 (2014), Article ID 29.

[15] K. Deimling, Zeros of accretive operators, Manus. Math. 13 (1974), 365-374.

[16] R. Glowinski, J. L. Lions, R. Tremolieres, Numerical Analysis and Variational Inequalities, North-Holland, Amsterdam, 1981.

[17] J.P. Gossez, E. Lami Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, Pacific J. Math. 40 (1972), 565-573.

[18] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, in: Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, 1990.

[19] M. Kikkawa, W. Takahashi, Strong convergence theorems by the viscosity approximation methods for a countable family of nonexpansive mappings, Taiwan. J. Math. 2 (2008), 583-598.

[20] M. Kikkawa, W. Takahashi, Viscosity approximation methods for countable families of nonexpansive mappings in Hilbert spaces, RIMS Kokyuroku, 148 (2006), 105-113.

[21] N. Kikuchi, J. T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, SIAM, Philadelphia, 1988.

[22] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1965), 1004-1006.

[23] I.V. Konnov, Combined Relaxation Methods for Variational Inequalities, Lecture Notes in economics and Mathematical Systems, Vol. 495, Springer-Verlag, Berlin, Heiderberg, New York, 2001.

[24] G.M. Korpelevich, An extragradient method for finding saddle points and for other problems, Ekonom. Mat. Metody, 12 (1976), 747-756.

[25] A. Moudafi, Viscosity approximation methods for fixed-point problems, J. Math. Anal. Appl. 241 (2000), 46-55.

[26] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 595-597.

[27] X. Qin, S.Y. Cho, L. Wang, Iterative algorithms with errors for zeros points of m-accretive operators, Fixed Point Theory Appl. 2013 (2013), Article ID 148.

[28] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979), 274-276.

[29] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwaness J. Math. 5 (2001), 387-404.

[30] N. Shioji, W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc. 125 (1997), 3641-3645.

[31] Y. Song, R. Chen, Convergence theorems of iterative algorithms for continuous pseudocontractive mappings, Nonlinear Anal. 67 (2007), 486-497.

[32] W. Takahashi, Weak and strong convergence theorems for families of nonexpansive mappings and their applications, Ann. Univ. Mariae Curie-Sklodowska (Sect. A), 51 (1997), 277-292.

[33] W. Takahashi, Nonlinear Functional Analysis-Fixed Point Theory and Its Application, Yokohama Publishers, Yokohama, 2000.

[34] D. van Dulst, Equivalent norms and the fixed point property for nonexpansive mappings, J. London Math. Soc. 25 (1982), 139-144.

[35] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2002), 240-256.

[36] I. Yamada, The hybrid steepest-descent method for the variational inequality problems over the intersection of the fixed-point sets of nonexpansive mappings, in: Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, D. Batnariu, Y. Censor, and S. Reich, eds., Amsterdam, North-Holland, Holland, 2001, pp. 473–504.

[37] L.C. Zeng, J.C. Yao, Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings, Nonlinear Anal. 64 (2006), 2507-2515.

[38] L.C. Zeng, S. Schaible, J.C. Yao, Hybrid steepest descent methods for zeros of nonlinear operators with applications to variational inequalities, J. Optim. Theory Appl. 141 (2009), 75-91.

[39] H. Zhou, L. Wei, Y.J. Cho, Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings in reflexive Banach spaces, Appl. Math. Comput. 173 (2006), 196-212.