HADAMARD WELL-POSEDNESS FOR VECTOR PARAMETRIC EQUILIBRIUM PROBLEMS

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Abstract. We consider some extensions of the Hadamard well-posedness notion for vector parametric equilibrium problems. We obtain some necessary and sufficient conditions for Hadamard well-posedness of these problems and also obtain the equivalence between their Hadamard well-posedness and their scalarizations.

Keywords. Extended Hadamard well-posedness; Generalized Hadamard well-posedness; Topological vector space; Vector equilibrium problem.

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1. INTRODUCTION

Well-posedness has played a crucial role in optimization theory. This fact has motivated many authors to study the well-posedness of optimization problems. The first concept of well-posedness is due to Tykhonov [26] for unconstrained optimization problems. Tykhonov well-posedness [26], requires the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Another fundamental generalization of Tykhonov well-posedness for an optimization problem (in the scalar case) is the well-posedness by perturbations due to Dontchev and Zolezzi [7] and Zolezzi [28, 29]. Levitin and Polyak [15] extended the notion to the constrained case. Levitin-Polyak well-posedness requires the existence of the solution and convergence of each Levitin-Polyak minimizing sequence to the solution, where \((x_n)\) is Levitin-Polyak minimizing sequence iff, for all \(n\), \(x_n\) to be outside of the feasible set and \(g\) is continuous map and \(K\) is a closed set and distance between \(g(x_n)\) and \(K\) tends to zero. There are various notions and generalizations of the concepts of well-posedness, see [4, 6, 12, 13, 14, 27].

The concept of Hadamard well-posedness is due to Hadamard [10]. Another fundamental generalization of Hadamard well-posed was given in [18, 21, 24]. Motivated and inspired by the above works, in this paper, we are investigated some new versions of Hadamard well-posedness for vector parametric equilibrium problems. The outline of the paper is as follows: In the first Section, we introduce a generalized parametric vector equilibrium problem and some preliminary results which are used in the sequel. Section 2 deals with notions of Hadamard well-posedness under perturbations for vector equilibrium

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problem and obtain a sufficient condition from Hadamard well-posedness. In Section 3, by introducing a gap function of our problem, we deduce the equivalent between its Hadamard well-posedness of our problem with the Hadamard well-posedness of its gap function.

Let $X$ be a metric topological vector space, $Y$ be a Hausdorff topological vector space, $W$ and $Z$ be topological vector spaces and $P$ be a metric space. We denote the family of neighborhood of $x \in X$ by $\Omega(x)$. Let $A$, $B$ and $D$ be nonempty sets of $X$, $W$ and $Z$, respectively and $C : X \times P \rightarrow 2^Y$ be a set-valued mapping such that for any $x \in X$ and for any $p \in P$, $C(x, p)$ is a closed, convex and pointed cone in $Y$ such that $\text{int}C(x, p) \neq \emptyset$. Assume that $\varepsilon : X \times P \rightarrow Y$ is a continuous vector valued mapping satisfying $\varepsilon(x, p) \in \text{int}C(x, p)$. Suppose that $K_1 : A \times P \rightarrow 2^A$, $K_2 : A \times P \rightarrow 2^B$ and $K_3 : A \times P \rightarrow 2^D$ are defined. Let the machinery of the problems be expressed by $F : A \times B \times D \times P \rightarrow 2^Y$.

For any subsets $A$ and $B$, we adopt the following notations

$$(u, v) \ r_1 \ A \times B \text{ means } \forall u \in A, \ \forall v \in B,$$

$$(u, v) \ r_2 \ A \times B \text{ means } \forall u \in A, \ \exists v \in B,$$

$$(u, v) \ r_3 \ A \times B \text{ means } \exists u \in A, \ \forall v \in B.$$

For $r \in \{r_1, r_2, r_3\}$, we consider the following vector parametric quasi-equilibrium problem, for given $\varepsilon \in \mathbb{R}^+$ and $p \in P$:

$$(P_r(F, p, \varepsilon)) \quad \exists \bar{x} \in \text{cl}K_1(\bar{x}, p) : (y, z) \ r K_2(\bar{x}, p) \times K_3(\bar{x}, p), \quad F(\bar{x}, y, z, p) + \varepsilon e(\bar{x}, p) \subseteq C(\bar{x}, p).$$

We denote the solution set of the above problem by $S_r(F, p, \varepsilon)$.

**Remark 1.1.** (a) In fact, we deduce those problems that were considered in [3, 24].

(b) If $\varepsilon = 0$, then the solution set of Problem $(P_r(F, p, \varepsilon))$, is the efficient solution of equilibrium problem for set-valued map $F$, that we denote its efficient solution set with $S_r(F, p)$.

(c) If $D : X \times P \rightarrow 2^Y$ is a set-valued mapping such that for any $x \in X$ and for any $p \in P$, $D(x, p)$ is a closed, convex and pointed cone in $Y$ such that $\text{int}D(x, p) \neq \emptyset$, then solutions of Problem $(P_r(F, p, \varepsilon))$, with assumption $C(x, p) = Y \setminus \text{int}D(x, p)$ is weakly $\varepsilon$-efficient solutions of $F$ that were considered in works [8, 18]. Furthermore, if $\varepsilon = 0$, then weakly $\varepsilon$-efficient solutions are weakly efficient solutions, see [1, 11, 17, 19] and references therein.

