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ON SOME FIXED POINT THEOREMS FOR MULTI-VALUED OPERATORS BY ALTERING DISTANCE TECHNIQUE

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Abstract. The aim of this paper is to present some sufficient conditions for the existence and uniqueness of fixed points for $(\varphi - \psi)$ type contractive multi-valued operators defined by altering distances. Furthermore, our main result consists of two theorems, one involving the convergence of the Picard successive approximation sequence to a fixed point of the multivalued $\varphi - \psi$ operator, and a theorem concerning a more general form for a fixed point result for this type of mappings.

Keywords. Fixed point; $(\varphi - \psi)$-contraction; Altering distance; Multivalued operator; Weakly contractive mapping.

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1. Preliminaries

In this section, we will present some preliminary notions and fixed point results for single-valued self-mappings satisfying some altering distance type conditions in a complete metric space.

In [6], Khan, Swaleh and Sessa gave sufficient conditions such that an operator has a unique fixed point. This contractive-type operator satisfy the condition

$$\psi(d(Tx,Ty)) \leq k\psi(d(x,y))$$

for each elements x,y of a complete metric space (X,d), where $\psi : [0,\infty) \to [0,\infty)$ is endowed with the following properties

$$\psi(t) = 0 \text{ if and only if } t = 0,$$
$$\psi \text{ is continuous and nondecreasing.}$$

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Furthermore, Alber and Guerre-Delabriere in [1] gave a different generalization, for mappings satisfying the assumption

$$\begin{aligned}
&d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)), \text{ where}\\
&\varphi : [0,\infty) \to [0,\infty) \text{ is also nondecreasing and continuous}\\
&\varphi(t) = 0 \text{ if and only if } t = 0,\\
&\lim_{t\to\infty} \varphi(t) = \infty.
\end{aligned}$$

Then in [19], Rhoades showed that the last assumption is not necessary for the existence and uniqueness of the fixed points of the above self-mappings. Generalizations for this type of mappings were done by Dutta *et al.* in [4] for self-mappings f defined on a complete metric space (X,d), satisfying

$$\begin{aligned}
&\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)), \text{ where}\\
&\psi,\varphi : [0,\infty) \to [0,\infty) \text{ are both nondecreasing and continuous functions,}\\
&\psi(t) = \varphi(t) = 0 \text{ if and only if } t = 0.
\end{aligned}$$

A very interesting approach was done by Amini-Harandi and Petruşel in [2], where the authors studied sufficient conditions for the existence and uniqueness of the fixed points for an operator T satisfing the following assumption

$$u(d(Tx,Ty)) \leq v(d(x,y)),$$

where the self-mappings u and v defined on $[0,\infty)$ satisfy some relaxed conditions. The authors also gave some interesting corollaries showing that their theorem is a real generalization of the already presented type of mappings.

Moreover, regarding the weakly contractive condition for a single-valued operator, Rhoades *et al.* [20] presented other types of generalizations in the framework of partially ordered metric spaces. As an example for this type of mappings we suppose that

$$d(fx,fy) \leq \varphi(d(x,y)) - \psi(d(x,y)),$$

where the operators ψ,φ have the following properties:

i) φ,ψ are both positive on $(0,\infty)$ with $\psi(0) = \varphi(0)$,

ii) $\varphi(t) - \psi(t) < t$,

iii) φ upper semicontinuous and nondecreasing,

iv) ψ lower semicontinuous and nonincreasing.

Last, but not least, for fixed point results involving other generalizations of the weakly contractive condition for single-valued mappings we refer to [10], [17] and [23].

Concerning the case of multi-valued operators, T. Lazar *et al.* [8] presented an exhaustive study of some qualitative properties concerning Reich type multi-valued operators. Moreover, V. Lazăr [7] extended the results concerning the case of multi-valued φ-type contractions.

T.P. Petru and M. Boriceanu [12] gave some fixed point results for φ-contractions in a set endowed with two metrics.

In all the articles [8], [7] and [12], the comparison function φ used for the case of φ-contractions satisfy the following properties:

i) $\varphi(0) = 0$,

ii) $\varphi(t) > 0$ for $t > 0$,

iii) $\varphi^k(t) \to 0,$ for each $t > 0$ for $k \to \infty$.

Notice that φ is not necessarily continuous on $[0,\infty)$, but in [12] the continuity of the mapping was additionally assumed. Furthermore, an important property of comparison functions is the fact that $\varphi(t) < t,$ for each $t > 0$.

As a conclusion, there are two distinct classes of mappings involved in the generalizations of contractive-type operators. There are comparison functions, on one hand, in many cases denoted by φ. On the other, there is the case of altering distance functions for which the most important conditions are continuity or semicontinuity properties and a certain monotonicity. We also notice that the weakly contractive mappings used in [20] are combination of these types of self-mappings.

Regarding the case of multi-valued operators, in 2011, Kamran and Kiran [5] presented some rezults involving altering distance type functionals. In this article, more precisely, in [Theorem 4.2] in [5], a special type of altering distance function denoted by θ was used. This mapping satisfies the following conditions on an interval $[0,A)$, where A is real number strictly greater than 0, i.e.,

$$(i)\ \theta \text{ is nondecreasing on } [0,A),$$

$$(ii)\ \theta(t) > 0, \text{ for each } t \in (0,A),$$

$$(iii)\ \theta \text{ subadditive on } (0,A) \text{ and}$$

$$(iv)\ \theta(at) \leq a\theta(t), \text{ for each } a > 0 \text{ and } t \in [0,A).$$