**Definition 1.1.** Let $T : X \rightarrow 2^Y$ be a set-valued map. Then,

(a) $T$ is said to be upper semi continuous (u.s.c.), iff for each closed set $B \subset Y$,

$$T^{-}(B) := \{x \in X : T(x) \cap B \neq \emptyset\} \text{ is closed in } X.$$  

(b) $T$ is said to be lower semi continuous (l.s.c.), iff for each open set $B \subset Y$,

$$T^{-}(B) := \{x \in X : T(x) \cap B \neq \emptyset\} \text{ is open in } X.$$ 

(c) $T$ is said to be closed iff $Gr(T) = \{(x, y) \in X \times Y : y \in T(x), x \in X\}$ is closed in $X \times Y$.

It is often convenient to characterize the upper and lower semicontinuity in terms of nets, as in the following lemma (see, for example, [[2], Theorems 17.16 and 17.19]).
Lemma 1.1. [2] Let $X$ and $Y$ be topological spaces and $T : X \to 2^Y$ be a set-valued map.

(i) If $T$ has compact values, then $T$ is u.s.c. iff for every net $x_{\alpha}$ in $X$ converging to $x \in X$ and for any net $y_{\alpha}$ with $y_{\alpha} \in T(x_{\alpha})$, there exist $y \in T(x)$ and a subnet $y_{\alpha}$, of $y_{\alpha}$ converging to $y$.

(ii) $T$ is l.s.c. iff for any net $x_{\alpha}$ in $X$ converging to $x \in X$ and each $y \in T(x)$, there exists a net $y_{\alpha}$ converging to $y$, with $y_{\alpha} \in T(x_{\alpha})$, for all $\alpha$. □

Motivated by Definition 4.1 in [22], here we define outer converge continuously, inner converge continuously and converge continuously for a sequence of set-valued maps.

Definition 1.2. Let $X$ be a metric topological vector space, $Y$ be a Hausdorff topological vector space, $G_n : X \to 2^Y$ be a sequence of set-valued maps and $G : X \to 2^Y$ be a set-valued map. The sequence $G_n$ is said to be outer converge continuously (resp. inner converge continuously) to $G$ at $x_0$ if

$$\limsup_{n} G_n(x_n) \subseteq G(x_0), \text{ (resp. } G(x_0) \subseteq \liminf_{n} G_n(x_n)) \text{ for all } x_n \to x_0,$$

where

$$\liminf_{n} G_n(x_n) := \{ y \in Y : y = \lim_{n} y_n, y_n \in G_n(x_n) \text{ for sufficiently large } n \},$$

$$\limsup_{n} G_n(x_n) := \{ y \in Y : y = \lim_{n} y_n, y_n \in G_n(x_n), \{ n \} \text{ is a subsequence of } \{ n \} \}.$$

The sequence $G_n$ is said to be converge continuously to $G$ at $x_0$ if $\limsup_{n} G_n(x_n) \subseteq G(x_0) \subseteq \liminf_{n} G_n(x_n)$, for all $x_n \to x_0$. If $G_n$ converges continuously to $G$ at every point $x \in X$, then it is said that $G_n$ converges continuously to $G$ on $X$.

2. HADAMARD WELL-POSEDNESS

Here we define the notion of $\Gamma_{Crp}$-convergence for a sequence of set-valued functions similar to the Definition 2.1 in [20].

Definition 2.1. (Definition 2.1 in [20]) Let $X$ be a topological space, $Y$ be a topological vector space and $C \subseteq Y$ be a closed, convex and pointed cone such that $\text{int} C \neq \emptyset$ and $\mathfrak{M}(x)$ be the family of neighborhoods of $x$. Let $f_n, f : X \to \mathbb{F}, n \in \mathbb{N}$ be given functions. We say that $(f_n)$ is $\Gamma_{C}$-convergence to $f$ and we shall write $f_n \Gamma_{C} f$, if for every $x \in X$, the following statements are true:

(i) for all $U \in \mathfrak{M}(x)$ and for all $\varepsilon \in \mathbb{R}^+$, there exists $n_{\varepsilon, U} \in \mathbb{N}$ such that

$$\forall n \geq n_{\varepsilon, U} \exists x_n \in U \ f_n(x_n) \leq f(x) + \varepsilon \varepsilon,$$

(ii) for all $\varepsilon \in \mathbb{R}^+$ there exist $U_\varepsilon \in \mathfrak{M}(x)$ and $n_{\varepsilon}$ such that

$$\forall \bar{x} \in U_\varepsilon, \forall n \geq n_{\varepsilon} \ f_n(\bar{x}) \geq f(x) - \varepsilon \varepsilon.$$

Definition 2.2. Suppose for all $n \in \mathbb{N}$, $F_n, F : A \times B \times D \times P \to 2^Y$ are defined. Then, the sequence $(F_n)$ is said to be $\Gamma_{Crp}$- convergence to $F$ and we denote by $F_n \Gamma_{Crp} F$, iff for every $x \in X$ the following statements hold,
(i) for all $U \in \mathcal{U}(x)$ and for all $\varepsilon \in \mathbb{R}^+$, there exists $n_{\varepsilon,U} \in \mathbb{N}$ such that for all $n \geq n_{\varepsilon,U}$ there exists $x_n \in U$ such that
\[
(y,z) rK_2(x,p) \times K_3(x,p), \quad F_n(x_n,y,z,p) - F(x,y,z,p) - \varepsilon e(x,p) \subseteq -C(x,p),
\]
(ii) for all $\varepsilon \in \mathbb{R}^+$ there exist $U_\varepsilon \in \mathcal{U}(x)$ and $n_\varepsilon \in \mathbb{N}$ such that for all $x \in U_\varepsilon$ and for all $n \geq n_\varepsilon$, we obtain
\[
(y,z) rK_2(x,p) \times K_3(x,p), \quad F(x,y,z,p) - F_n(x,y,z,p) - \varepsilon e(x,p) \subseteq -C(x,p).
\]

Now, we can define some new notions of Hadamard well-posedness for vector parametric equilibrium problem that include Definitions 2.5 and 2.6 in [16], Definition 4 in [24] and Definitions 4.1 and 4.2 in [9].