Also, in 2012, Liu *et al.* [9] gave a similar theorem, namely [Theorem 2.3], were the functional φ is similar to the functional α from [5]. From the same article we observe that the conditions put upon the altering distance mapping θ are somewhat different. For the sake of completeness, we recall them here

$$(a)\ \theta \text{ is nondecreasing on } \mathbb{R}^+,$$

$$(b)\ \theta(t) > 0, \text{ when } t \in (0,\infty),$$

$$(c)\ \theta \text{ is subadditive on } (0,\infty),$$

$$(d)\ \theta(\mathbb{R}^+) = \mathbb{R}^+ \text{ and}$$

$$(e)\ \theta \text{ is strictly inverse on } \mathbb{R}^+.$$

Finally, concerning weakly contractive $(\varphi - \psi)$ contractive type multivalued operators, G. Petruşel et.al. in [16] presented a fixed point result for this kind of operators in the context of complete ordered b-metric spaces with coefficient $s \geq 1$, along with some theorems involving coupled fixed points. From [Theorem 2.2] of [16], a self multivalued operator $T : X \to P_{cl}(X)$ was defined by a contractive-type inequality, i.e.

$$\varphi(H(Tx,Ty)) \leq \varphi(d(x,y)) - \psi(d(x,y))$$

and sufficient conditions for the existence of fixed points for this kind of operators were studied. Here, the altering distance function $\varphi : [0,\infty) \to [0,\infty)$ satisfy the following

$$(i_\varphi)\ \varphi \text{ continuous and strictly increasing,}$$
$$(ii_\varphi)\ \varphi(t) < t,\ \text{for each } t > 0,$$
$$(iii_\varphi)\ \varphi(a+b) \le \varphi(a) + b,\ \text{for } a,b \in [0,\infty),$$
$$(iv_\varphi)\ \varphi(st) \le s\varphi(t),\ \text{for each } t \in [0,\infty).$$

Also, the other altering distance function $\psi : [0,\infty) \to [0,\infty)$ satisfy

$$(i_\psi)\ \limsup_{t\to r} \psi(t) > 0,\ \text{for all } r > 0 \text{ and}$$
$$(ii_\psi)\ \lim_{t\to 0^+} \psi(t) = 0.$$

Moreover, we recognize that in contrast to the original usage of altering functions as in [4], the conditions from [16] on ψ were relaxed and the conditions on the mapping φ were made more restrictive, since the condition (ii_φ) is a comparison type condition. So, in this sense, these weakly contractive-type self-mappings are a combination of altering distances and comparison functions as the operators defined in [20].

Finally, in [3] the authors presented some fixed point theorems for multivalued operators and in [18] Popescu et.al. extended these types of comparison based multivalued operators, i.e. $T : X \to P_{b,cl}(X)$, such that $H(Tx,Ty) \le \varphi(d(x,y))$, for each x,y from the complete metric space (X,d). Here, the mapping φ satisfies the following assumptions

$$(1)\ \varphi(x) \le x,\ \text{for each } x \in [0,\infty),$$
$$(2)\ \varphi(x+y) \le \varphi(x) + \varphi(y),\ \text{for all } x,y \in [0,\infty),$$
$$(3)\ \varphi(x) = 0 \text{ if and only if } x = 0 \text{ and}$$
$$(4)\ \text{for any } \varepsilon > 0,\ \text{there exists } \delta > \varepsilon \text{ such that } \varepsilon < t < \delta \text{ implies } \varphi(t) \le \varepsilon.$$

Furthermore, condition (4) defined for the above mapping φ can be considered as a local type comparison function. Also, the authors in [18] extended the results of the authors of [3] from the case of hyperconvex metric spaces to the usual metric spaces. Furthermore, since they worked in a less restrictive framework, they have put an important assumption regarding the well known diameter functional and so they used the condition that the multivalued operator has bounded values.

In the last part of this section, fundamental notions and concepts for the fixed point theory of multi-valued operators are used. For a general perspective regarding these terminologies for multi-valued mappings, we refer to [7], [11], [13], [15], [21] and [22].

Let X be a nonempty set. First of all, we shall use the following class of sets
$\mathscr{P}(X) := \{Y / Y \subset X\}, \qquad P(X) := \{Y \in \mathscr{P}(X) / Y \neq \emptyset\}$.
Now, for the case when (X,d) is a metric space, we recall that $Y \subseteq X$ is bounded if and only if $\delta(Y) := sup\{d(a,b)/a,b \in Y\} < +\infty$, where δ is the usual diameter functional on $[0,\infty)$.

Now, we define the following families of sets:
$P_b(X) := \{Y \in P(X)/Y \text{ bounded }\}$, $\quad P_{cl}(X) := \{Y \in P(X)/Y \text{ closed }\}$,
$P_{cp}(X) := \{Y \in P(X)/Y \text{ compact }\}$, $\quad P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$.

Moreover, if $Y \subseteq X$ and $T : Y \to P(X)$ is a multi-valued operator, then we define the following useful symbols. $F_T := \{x \in Y/x \in Tx\}$ is the fixed point set of the operator T, $(SF)_T := \{x \in Y/\{x\} = Tx\}$ denotes the strict fixed point set of the multi-valued operator T and by $Graph(T) := \{(x,y) \in Y \times X/y \in Tx\}$ we denote the graph of the multivalued operator T. Also, if $T : X \to P(X)$, then by $T^0 := 1_X$, $T^1 := T$, ..., $T^{n+1} := T \circ T^n$, for $n \in \mathbb{N}$ we denote the iterates of the operator T, where $T(A) := \bigcup_{a\in A} Ta$, for $A \subset X$.

Furthermore, we shall frequently use the following generalized functionals in the next section, so we shall recall them.

The gap functional $D : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as

$$D(A,B) = \begin{cases} inf\{d(a,b)/a \in A, b \in B\}, & A \neq \emptyset \text{ and } B \neq \emptyset, \\ 0, & A = \emptyset \text{ and } B = \emptyset, \\ +\infty, & \text{otherwise.} \end{cases}$$

The excess generalized functional $\rho : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as

$$\rho(A,B) = \begin{cases} sup\{D(a,B)/a \in A\}, & A \neq \emptyset \text{ and } B \neq \emptyset, \\ 0, & A = \emptyset, \\ +\infty, & A \neq \emptyset \text{ and } B = \emptyset. \end{cases}$$

The Hausdorff-Pompeiu generalized functional $H : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as

$$H(A,B) = \begin{cases} max\{\rho(A,B), \rho(B,A)\}, & A \neq \emptyset \text{ and } B \neq \emptyset, \\ 0, & A = \emptyset \text{ and } B = \emptyset, \\ +\infty, & \text{otherwise.} \end{cases}$$

For the sake of completeness, we recall now a very useful concept in fixed point theory for multi-valued operators. This is a basic concept known by the name of multivalued weakly Picard operator. It is defined as follows.