**Definition 2.3.** Let $(\lambda_n, p_n) \subseteq \Lambda \times P$ be a sequence converging to $(\lambda_0, p_0)$ and $F_n \xrightarrow{r_{\text{C}} g} F_{\lambda_0}$. The Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$, is said to be

(a) Hadamard well-posed corresponding to $(F_{\lambda_0})$, iff:
(i) there exists only one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) for all sequence $(\varepsilon_n) \subseteq \mathbb{R}^+$ that $\varepsilon_n \longrightarrow \varepsilon$,
\[
\limsup_{n} [S_r(F_{\lambda_0}, p_n, \varepsilon_n)] \subseteq S_r(F_{\lambda_0}, p_0, \varepsilon).
\]

(b) Generalized Hadamard well-posed corresponding to $(F_{\lambda_0})$, iff:
(i) there exists one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) for all sequence $(\varepsilon_n) \subseteq \mathbb{R}^+$ that $\varepsilon_n \longrightarrow \varepsilon$, condition (ii) of part (a) holds.

(c) Extended Hadamard well-posed corresponding to $(F_{\lambda_0})$, iff
(i) there exists one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) there exists $\varepsilon_0 \in \mathbb{R}^+$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$,
\[
\limsup_{n} [S_r(F_{\lambda_0}, p_\alpha, \varepsilon_n)] \subseteq S_r(F_{\lambda_0}, p_0, \varepsilon).
\]

**Example 2.1.** Let $X = P = \mathbb{R}$, $Y = \mathbb{R}^2$, $K_1(x,p) = K_2(x,p) = K_3(x,p) = [-1,0]$, for all $x \in X$ and $p \in P$, $r = r_1$, $C(x,p) = \mathbb{R}^2_+$, $e(x,p) = (1,1)$ and $F : X \times X \times X \times P \longrightarrow 2^Y$ be defined by $F(x,y,z,p) = \{(x,x)\}$ and for all $n \in \mathbb{N}$, $F_n : X \times X \times X \times P \longrightarrow 2^Y$ be defined by $F_n(x,y,z,p) = \{(x+\frac{1}{n},t) : t \in [x,x+\frac{1}{n}]\}$. We show that $F_n \xrightarrow{r_{\text{C}} g} F$. In fact for all $x \in X$, $U \in \mathcal{U}(x)$ and $\varepsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$ and for all $n \geq N$, $x - \frac{1}{n} \in U$. Therefore, for $x_n = x - \frac{1}{n}$, we have
\[
F_n(x_n,y,z,p) - F(x,y,z,p) - \varepsilon(1,1) = (x_n + \frac{1}{n},t) - (x,x) - (\varepsilon,\varepsilon) = (x - \frac{1}{n} + \frac{1}{n},t) - (x,x) - (\varepsilon,\varepsilon) = (0,t-x) - (\varepsilon,\varepsilon) = (-\varepsilon,t-x-\varepsilon),
\]
that is, $t \in [x_n,x_n+\frac{1}{n}]= [x-\frac{1}{n},x]$ and
\[
t - x - \varepsilon < x - x - \varepsilon < -\varepsilon < 0.
\]
Therefore, \( F_n(x_n, y, z, p) - F(x, y, z, p) - \varepsilon (1, 1) \subseteq -C(x, p) \) and condition (i) of Definition 2.2 holds. For condition (ii) of Definition 2.2, for each \( x \in X \) and \( \varepsilon \in \mathbb{R}^+ \), we suppose that \( U_\varepsilon = B_\varepsilon(x) \) (where \( B_\varepsilon(x) \) is the ball with center \( x \) and radius \( \varepsilon \)). Since there exists \( N \in \mathbb{N} \) such that \( \frac{2}{N} < \varepsilon \), for all \( n \geq N \) and \( x' \in U_\varepsilon \), we have

\[
F(x, y, z, p) - F_n(x', y, z, p) - \varepsilon (1, 1) = (x, x) - (x' + \frac{1}{n}, t) - (\varepsilon, \varepsilon)
\]

(2.6)

\[
= (x - x' - \frac{1}{n} - \varepsilon, x - t - \varepsilon),
\]

(2.7)

but \( x - x' - \frac{1}{n} - \varepsilon = (x - x' - \varepsilon) - \frac{1}{n} < -\frac{1}{n} < 0 \). On the other hand \( \frac{1}{n} < \frac{2}{N} < \varepsilon \), we have

\[
x - t - \varepsilon < x - x' - \varepsilon < 0,
\]

and for all \( n \geq N \) and \( x' \in U_\varepsilon \), we obtain \( F(x, y, z, p) - F_n(x', y, z, p) - \varepsilon (1, 1) \subseteq -C(x, p) \).

Obviously, Problem \((P_n(F, p, \varepsilon))\) is extended Hadamard well-posed corresponding to \((F_n)\). Since, for all \( \varepsilon \in \mathbb{R}^+ \), \( S_{\varepsilon_n p}(F, p, \varepsilon) = S_{\varepsilon_n p}(F_n, p, \varepsilon) = [-\varepsilon, 0] \). It follows that

\[
\limsup_{n}[S_{\varepsilon_n p}(F_n, p, \varepsilon)] \subseteq S_{\varepsilon_n}(F, p, \varepsilon).
\]

Note that, if Problem \((P_n(F, p, \varepsilon))\) is Hadamard well-posed corresponding to \((F_n)\), it is also generalized Hadamard well-posed corresponding to \((F_{\lambda_n})\). In the following result, we show that if Problem \((P_n(F, p, \varepsilon))\) is extended Hadamard well-posed corresponding to \((F_{\lambda_n})\), it is also generalized Hadamard well-posed corresponding to \((F_{\lambda_n})\).