Definition 1.1. Let (X,d) be a metric space and $T : X \to P(X)$ is called a multivalued weakly Picard operator, briefly a MWP operator, if for each $x \in X$ and for each $y \in Tx$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \in X$, such that:
(i) $x_0 = x$ and $x_1 = y$,
(ii) $x_{n+1} \in Tx_n$, for each $n \in \mathbb{N}$ and
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point of T.

Finally, a sequence $(x_n)_{n\in\mathbb{N}}$ defined by the properties (i) and (ii) is called a sequence of successive approximations.

2. Main results

The first main result of this section concerns the fixed points of $(\varphi - \psi)$ multivalued operators. Let us mention first that, in [16], G. Petruşel et.al. presented some fixed point results for this type of multivalued operators on complete ordered b-metric spaces. In the same article, the authors constructed a sequence of successive approximations and showed that this sequence is convergent to a fixed point of the multivalued operator. The proof of [Theorem 2.2] in [16] contains a certain gap. The authors consider an $\tilde{\varepsilon}$ at each step and then, using the reductio ad absurdum argument, chose $\tilde{\varepsilon}$, such that $\tilde{\varepsilon} < \lim\limits_{n\to\infty} \psi(\delta_n)$. Since $\tilde{\varepsilon}$ was already constructed at each step (as $\tilde{\varepsilon_n}$) and the sequence δ_n was not given there, this technique is not valid. Furthermore, trying to show that the sequence (x_n) is Cauchy, the authors used the fact that $d(x_{m(k)+1}, x_{n(k)+1}) \le H(T(x_{m(k)}), T(x_{n(k)})) + \tilde{\varepsilon}$, which, by the well known lemma of Nadler, is not necessarily true.

Our first purpose is to correct these arguments, by imposing the additional assumption that $\delta(T^n x) \to 0$ as $n \to \infty$. Our assumption is inspired by an idea from [3] and [18]. Based on this assumption and using the technique from [16], we give a fixed point theorem for $(\varphi - \psi)$ multivalued operators. At the same time, we relax some conditions on the altering distance functions (such as continuity) and we get rid off the property of comparison functions, i.e. $\varphi(t) < t$.

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X \to P_{b,cl}(X)$ a multivalued operator, that satisfies the following*

$$\varphi(H(Tx,Ty)) \le \varphi(d(x,y)) - \psi(d(x,y)), \text{ for each } x,y \in X,$$

where the mappings φ and ψ satisfy

$(H1)$ $\varphi, \psi : [0,\infty) \to [0,\infty)$,

$(H2)$ φ *is usc and* ψ *is lsc*

$(H3)$ $\varphi(0) = \psi(0) = 0$ *and* $\varphi(t), \psi(t) > 0$, *for each* $t > 0$,

$(H4)$ φ *is (strictly) increasing* ,

$(H5)$ $\varphi(a+b) \le \varphi(a) + b$, *for each* $a > 0$ *and* $b \ge 0$.

Also, suppose that $\delta(T^n x) \to 0$ as $n \to \infty$, for each $x \in X$. Then, the multi-valued operator T has at least one fixed point $x^* \in F_T$. Moreover, if $(SF)_T \neq \emptyset$, then $F_T = (SF)_T = \{x^*\}$.

Proof. (i) Let $x_0 \in X$ be arbitrary taken and let $x_1 \in Tx_0$. Also, consider the sequence $(\delta_n)_{n\in\mathbb{N}}$, with $\delta_n > 0$, for each $n \in \mathbb{N}$, with $\lim\limits_{n\to\infty} \delta_n = 0$. Define $\varepsilon_1 \le min\{\delta_1, t_1\}$, where

$$t_1 := \varphi(d(x_0,x_1)) - \varphi(H(Tx_0,Tx_1)).$$

Now, t_1 is well defined, because x_0 and x_1 were already constructed. Furthermore, we can suppose that $x_0 \neq x_1$, because, if x_0 and x_1 were identical, then $x_0 \in Tx_0$, so x_0 was a fixed point for the operator T. Now, for $x_1 \in Tx_0$ and ε_1, there exists $x_2 \in Tx_1$, such that

$$d(x_2,x_1) \le H(Tx_1,Tx_0) + \varepsilon_1.$$

By induction, we can create the sequence $(x_n)_{n\in\mathbb{N}}$ as follows.

Let $x_{n-1} \in Tx_{n-2}$ and $x_n \in Tx_{n-1}$ already constructed. For $x_n \in Tx_{n-1}$ and $\varepsilon_n > 0$, there exists $x_{n+1} \in Tx_n$, such that

$$\rho_n := d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + \varepsilon_n.$$

So, we obtain that

$$\varphi(\rho_n) < \varphi(\rho_{n-1}) - \psi(\rho_{n-1}) + \varepsilon_n.$$

Also, we use ε_n, such that $\varepsilon_n \leq \min\{\delta_n, t_n\}$, with

$$t_n := \varphi(\rho_{n-1}) - \varphi(H(Tx_{n-1}, Tx_n)).$$

Now, if there exists $n \in \mathbb{N}$, such that $x_{n-1} = x_n$, then $x_{n-1} \in Tx_{n-1}$, i.e. $x_{n-1} \in F_T$ and the proof is over. So, we can suppose that $x_{n-1} \neq x_n$, for each $n \in \mathbb{N}$, so $\rho_n > 0$, for each $n \in \mathbb{N}$. Furthermore, at each step we can suppose that $H(Tx_{n-1}, Tx_n) \neq 0$. Assuming the contrary, we obtain that $Tx_{n-1} = Tx_n$, so $x_n \in F_T$ and the proof is over. In this way, we applied in a valid way assumption (H5). We know that

$$\varphi(H(Tx_{n-1}, Tx_n)) \leq \varphi(d(x_{n-1}, x_n)) - \psi(d(x_{n-1}, x_n)) < \varphi(d(x_{n-1}, x_n)).$$