**Proposition 2.1.** Let \((\lambda_n, p_n) \subseteq \Lambda \times P\) be a sequence converging to \((\lambda_0, p_0)\) and \( F_{\lambda_n} \xrightarrow{\Gamma_{C(p_n)}} F_{\lambda_0} \). If Problem \((P_{\varepsilon_n}(F_{\lambda_0}, p_0, \varepsilon))\), is extended Hadamard well-posed corresponding to \((F_{\lambda_n})\), then Problem \((P_{\varepsilon_n}(F_{\lambda_0}, p_0, \varepsilon))\), is generalized Hadamard well-posed corresponding to \((F_{\lambda_n})\).

**Proof.** The proof with minor modifications is similar to the proof of Proposition 2.2 in [16], therefore, it is omitted.

Motivated by an idea of Tam and Loan in [25] and by using Proposition 2.1, we can deduce some sufficient conditions for generalized Hadamard well-posedness.

**Theorem 2.1.** Suppose that \( X \) is a metric vector topological space, \( Y \) is a topological vector space and \((\lambda_n, p_n) \subseteq \Lambda \times P\) is a sequence converging to \((\lambda_0, p_0)\), \( F_{\lambda_n}, F_{\lambda_0} : X \rightarrow 2^Y\) and the following conditions hold:

(i) \( F_{\lambda_n} \xrightarrow{\Gamma_{C(p_n)}} F_{\lambda_0} \);

(ii) for all \( \varepsilon \in \mathbb{R}^+ \), the solution set of Problem \((P_{\varepsilon_n}(F_{\lambda_0}, p_0, \varepsilon))\) is nonempty;

(iii) \( \text{cl} K_1 \) is upper semi continuous and compact valued;

(iv) \( K_2 \) and \( K_3 \) are lower semi continuous;

(v) \( F_{\lambda_n} \) is inner converge continuously to \( F_{\lambda_0} \).

Then, Problem \((P_{\varepsilon_n}(F_{\lambda_0}, p_0, \varepsilon))\) is extended Hadamard well-posed corresponding to \((F_{\lambda_n})\).
Proof. We show for all $\varepsilon \in \mathbb{R}^+$

$$\limsup_n [S_{r_1}(F_{\lambda_n}, p_n, \varepsilon)] \subseteq S_{r_1}(F_{\lambda_0}, p_0, \varepsilon).$$

Suppose that $w \in \limsup_n [S_{r_1}(F_{\lambda_n}, p_n, \varepsilon)]$ and $w \not\in S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$. Then, there exist a subsequence of $(S_{r_1}(F_{\lambda_n}, p_n, \varepsilon))$ and sequence $w_{n_k}$ such that for all $n_k$, $w_{n_k} \in S_{r_1}(F_{\lambda_{n_k}}, p_{n_k}, \varepsilon)$ and $w_{n_k} \to w$. Since $w_{n_k} \in S_{r_1}(F_{\lambda_{n_k}}, p_{n_k}, \varepsilon)$, one sees that $w_{n_k} \in \text{clK}_1(w_{n_k}, p_{n_k})$ and for all $(y, z) \in K_2(w_{n_k}, p_{n_k}) \times K_3(w_{n_k}, p_{n_k})$

$$F_{\lambda_{n_k}}(w_{n_k}, y, z, p_{n_k}) + \varepsilon e(w_{n_k}, p_{n_k}) \subseteq C(w_{n_k}, p_{n_k}).$$

(2.8)

But $w_{n_k} \in \text{clK}_1(w_{n_k}, p_{n_k})$ and $\text{clK}_1$ is upper semi continuous and compact valued. Then $w \in \text{clK}_1(w, p)$. Since $w \not\in S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$, one sees that

$$\exists (y_0, z_0) \in K_2(w, p_0) \times K_3(w, p_0) : F_{\lambda_0}(w, y_0, z_0, p_0) + \varepsilon e(w, p_0) \not\subseteq C(w, p_0).$$

(2.9)

Therefore, there exists $u_0 \in F_{\lambda_0}(w, y_0, z_0, p_0)$ such that $u_0 + \varepsilon e(w, p_0) \not\subseteq C(w, p_0)$. Since $K_2$ and $K_3$ are lower semi continuous, one finds that there exists a sequence

$$(y_{n_k}, z_{n_k}) \in K_2(w_{n_k}, p_{n_k}) \times K_3(w_{n_k}, p_{n_k})$$

such that $(y_{n_k}, z_{n_k}) \to (y_0, z_0)$. From (2.8),

$$F_{\lambda_{n_k}}(w_{n_k}, y_{n_k}, z_{n_k}, p_{n_k}) + \varepsilon e(w_{n_k}, p_{n_k}) \subseteq C(w_{n_k}, p_{n_k}),$$

(2.10)

as $(F_{\lambda_{n_k}})$ is inner converge continuously to $F_{\lambda_0}$ at $(w, y_0, z_0, p_0)$, therefore

$$u_0 \in F_{\lambda_0}(w, y_0, z_0, p_0) \subseteq \liminf_{n_k} F_{\lambda_{n_k}}(w_{n_k}, y_{n_k}, z_{n_k}, p_{n_k}).$$

So, there exists $(u_{n_k})$ such that $u_{n_k} \in F_{\lambda_{n_k}}(w_{n_k}, y_{n_k}, z_{n_k}, p_{n_k})$ and $u_{n_k} \to u_0$. By using (2.10), $u_{n_k} + \varepsilon e(w_{n_k}, p_{n_k}) \subseteq C(w_{n_k}, p_{n_k})$, and since $\varepsilon$ is continuous and $C$ is closed, one find that $u_0 + \varepsilon e(w, p_0) \in C(w, p_0)$, which is a contradiction. This completes the proof. \hfill \Box

Remark 2.1. (a) In the previous theorem, if we replace condition (iv) by the following condition:

(iv)’ $K_2$ is lower semi continuous on $A \times P$ and $K_3$ is upper semi continuous and compact valued on $A \times P$.

Then, with minor modifications in the proof, one can conclude the extended Hadamard well-posedness of the Problem $(P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))$.

(b) In the previous theorem, if we replace condition (iv) by the following condition:

(iv)’’ $K_2$ is upper semi continuous and compact valued on $A \times P$ and $K_3$ is lower semi continuous on $A \times P$.

Then, with minor modifications in the proof, one can deduce the extended Hadamard well-posedness of the Problem $(P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))$.

The next example shows that our assumptions in Theorem 2.3 is weaker than Salamon’s assumptions in [24].
Example 2.2. For $r = 1$ and fixed $p$, let $K_1, K_2 : [-1, 1] \times \{p\} \to [-1, 1]$, $K_1(x, p) = K_2(x, p) = [-1, 1]$ and $K_3 : [-1, 1] \times \{p\} \to \{1\}$ and let $F : [-1, 1] \times [-1, 1] \times \{1\} \times \{p\} \to \{0, 1\}$ and for all $n \in \mathbb{N}$, $F_n : [-1, 1] \times [-1, 1] \times \{1\} \times \{p\} \to \{0, 1\}$ be defined by

$$F(x, y, 1, p) = \begin{cases} 
0 & x = 0, \\
1 & \text{o.w.,}
\end{cases}$$

$$F_n(x, y, 1, p) = \begin{cases} 
0 & x \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\
1 & \text{o.w.,}
\end{cases}$$

For all $x, p \in [-1, 1]$, let $C(x, p) = [0, +\infty]$. Since condition (iii) of Theorem 1 in [24] doesn’t hold, therefore we can not achieve any results. But the conditions of Theorem 2.3 hold, therefore this problem is extended Hadamard well-posed corresponding by $(F_n)$.

The next theorem and its corollary show some alternative characterization for extended Hadamard well-posedness and Hadamard well-posedness of Problem $(P_r(F, p, \varepsilon))$. In fact, we obtain a relation between Hadamard well-posed of Problem $(P_r(F, p, \varepsilon))$ and its approximate solutions. For this idea, we need to define approximate solutions of Problem $(P_r(F, p, \varepsilon))$, denoted by:

$$\Pi_r(F, p, \varepsilon, \delta) = \{x \in cK_1(x, p) : F(x, y, z, p) + \varepsilon e(x, p) + \delta e(x, p) \subseteq C(x, p), (y, z) \in K_2(x, p) \times K_3(x, p)\}.$$  

Obviously, if $\delta_1 \leq \delta_2$, then $\Pi_r(F, p, \delta_1) \subseteq \Pi_r(F, p, \delta_2)$ and

$$S_r(F, p, \varepsilon) = \cap_{\delta > 0} \Pi_r(F, p, \varepsilon, \delta) = \Pi_r(F, p, \varepsilon, 0).$$  

The following theorem and its corollary improve Theorems 4.8 and 4.10 in [9].

**Theorem 2.2.** Let $X$ be a metric topological vector space, $Y$ be a Hausdorff topological vector space and $(\lambda_n, p_n)$ be a sequence converges to $(\lambda_0, p_0)$, $\tilde{E}$ (where, $\tilde{E}(p_0) = \{x \in X : x \in cK_1(x, p_0)\}$) be a compact-valued and upper semi continuous set-valued map and $F_{\lambda_0} \xrightarrow{\Gamma_{\epsilon}^{(0)}} F_{\lambda_n}$. If Problem $(P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))$ is extended Hadamard well-posed corresponding to $(F_{\lambda_n})$, then there exists a nonempty and compact subset $M$ of $S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$ such that for every neighborhood $U$ of $0_X$, there exists $\delta > 0$ such that

$$x \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta) \implies x \in M + V.$$

Conversely, if

(i) there exists a nonempty compact subset $M$ of $S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$ such that for every neighborhood $V$ of $0$, there exists $\delta > 0$ such that

$$x \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta) + V \implies x \in M + V;$$

(ii) $\Pi_{r_1}$ is upper semi continuous in its third argument.

Then, Problem $(P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))$, is extended Hadamard well-posed corresponding to $(F_{\lambda_n})$.  