It follows that

$$\varphi(H(Tx_{n-1}, Tx_n)) \leq \varphi(\rho_{n-1}) - \psi(\rho_{n-1}) < \varphi(\rho_{n-1}).$$

The last inequality is strict, because if it was not a strict inequality, then $\psi(\rho_{n-1}) = 0$, that lead to $\rho_{n-1} = 0$, which is false. So, defining t_n as above, it follows that $t_n > 0$, for each $n \in \mathbb{N}$. Furthermore,

$$\varphi(\rho_n) < \varphi(H(Tx_{n-1}, Tx_n)) + t_n,$$

which lead to the fact that $\varphi(\rho_n) < \varphi(\rho_{n-1})$, which implies $\varphi(\rho_n) \leq \varphi(\rho_{n-1})$, for each $n \in \mathbb{N}$. It follows that sequence $(\varphi(\rho_n))_{n\in\mathbb{N}}$ is decreasing and bounded below by 0, because φ takes values in $[0, \infty)$. So, we get that there exists $\tau \geq 0$, such that $\tau = \lim_{n\to\infty} \varphi(\rho_n)$. Furthermore, since $\varphi(\rho_n) \leq \varphi(\rho_{n-1})$, for each $n \in \mathbb{N}$, we obtain that $\rho_n \leq \rho_{n-1}$. So $(\rho_n)_{n\in\mathbb{N}}$ is a decreasing sequence and is also bounded below by 0. Hence there exists $\rho^* \geq 0$, such that $\rho^* = \lim_{n\to\infty} \rho_n$. We have that

$$\varphi(\rho_n) \leq \varphi(\rho_{n-1}) - \psi(\rho_{n-1}) + \varepsilon_n,$$

which implies that $\varphi(\rho_n) \leq \varphi(\rho_{n-1}) - \psi(\rho_{n-1}) + \delta_n$. Taking $\limsup_{n\to\infty}$, one sees that

$$\begin{aligned}
&\limsup_{n\to\infty} \varphi(\rho_n) \leq \limsup_{n\to\infty} \varphi(\rho_{n-1}) + \limsup_{n\to\infty} [-\psi(\rho_{n-1})] + \limsup_{n\to\infty} \delta_n \\
&\lim_{n\to\infty} \varphi(\rho_n) \leq \lim_{n\to\infty} \varphi(\rho_{n-1}) - \liminf_{n\to\infty} \psi(\rho_{n-1}) + \lim_{n\to\infty} \delta_n \\
&\tau \leq \tau - \liminf_{n\to\infty} \psi(\rho_{n-1}) \\
&\liminf_{\rho_{n-1}\to\rho^*} \psi(\rho_{n-1}) = 0 \Rightarrow \rho^* = 0, \text{ i.e. } \rho_n \to 0 \text{ as } n \to \infty.
\end{aligned}$$

In this way, $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular, namely a pseudo-Cauchy sequence.

The next step is to show that the Picard sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let us suppose the contrary that the sequence $(x_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$, there exists $n_k, m_k > k$, such that $n_k > m_k > k$, where n_k is the lowest element with property that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } d(x_{m_k}, x_{n_k-1}) < \varepsilon$$

Define $\tilde{\varepsilon} > 0$, such that $\tilde{\varepsilon} < \psi(\varepsilon)$. Also, from Nadler's lemma, we know that, for $\tilde{\varepsilon}$ and $x_{m_k} \in Tx_{m_k-1}$, there exists $v \in Tx_{n_k-1}$, such that

$$d(x_{m_k}, v) \leq H(Tx_{m_k-1}, Tx_{n_k-1}) + \tilde{\varepsilon}.$$

Without loss of generality, we may assume that $v \neq x_{n_k}$, with both v and $x_{n_k} \in Tx_{n_k-1}$. Applying functional φ, one obtains that

$$\varphi(d(x_{m_k}, v)) < \varphi(H(Tx_{m_k-1}, Tx_{n_k-1})) + \tilde{\varepsilon},$$
$$\varphi(d(x_{m_k}, v)) < \varphi(d(x_{m_k-1}, x_{n_k-1})) - \psi(d(x_{m_k-1}, x_{n_k-1})) + \tilde{\varepsilon}.$$

Also, we know that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, so $\varphi(d(x_{m_k}, x_{n_k})) \geq \varphi(\varepsilon)$.

Now, since we used assumption (H5), we supposed that $H(Tx_{m_k-1}, Tx_{n_k-1}) \neq 0$. Assuming the contrary, we obtain that $Tx_{m_k-1} = Tx_{n_k-1}$, which shows that x_{m_k}, x_{n_k} are both in Tx_{n_k-1}. Then, we have that

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq \delta(Tx_{n_k-1}) \leq \delta(T^{n_k}x_0) \to 0,$$

as $k \to \infty$ and we get a contradiction. The usage of diameter functional δ is also explained below, in a more detailed manner. Applying the triangle inequality, we have that

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, v) + d(v, x_{n_k})$$
$$\varphi(d(x_{m_k}, x_{n_k})) < \varphi(d(x_{m_k}, v)) + d(v, x_{n_k})$$
$$\Rightarrow \varphi(\varepsilon) < \varphi(d(x_{m_k-1}, x_{n_k-1})) - \psi(d(x_{m_k-1}, x_{n_k-1})) + \varepsilon d(v, x_{n_k}) + \tilde{\varepsilon}.$$

Now, we make the following remark : using (H5), we have assumed that $d(x_{m_k}, v) \neq 0$. Assuming the contrary, we let $d(x_{m_k}, v) = 0$. So, $x_{m_k} = v \in Tx_{n_k-1}$. Then x_{m_k} and $x_{n_k} \in Tx_{n_k-1}$. At the same time, it follows that

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq \delta(Tx_{n_k-1}) \leq \delta(T^{n_k}x_0) \to 0$$

as $k \to \infty$. Hence, we get a contradiction. Since v and x_{n_k} are in Tx_{n_k-1}, one finds that $d(v, x_{n_k}) \leq \delta(Tx_{n_k-1})$. It follows that

$$\varphi(\varepsilon) < \varphi(d(x_{m_k-1}, x_{n_k-1})) - \psi(d(x_{m_k-1}, x_{n_k-1})) + \tilde{\varepsilon} + \delta(Tx_{n_k-1}).$$

Since we want to use the assumption regarding the diameter functional δ, we make the following remark, that for each $n \in \mathbb{N}$, $Tx_n \subseteq T^{n+1}x_0$.