Proof. Suppose that Problem \((P_{r_2}(F_{\lambda_0}, p_n, \varepsilon))\) is extended Hadamard well-posed corresponding to \((F_{\lambda_0})\). Let \(M = S_{r_1}(F_{\lambda_0}, p_0, \varepsilon) \neq \emptyset\). We will show that \(S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)\) is a compact set. If \((x_n)\) is a sequence in \(S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)\), then
\[
(y, z) \in K_2(x_n, p_0) \times K_3(x_n, p_0), \quad F_{\lambda_0}(x_n, y, z, p_0) + \varepsilon e(x_n, p_0) \subseteq C(x_n, p_0). \tag{2.11}
\]
Since \(F_{\lambda_n} \xrightarrow{\Gamma_{C_1(\mathbf{R}^+)}} F_{\lambda_0}\), one sees that if \((\delta_n) \subseteq \mathbf{R}^+\) and \(\delta_n \rightarrow 0\). For any \(\delta_n\), \((y, z) \in K_2(x_n, p_0) \times K_3(x_n, p_0)\)
\[
F_{\lambda_0}(x_n, y, z, p_0) - F_{\lambda_n}(x_n, y, z, p_0) - \delta_n e(x_n, p_0) \subseteq -C(x_n, p_0). \tag{2.12}
\]
By summing (2.11) and (2.12), we obtain
\[
(y, z) \in K_2(x_n, p_0) \times K_3(x_n, p_0), \quad F_{\lambda_0}(x_n, y, z, p_0) + \varepsilon e(x_n, p_0) + \delta_n e(x_n, p_0) \subseteq C(x_n, p_0). \tag{2.13}
\]
Therefore \(x_n \in S_{r_1}(F_{\lambda_0}, p_0, \varepsilon + \delta_n)\). Since \(x_n \in \bar{E}(p_0)\), one sees that \((x_n)\) has a subsequence converging to some \(x_0\) of \(\bar{E}(p_0)\). Let for all \(n, p_n = p_0\) and \(\varepsilon_n = \varepsilon + \delta_n\). Since Problem \((P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))\) is generalized Hadamard well-posed corresponding to \((F_{\lambda_0})\), one finds that
\[
\limsup_n [S_{r_1}(F_{\lambda_0}, p_0, \varepsilon_n)] \subseteq S_{r_1}(F_{\lambda_0}, p_0, \varepsilon),
\]
and \(x_0 \in S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)\). Therefore \(S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)\) is a compact set.

Now, we show the existence of \(\delta\) for any neighborhood of \(0_X\). On the contrary, there exists a neighborhood \(U\) of \(0_X\) and a sequence \((\delta_n) \subseteq \mathbf{R}^+\) such that \(\delta_n \rightarrow 0\) and \((x_n) \subseteq X\) such that for all \(n, x_n \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta_n)\), but \(x_n \notin S_{r_1}(F_{\lambda_0}, p_0, \varepsilon) + U\). Since \(x_n \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta_n)\), one obtains that \(x_n \in S_{r_1}(F_{\lambda_n}, p_n, \varepsilon + \delta_n)\). On the other hand, \(x_n \in \bar{E}(p_n)\). Using our assumption, one obtains that there exists \(x_0 \in \bar{E}(p_0)\) and subsequence \(x_{n_k}\) such that \(x_{n_k} \rightarrow x_0\). But Problem \((P_{r_1}(F_{\lambda_0}, p, \varepsilon))\) is generalized Hadamard well-posed corresponding to \((F_{\lambda_0})\).

Then
\[
\limsup_n [S_{r_1}(F_{\lambda_n}, p_{n_k}, \varepsilon + \delta_{n_k})] \subseteq S_{r_1}(F_{\lambda_0}, p_0, \varepsilon),
\]
and \(x_0 \in S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)\), which is a contradiction.

Conversely, we show that for any sequence \((\lambda_n, p_n) \subseteq \Lambda \times P\) converges to \((\lambda_0, p_0), \varepsilon_n \rightarrow \varepsilon\) and \(F_{\lambda_n} \xrightarrow{\Gamma_{C_1(\mathbf{R}^+)}} F_{\lambda_0}\). Note that
\[
\limsup_n [S_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n)] \subseteq S_{r_1}(F_{\lambda_0}, p_0, \varepsilon).
\]
Let \(\bar{x} \in \limsup_n [S_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n)]\). Hence, there exists a sequence \((x_n)\) such that \(x_n \rightarrow \bar{x}\) and \(x_n \in S_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n)\).

For completing the proof, we show that \((x_n)\) contains a subsequence converges to a point \(x_0 \in M\). Suppose on the contrary that for all \(x \in M\) there exists a neighborhood \(U_x\) of 0 such that \((x) + U_x\) does not contain any subsequence of \(x_n\). Also, for all \(x \in M\), there exists a neighborhood \(V_x\) of 0 such that \(V_x + V_x \subseteq U_x\). Since \(M \subseteq \bigcup_{x \in M} (\{x\} + V_x)\) and \(M\) is compact, we see that there exists \(n \in \mathbb{N}\) such that
\[
M \subseteq \bigcup_{i=1}^{n} (\{x_i\} + V_{x_i})
\]
Letting \(\bar{V} = \bigcap_{i=1}^{n} V_{x_i}\), we find from the assumption that there exists \(\delta > 0\) such that
\[
x \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta) + \bar{V} \Rightarrow x \in M + \bar{V}.
\]
Obviously, there exists \( n_1 \) such that for all \( n \geq n_1 \),
\[
\Pi_{r_1}(F_{\lambda_n}, p_n, \delta_n) \subseteq \Pi_{r_1}(F_{\lambda_n}, p_n, \delta_n).
\]
On the other hand, \( x_n \in S_{r_1}(F_{\bar{\lambda}_n}, p_n, \epsilon_n) \). Then \( x_n \in \Pi_{r_1}(F_{\lambda_n}, p_n, \epsilon_n, 0) \subseteq \Pi_{r_1}(F_{\lambda_n}, p_n, \epsilon_n, \delta_n) \). Since \( \Pi_{r_1} \) is upper semi continuous in its third argument and \( \epsilon_n \rightarrow \epsilon \), we see that there exists \( n_2 \) such that for all \( n \geq n_2 \), \( \Pi_{r_1}(F_{\lambda_n}, p_n, \epsilon_n, \delta_n) \subseteq \Pi_{r_1}(F_{\lambda_n}, p_n, \epsilon, \delta) + V \). Now, if \( n \geq \max\{n_1, n_2\} \), then \( x_n \in \Pi_r(F_{\lambda_n}, p_0, \delta) + V \). Therefore, \( x_n \in M + V \). But
\[
M + V \subseteq \bigcup_{i=1}^{n}(\{x_i\} + V_i) + V = \bigcup_{i=1}^{n}(\{x_i\} + V_i) + V
\]
\[
\subseteq \bigcup_{i=1}^{n}(\{x_i\} + V_i + V_i) \subseteq \bigcup_{i=1}^{n}(\{x_i\} + U_i).
\]
Hence, for those \( n \geq \max\{n_1, n_2\} \), \( x_n \in \bigcup_{i=1}^{n}(\{x_i\} + U_i) \), which is a contradiction, since for all \( x \in M \), \( \{x\} + U_i \) does not contain any subsequence of \( x_n \), but as the limit is unique. Therefore, \( x_0 = \bar{x} \). \( \square \)

Now, we obtain the Tykhonov well-posedness and the Levitin-Polyak well-posedness from the Hadamard well-posedness.

**Definition 2.4.** [5] The Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \), is said to be
(a) Tykhonov wellposed iff
(i) there exists only one solution for Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \);
(ii) for any sequence \( (\lambda_n, p_n) \subseteq \Lambda \times P \) converging to \( (\lambda_0, p_0) \), every asymptotically solving sequence for Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \) corresponding to \( (\lambda_n, p_n) \), converges to \( S_r(F_{\lambda_0}, p_0, \epsilon) \).
(b) Tykhonov well-posed in the general sense iff
(i) there exists one solution for Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \);
(ii) for any sequence \( (\lambda_n, p_n) \subseteq \Lambda \times P \) converging to \( (\lambda_0, p_0) \), every asymptotically solving sequence for Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \) corresponding to \( (\lambda_n, p_n) \), contains a subsequence converges to some point of \( S_r(F_{\lambda_0}, p_0, \epsilon) \).

**Definition 2.5.** [4] Let \( X \) and \( Y \) be two metric spaces, \( (\lambda_n, p_n) \subseteq \Lambda \times P \) be a sequence converging to \( (\lambda_0, p_0) \). A sequence \( \{x_n\} \subset A \) is said to be
(a) type I LP asymptotically solving sequence corresponding to \( (\lambda_n, p_n) \), for Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \), if \( x_n \in cK_1(x_n, p_n) \) and there exists a sequence \( (\epsilon_n) \subseteq \mathbb{R}^+ \) with \( \epsilon_n \rightarrow 0 \) such that
\[
(y, z) \rightarrow K_2(x_n, p_n) \times K_3(x_n, p_n), \quad F(x_n, y, z, p_n) + \epsilon_n e(x_n, \lambda_n, p_n) \subseteq C(x_n, p_n).
\]
(b) type II LP asymptotically solving sequence corresponding to \( (\lambda_n, p_n) \), for Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \), if there exists a sequence \( (\epsilon_n) \subseteq \mathbb{R}^+ \) with \( \epsilon_n \rightarrow 0 \) such that
\[
d(x_n, K_1(x_n, p_n)) \leq \epsilon_n
\]
and (2.14) holds.
(c) The Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \) is LP well-posed of type I (resp. type II) if and only if
(i) there is only one solution for Problem \( (P_r(F_{\lambda_0}, p_0, \epsilon)) \);
(ii) for any sequence \((\lambda_n, p_n) \subseteq \Lambda \times P\), which converges to \((\lambda_0, p_0)\), every LP of type I (resp. type II) asymptotically solving sequence for Problem \((P_r(F_{\lambda_0}, p_0, \epsilon))\) corresponding to \((\lambda_n, p_n)\), converges to \(S_r(F_{\lambda_0}, p_0, \epsilon)\).

In the following result, we obtain a generalization of Theorem 2.2 in [21]. As a matter of fact, Revalski in [21], have shown that if \(X\) is a real Banach space and \(f : X \to \mathbb{R} \cup \{+\infty\}\) is a convex lower semi-continuous extended real-valued function, then the Tykhonov well-posedness and the Levitin-Polyak well-posedness are deduced from Hadamard well-posedness. Here, we obtain those results for set-valued maps in topological spaces.

**Theorem 2.3.** Let \((\lambda_n, p_n) \subseteq \Lambda \times P\) be a sequence converging to \((\lambda_0, p_0)\) for all \(n, F_{\lambda_n} = F_{\lambda_0}\). If Problem \((P_r(F_{\lambda_0}, p_0, \epsilon))\) is Hadamard well-posed corresponding to \((F_{\lambda_0})\) and one of the following conditions holds:

(i) \(clK_1\) is compact valued and upper semi-continuous in the first argument;
(ii) \(clK_1\) is closed map.

Then, Problem \((P_r(F_{\lambda_0}, p_0, \epsilon))\) is Tykhonov well-posed, type I Levitin-Polyak well-posed and type II Levitin-Polyak well-posed.

**Proof.** (i) It suffices to prove for generalized Hadamard well-posed. Other case obtains from Proposition 2.1. Suppose Problem \((P_r(F_{\lambda_0}, p_0, \epsilon))\) is generalized Hadamard well-posed and \(clK_1\) is compact valued and upper semi-continuous. If \((x_n)\) is an asymptotically solving sequence, (resp. type I LP asymptotically solving sequence and type II LP asymptotically solving sequence), since for all \(n, x_n \in clK_1(x_n, p_0)\), then there exists \(x_0 \in clK_1(x_0, p_0)\) such that \(x_n \to x_0\). On the other hand, \((x_n)\) is an asymptotically solving sequence (resp. type I LP asymptotically solving sequence and type II LP asymptotically solving sequence) then for all \(n, x_n \in S_{r_1}(F_{\lambda_0}, p_n, \epsilon_n)\). Since for all \(n, F_{\lambda_n} = F_{\lambda_0}\) and \(x_0 \in \limsup\{S_{r_1}(F_{\lambda_0}, p_n, \epsilon_n)\} \subseteq S_{r_1}(F_{\lambda_0}, p_0)\), so the proof is complete.