For $n = 0$, we have $Tx_0 \subseteq T^1x_0 = Tx_0$, which is valid.

For $n = 1$, it follows that $Tx_1 \subseteq T^2x_0 = (T \circ T)x_0 = T(Tx_0) = T(A) = \bigcup_{x\in A} Tx = \bigcup_{x\in Tx_0} Tx$, where $A := Tx_0$. Since $x_1 \in Tx_0$ and $Tx_1 \subseteq Tx_1$, the the affirmation is valid also for $n = 1$. By induction, one can easily show that, for each $n \in \mathbb{N}$, $Tx_n \subseteq T^{n+1}x_0$. Now, using the above property, it follows that $\delta(Tx_n) \leq$

$\delta(T^{n+1}x_0) \to 0$ as $n \to \infty$, i.e. $\lim_{n\to\infty} \delta(Tx_n) = 0$. Hence, we obtain that $\lim_{k\to\infty} \delta(Tx_{n_k}) = 0$, because $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. As in [2], by an easy argument based on triangle inequality, one can show that $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$. So

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \varepsilon,$$

since $d(x_{m_k-1}, x_{n_k-1})_{k\in\mathbb{N}}$ is a subsequence of $d(x_{m_k}, x_{n_k})_{k\in\mathbb{N}}$. By hypothesis (H2), i.e. φ is usc and ψ is lsc, it follows the following chain of inequalities

$$\varphi(\varepsilon) \le \limsup_{k\to\infty} \varphi(d(x_{m_k-1}, x_{n_k-1})) + \limsup_{k\to\infty} [-\psi(d(x_{m_k-1}, x_{n_k-1}))] + \tilde{\varepsilon} + \lim_{k\to\infty} \delta(Tx_{n_k-1})$$

$$\varphi(\varepsilon) \le \limsup_{k\to\infty} \varphi(d(x_{m_k-1}, x_{n_k-1})) - \liminf_{k\to\infty} \psi(d(x_{m_k-1}, x_{n_k-1})) + \tilde{\varepsilon}$$

$$\varphi(\varepsilon) \le \limsup_{d(x_{m_k-1}, x_{n_k-1})\to\varepsilon} \varphi(d(x_{m_k-1}, x_{n_k-1})) - \liminf_{d(x_{m_k-1}, x_{n_k-1})\to\varepsilon} \psi(d(x_{m_k-1}, x_{n_k-1})) + \tilde{\varepsilon}$$

$$\varphi(\varepsilon) \le \varphi(\varepsilon) - \psi(\varepsilon) + \tilde{\varepsilon} \Rightarrow \psi(\varepsilon) \le \tilde{\varepsilon}.$$

Since $\tilde{\varepsilon} < \psi(\varepsilon)$, we obtain a contradiction. S $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Because (X,d) is complete, one sees that there exists $x^* \in X$, such that $\lim_{n\to\infty} x_n = x^*$.

The next step is to show that $x^* \in F_T$. Using triangle inequality and the fact that the gap functional is nonexpansive (see [14]), one sees that

$$D(x^*, Tx^*) \le d(x^*, x_n) + D(x_n, Tx^*) \le d(x^*, x_n) + H(Tx_{n-1}, Tx^*) \Rightarrow$$

$$\varphi(D(x^*, Tx^*)) < d(x^*, x_n) + \varphi(H(Tx_{n-1}, Tx^*))$$

$$\varphi(D(x^*, Tx^*)) < d(x^*, x_n) + \varphi(d(x_{n-1}, x^*)) - \psi(d(x_{n-1}, x^*))$$

$$\varphi(D(x^*, Tx^*)) \le \lim_{n\to\infty} d(x^*, x_n) + \limsup_{d(x_{n-1}, x^*)\to 0} \varphi(d(x_{n-1}, x^*)) - \liminf_{d(x_{n-1}, x^*)\to 0} \psi(d(x_{n-1}, x^*)),$$

where we have taken $\limsup_{n\to\infty}$. It follows that

$$\varphi(D(x^*, Tx^*)) \le \varphi(0) - \psi(0) = 0.$$

So $\varphi(D(x^*, Tx^*)) = 0$, which implies that $D(x^*, Tx^*) = 0$. Because T has closed values, it follows that $x^* \in F_T$.

Finally, we make the observation that in the chain of inequalities from above we have used the fact that $H(Tx_{n-1}, Tx^*) \neq 0$. If $n \in \mathbb{N}$ for which $H(Tx_{n-1}, Tx^*) = 0$, then we can use just the simple inequality $D(x^*, Tx^*) \le d(x^*, x_n)$, because $D(x_n, Tx^*) \le H(Tx_{n-1}, Tx^*) = 0$ and the proof remains the same.

(ii) Now, for the second part of the proof, let's suppose that $(SF)_T \neq \emptyset$. So, there exists $y^* \in (SF)_T$. Furthermore, let's suppose that there exists $y \in (SF)_T$, such that $y \neq y^*$. Then, it follows that

$$d(y, y^*) = H(Ty, Ty^*) \Rightarrow$$

$$\varphi(d(y, y^*)) = \varphi(H(Ty, Ty^*)) \Rightarrow$$

$$\varphi(y, y^*) \le \varphi(d(y, y^*)) - \psi(d(y, y^*)), \text{ so } \psi(d(y, y^*)) \le 0.$$

It follows that $d(y,y^*) = 0$. So, we obtained a contradiction, i.e. $(SF)_T = \{y^*\}$. Also, we know that $(SF)_T \subseteq F_T$. At this time we can show that $F_T \subseteq (SF)_T$, i.e. $F_T = \{y^*\}$. Let $x^* \in F_T$. Suppose the contrary, i.e. $x^* \neq y^*$. Then $d(x^*,y^*) > 0$, so $\psi(d(x^*,y^*)) > 0$. We have that

$$d(x^*,y^*) = D(x^*,Tx^*) \leq H(Tx^*,Ty^*).$$

So

$$\varphi(d(x^*,y^*)) \leq \varphi(H(Tx^*,Ty^*)) \leq \varphi(d(x^*,y^*)) - \psi(d(x^*,y^*)).$$

Hence, $\psi(d(x^*,y^*)) \leq 0$, which is false. Then, the conclusion holds properly. □

Remark 2.1. In [5], the authors have used the fact that $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing, so it implies that the functional φ is strictly inverse isotone on $[0,\infty)$, i.e.

$$\varphi(t_1) < \varphi(t_2) \Rightarrow t_1 < t_2, \text{ where } t_1,t_2 \in [0,\infty).$$

In our case, the function is (strictly) increasing. This means that φ is indeed strictly increasing, so $t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$. By logical transposition, it follows that

$$\varphi(t_2) \leq \varphi(t_1) \Rightarrow t_2 \leq t_1.$$

Furthermore, we have used in the previous proof that for $\varphi(t_1) < \varphi(t_2) \Rightarrow t_1 \leq t_2$. This is equivalent to the following fact : if $t_1 > t_2 \Rightarrow \varphi(t_1) \geq \varphi(t_2)$. By the fact that φ is strictly increasing, we get that the last inequality is a particular case, i.e. a strict inequality. Also, some parts from the above proof, we have used that φ is only increasing. Now, since φ is strictly increasing, then this functional is one-to-one, so $\varphi(t_1) = \varphi(t_2) \Rightarrow t_1 = t_2$. Combining this with the fact that φ is strictly increasing, we have simplified the proof by taking $\varphi(t_1) \leq \varphi(t_2) \Rightarrow t_1 \leq t_2$. Also, since φ is strictly increasing we get that $t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$. Moreover, for $t_1 = t_2$, it follows that $\varphi(t_1) = \varphi(t_2)$ from the definition of a regular function. Combining both of these, it follows that for $t_1 \leq t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$.

Remark 2.2. Instead of using hypothesis (H5), similar theorems can be constructed using the assumption that φ is subadditive, i.e. $\varphi(t_1+t_2) \leq \varphi(t_1)+\varphi(t_2)$, for each $t_1,t_2 \geq 0$. We can manage this assumption through the following hypothesis.

(a) If we put the additional assumption such that φ is onto and using the fact that

$$\rho_n \leq H(Tx_{n-1},Tx_n) + \varepsilon',$$

we get the following: denote by $M_n := \min\{t_n,\delta_n\}$. Then, for $\frac{M_n}{2}$, there exists ε'_n, such that

$$\varphi(\varepsilon'_n) = \frac{M_n}{2} \leq M_n.$$

Also, since if φ is strictly increasing then it is one-to-one. We can get rid of this restrictive assumption, by remaking the proof just in the case when φ is only increasing.

(b) Another method when φ is only increasing and subadditive is similar to the proof of [9]. For example, using the assumptions available in [9], we can construct the sequence $(x_n)_{n\in\mathbb{N}}$ as follows : for $x_1 \in Tx_0$, there exists $x_2 \in Tx_1$, with

$$\varphi(d(x_1,x_2)) \leq q_1\varphi(D(x_1,Tx_1)).$$

Also, we can take

$$q_1 := \frac{\varepsilon_1 + D(x_1, Tx_1)}{D(x_1, Tx_1)}$$

when $\varepsilon_1 \leq min\{\delta_1, t_1\}$, and so on.

Remark 2.3. An important observation is that in assumption (H5) of the previous theorem we have that $\varphi(a+b) \leq \varphi(a) + b$, for each $b \geq 0$ and $a > 0$. If the assumption was defined for each $a \geq 0$, taking $a = 0$, it follows that $\varphi(a) \leq a$. So, we get the property of comparison functions. In this case, if (H5) seems restrictive, one can use the previous remark.

In what follows, we will present the second main result of this article. The next theorem gives a fixed point result concerning $(\varphi - \psi)$ multivalued operators. In fact, the next theorem is an improvement of the previous one, where an assumption involving the diameter functional δ is imposed. Using the technique introduced in [18], we can relax some hypothesis of this theorem. Finally, we show that, in addition to the above mentioned conclusions, we can obtain the uniqueness of the fixed point for the multi-valued operator.

Before the second main result of this paper, we present an important remark that shall be used further. The technique is based on the approach given in [18].

Remark 2.4. Let φ be a functional endowed with properties (H1), (H3), (H4) and (H5) from the previous theorem. Also, consider a sequence $(d_n)_{n\in\mathbb{N}}$, such that $d_n \geq d$ and with $\lim_{n\to\infty} d_n = d$. Then, $d_n - d \geq 0$. It follows that

$$\varphi(d_n) = \varphi(d + d_n - d) \leq \varphi(d) + [d_n - d],$$

which is valid since $d > 0$ and $d_n - d \geq 0$. It follows that

$$\varphi(d_n) \leq \varphi(d) + (d_n - d).$$

Taking the upper limit, we have that $\limsup_{n\to\infty} \varphi(d_n) \leq \varphi(d)$. We remark that if φ was usc, then $\limsup_{n\to\infty} \varphi(d_n) \leq \varphi(d)$. So, because of the assumptions on φ, it follows that the above property is more relaxed, in the sense that the functional φ must satisfy the usc condition only for the case when $d_n \geq d$.

Now, we are in a position to give the second fixed point theorem.

Theorem 2.2. *Let (X,d) be a complete metric space and $T : X \to P_{b,cl}(X)$ be a multivalued operator satisfying the following assumption*

$$\varphi(H(Tx,Ty)) \leq \varphi(d(x,y)) - \psi(d(x,y)), \text{ for each } x,y \in X,$$

where the mappings φ and ψ satisfy

$(H1)$ $\varphi, \psi : [0,\infty) \to [0,\infty)$,

$(H2)$ φ *is usc in 0 and* ψ *is lsc in* 0,

$(H3)$ $\varphi(0) = \psi(0) = 0$ *and* $\varphi(t), \psi(t) > 0$, *for each* $t > 0$,

$(H4)$ φ *is (strictly) increasing* ,

$(H5)$ $\varphi(a+b) \leq \varphi(a) + b$, *for each* $a > 0$ *and* $b \geq 0$.

We also suppose that $\delta(T^n x) \to 0$ *as* $n \to \infty$, *for each* $x \in X$. *Then, multivalued operator* T *has a unique fixed point.*

Proof. (i) Let $x_0 \in X$ and $x_1 \in Tx_0$ be arbitrary taken. Also, consider the sequence $(\delta_n)_{n\in\mathbb{N}}$, such that $\delta_n > 0$, satisfying $\lim_{n\to\infty} \delta_n = 0$. As in the proof of the previous theorem, we construct inductively the sequence $(x_n)_{n\in\mathbb{N}}$, as follows. For $\varepsilon_n > 0$ and $x_n \in Tx_{n-1}$, there exists $x_{n+1} \in Tx_n$, such that

$$d(x_n, x_{n+1}) < \varepsilon_n + D(x_n, Tx_n),$$

where $\varepsilon \leq min\{\delta_n, t_n\}$, with t_n defined as in the previous theorem, for each $n \in \mathbb{N}$. Also, denoting $d(x_n, x_{n+1})$ by ρ_n, it follows that

$$\begin{aligned}
&\rho_n < \varepsilon_n + D(x_n, Tx_n)\\
&\rho_n < \delta_n + D(x_n, Tx_n), \text{ for each } n \in \mathbb{N} \Rightarrow\\
&\rho_n \leq \delta_n + d(x_n, x_{n+1}).
\end{aligned}$$

Now,

$$\rho_n < \varepsilon + D(x_n, Tx_n) \leq \varepsilon_n + H(Tx_{n-1}, Tx_n).$$

So, as in the proof of the previous theorem, there exists $\rho^* = \lim_{n\to\infty} \rho_n$, with $\rho^* \geq 0$. At the same time, there exists $\tau = \lim_{n\to\infty} \varphi(\rho_n)$, where $\tau \geq 0$. It follows that

$$\lim_{n\to\infty} \rho_n \leq \lim_{n\to\infty} \delta_n + \lim_{n\to\infty} D(x_n, Tx_n) \leq \lim_{n\to\infty} \delta_n + \lim_{n\to\infty} \rho_n.$$

So, we have that

$$\rho^* \leq \lim_{n\to\infty} D(x_n, Tx_n) \leq \rho^*.$$

This means that $\lim_{n\to\infty} D_n = \rho^*$, where $D_n := D(x_n, Tx_n)$, for each $n \in \mathbb{N}$.

The next step of this proof is to show that $\rho^* = 0$. Moreover, it follows that

$$\begin{aligned}
&D_{n+1} = D(x_{n+1}, Tx_{n+1}) \leq H(Tx_{n-1}, Tx_n)\\
&\varphi(D_{n+1}) \leq \varphi(H(Tx_n, Tx_{n+1}))\\
&\varphi(D_{n+1}) \leq \varphi(\rho_n) - \psi(\rho_n) \leq \varphi(\rho_n).
\end{aligned}$$

Since $\rho_n < \delta_n + D_n \leq \delta_n + \rho_n$, for each $n \in \mathbb{N}$ and using the fact that $\lim_{n\to\infty} \delta_n = 0$ and

$$\tau = \lim_{n\to\infty} \varphi(\rho_n) = \liminf_{n\to\infty} \varphi(\rho_n) = \limsup_{n\to\infty} \varphi(\rho_n)$$

by taking the upper limit, it follows that

$$\tau \le \limsup_{n\to\infty} \varphi(D_n) \le \tau$$
$$\limsup_{n\to\infty} \varphi(D_n) = \limsup_{D_n\to\rho^*} \varphi(D_n) = \tau.$$

So, we have the following

$$\limsup_{D_n\to\rho^*} \varphi(D_n) \le \limsup_{n\to\infty} \varphi(\rho_n) - \liminf_{n\to\infty} \psi(\rho_n) \le \limsup_{n\to\infty} \varphi(\rho_n) = \tau.$$

So $\limsup_{\rho_n\to\rho^*} \varphi(\rho_n) - \liminf_{n\to\infty} \psi(\rho_n) = \tau$. This means that $\limsup_{\rho_n\to\rho^*} \varphi(\rho_n) = \tau + \liminf_{n\to\infty} \psi(\rho_n)$, i.e. $\lim_{\rho_n\to\rho^*} \varphi(\rho_n) = \tau + \liminf_{n\to\infty} \psi(\rho_n)$. Finally, we get that $\tau = \tau + \liminf_{n\to\infty} \psi(\rho_n)$, so $\liminf_{\rho_n\to\rho^*} \psi(\rho_n) = 0$. Because ψ is lsc in 0, it follows up that $\rho^* = 0$.

Now, we show that sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Also, as in the previous theorem, there exists $\varepsilon > 0$ and there exists $n_k, m_k > k$, with $d(x_{m_k}, x_{n_k}) \ge \varepsilon$. As in [18], we get that $\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$. At the same time, we shall use [Lemma 2.1] from [18], i.e. for each $x, y \in X$, we have that

$$d(x,y) \le D(x,Tx) + H(Tx,Ty) + D(y,Ty) + \delta(Ty).$$

Applying this to subsequences $(x_{m_k})_{k\in\mathbb{N}}$ and $(x_{n_k})_{k\in\mathbb{N}}$, we get that

$$d(x_{m_k}, x_{n_k}) - D(x_{n_k}, Tx_{n_k}) - D(x_{m_k}, Tx_{m_k}) - \delta(Tx_{m_k}) \le H(Tx_{n_k}, Tx_{m_k})$$
$$d(x_{m_k}, x_{n_k}) \le D_{n_k} + D_{m_k} + \delta(Tx_{m_k}) + H(Tx_{m_k}, Tx_{n_k}).$$

Applying the functional φ and using some assumptions on this mapping, we get that

$$\varphi(d(x_{m_k}, x_{n_k})) \le \varphi(D_{n_k} + D_{m_k} + H(Tx_{m_k}, Tx_{n_k})) + \delta(Tx_{m_k}) \le$$
$$\varphi(H(Tx_{m_k}, Tx_{n_k}) + D_{n_k}) + D_{m_k} + \delta(Tx_{m_k}) \le$$
$$\varphi(H(Tx_{m_k}, Tx_{n_k})) + D_{n_k} + D_{m_k} + \delta(Tx_{m_k}).$$

This chain of inequalities is valid if we suppose that for each $n \in \mathbb{N}$, $D_n \neq 0$. By the contrary, if there exists $n \in \mathbb{N}$, such that $D_n = 0$, then $x_n \in F_T$. So, without restraining the generality, we can suppose that $D_n \neq 0$, so all the sums used inside φ are nonzero. Furthermore, by this process we get at least a fixed point for the operator T. Proceeding as is shown below with using the diameter functional δ, we still get a unique fixed contractive fixed point. Also, in the last inequality we can suppose that $H(Tx_{m_k}, Tx_{n_k}) \neq 0$. Assuming the contrary, it follows that $Tx_{m_k} = Tx_{n_k}$, so x_{m_k+1}, x_{n_k+1} are both in Tx_{n_k}. Reasoning as in the proof of the first theorem, and using the fact that $d(x_{m_k+1}, x_{n_k+1}) \to \varepsilon$, because $(d(x_{m_k+1}, x_{n_k+1}))$ is a subsequence of $(d(x_{m_k}, x_{n_k}))$, the contradiction follows easily. Furthermore, we have shown that $D_n \to \rho^* = 0$. We have that

$$\varphi(\varepsilon) \le \varphi(d(x_{m_k}, x_{n_k})) \le \varphi(d(x_{m_k}, x_{n_k})) - \psi(d(x_{m_k}, x_{n_k})) + D_{n_k} + D_{m_k} + \delta(Tx_{m_k})$$
$$\varphi(\varepsilon) \le \limsup_{d(x_{m_k}, x_{n_k})\to\varepsilon} \varphi(d(x_{m_k}, x_{n_k})) - \liminf_{d(x_{m_k}, x_{n_k})\to\varepsilon} \psi(d(x_{m_k}, x_{n_k})).$$

Now, since $d(x_{m_k},x_{n_k}) \to \varepsilon$ and $d(x_{m_k},x_{n_k}) \geq \varepsilon$, using the previous remark, it follows that

$$\varphi(\varepsilon) \leq \varphi(\varepsilon) - \liminf_{k\to\infty} \psi(d(x_{m_k},x_{n_k}))$$

$$\liminf_{k\to\infty} \psi(d(x_{m_k},x_{n_k})) = 0.$$

So, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Now, for the existence of the fixed point, one can reason as in the previous proof. So, we denote by x^* the fixed point of the multivalued operator T.

(ii) Now, we can show that F_T is a singleton, even when $(SF)_T \neq \emptyset$ is not satisfied. We reason as in [18]. So, let $x^*, y^* \in F_T$. We consider the following sequence :

$y_1 \in Tx^*$, $y_2 \in Ty_1$,...,$y_n \in Ty_{n-1}$, with $d(y_n,y^*) \leq H(Ty_{n-1},Tx^*) + \delta_n$, with $\delta_n \to 0$ as $n \to \infty$. This can be done by a successive application of Nadler's lemma.

Furthermore, one can show that $y_n \in Ty_{n-1} \subseteq .... \subseteq T^{n-1}y_1 \subseteq T^n x^*$. We estimate

$$d(x^*,y^*) \leq d(x^*,y_n) + d(y_n,y^*) \leq \delta(T^n x^*) + H(Ty_{n-1},Tx^*) + \delta_n. \text{ Thus}$$

$$\varphi(d(x^*,y^*)) \leq \delta_n + \varphi(\delta(T^n x^*) + H(Ty_{n-1},Tx^*)) \leq$$

$$\delta_n + \delta(T^n x^*) + \varphi(H(Ty_{n-1},Tx^*)) \leq$$

$$\delta_n + \delta(T^n x^*) + \varphi(d(y_{n-1},x^*)) - \psi(d(y_{n-1},x^*)) \leq$$

$$\delta_n + \delta(T^n x^*) + \varphi(\delta(T^n x^*)).$$

In the above argument, we have used hypothesis (H5) with $H(Ty_{n-1},Tx^*) \neq 0$. If this was not true, we can proceed as in the proof of the previous theorem.

Now, we can use the fact that $\limsup\limits_{\delta(T^n x^*)\to 0} \varphi(\delta(T^n x^*)) = 0$. Taking the upper limit, it follows that $\varphi(d(x^*,y^*)) = 0$, so $d(x^*,y^*) = 0$, i.e., F_T is a singleton. □

Conclusions

In this article, in connection to some theorems given in [16] and [18], we present two results concerning the fixed points of $(\varphi - \psi)$ multi-valued operators. The additional assumption imposed here involves a certain behavior of the diameter of the multi-valued iterates. The second theorem is an existence and uniqueness result for the fixed point of a $(\varphi - \psi)$ multi-valued operator. Finally, in the same context, an open problem is presented.

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