If condition (ii) holds, then \((x_n)\) has a subsequence converges to \(x_0\). The remaining proof is similar to the proof of part (i).

### 3. A Scalarization for Hadamard Well-Posedness

In this section, let \((\lambda_n, p_n) \subseteq \Lambda \times P\) be a sequence converging to \((\lambda_0, p_0)\). We shall show that the Hadamard well-posedness of Problem \((P_r(F_{\lambda_0}, p_0, \epsilon))\) reaches from the Hadamard wellposed of scalar optimization problem. In this Section, we suppose that maps \(C, e, K_2, K_3\) are constant maps and therefore, for all \(x \in X\) and \(p \in P, C(x, p) = C\) and \(e(x, p) = e \in \text{int}C, K_2(x, p) = B\) and \(K_3(x, p) = D\).

Here, we use a modified version of a result of Sach [23] for obtaining a nonlinear scalarization function to define a gap function for problem \((P_r(F_{\lambda_0}, p_0, \epsilon))\).

Following the idea in [23], we need to define the following notations.

**Definition 3.1.** Let \(Q \subset Y, C\) be a closed convex cone in a topological vector space \(Y\). Then

(i) \(Q\) is called \(C\)-bounded if for each neighborhood \(U\) of the origin of \(Y\) there exists a positive \(t, \) such that
$Q \subset C + tU$.

(ii) $Q$ is called $-C$-closed if $Q - C$ is closed.

**Remark 3.1.** One can show that when $Q$ is $C$-compact, then $Q$ is $-C$-closed and $C$-bounded. If the set-valued function $F$ satisfies condition (i) (resp. (ii)) of Definition 3.1 at each point of $A \times B \times D \times P$, then we say that $F$ is $C$-bounded (resp. $-C$-closed). It is evident that if $F$ has bounded values in $Y$, then it is $C$-bounded and furthermore, if $F$ has compact values in $Y$, then $F$ is simultaneously $C$-bounded and $-C$-closed.

The proof of the following results are similar to the corresponding ones in [23], with replacing $C$ by $-C$, therefore it is omitted.

**Lemma 3.1.** Let $Q \subset Y$, $C$ be a closed convex cone in $Y$ with $\text{int} C \neq \emptyset$ and $e \in \text{int} C$. For a subset $Q$ of $Y$, $e \in \text{int} C$ and $e > 0$, we have

(i) If $Q$ is $C$-bounded, then $s_Q := \min \{t \geq 0 : Q + te \subset C\}$ is well-defined.

(ii) If $Q$ is $C$-bounded, then $s_Q = 0$ iff $Q \subset C$ and $s_Q \leq e$ iff $Q \subset C - e$.

**Remark 3.2.** Let $F$ be a set-valued map with compact values. Using the Remark 3.1, all of the consequences of Lemma 3.1 is valid for $\varphi_F : X \times P \times \mathbb{R} \rightarrow \mathbb{R}$ which is defined by

$$\varphi_F(x, p, e) := \min \{t \in \mathbb{R}^+ : F(x, y, z, p) + e e + te \subset C \forall (y, z) \in B \times D\}. \quad (3.1)$$

Obviously, $\bar{x} \in S_1(F, p, e)$ iff $\varphi_F(\bar{x}, p, e) = 0$.

**Lemma 3.2.** Let $(\lambda_n, p_n) \subset A \times P$ be a sequence converging to $(\lambda_0, p_0)$ and $F_{\lambda_n} \xrightarrow{\Gamma_{C^{CP}}} F_{\lambda_0}$. Then $-\varphi_{F_{\lambda_n}} \xrightarrow{\Gamma_{R^+}} -\varphi_{F_{\lambda_0}}$.

**Proof.** Suppose that $F_{\lambda_n} \xrightarrow{\Gamma_{C^{CP}}} F_{\lambda_0}$. For every $\bar{x} \in X$, conditions (i) and (ii) of Definition 2.2 hold. Since $F_{\lambda_n} \xrightarrow{\Gamma_{C^{CP}}} F_{\lambda_0}$, of condition (i) of Definition 2.2, for all $U \in \mathfrak{U}(\bar{x})$ and for all $e \in \mathbb{R}^+$, there exists $\alpha_{\epsilon, U}$ such that for all $n \geq n_{\epsilon, U}$ and there exists $x_n \in U$ such that for all $y, z \in B$

$$F_{\lambda_n}(x_n, y, z, p_0) - F_{\lambda_0}(\bar{x}, y, z, p_0) - e e \subset -C. \quad (3.2)$$

If $t$ belongs to

$$\{t \in \mathbb{R}^+ : F_{\lambda_n}(x_n, y, z, p_0) + e' e + te \subset C \forall (y, z) \in B \times D\}, \quad (3.3)$$

then by summing (3.2) and (3.3), we obtain $t + e$ belongs to

$$\{t \in \mathbb{R}^+ : F_{\lambda_0}(\bar{x}, y, z, p_0) + e' e + te \subset C \forall (y, z) \in B \times D\}. $$

Therefore, $\varphi_{\lambda_0}(\bar{x}, p_0, e') \leq t + e$ and $\varphi_{\lambda_0}(\bar{x}, p_0, e') \leq \varphi_{\lambda_n}(x_n, p_0, e') + e$. Then condition (i) of Definition 2.1 holds. From condition (ii) of Definition 2.2, we obtain for all $e \in \mathbb{R}^+$ there exist $U_\epsilon \in \mathfrak{U}(\bar{x})$ and $n_\epsilon$ such that for all $x \in U_\epsilon$, for all $n \geq n_\epsilon$, and for all $y, z \in S$, we have

$$F_{\lambda_0}(\bar{x}, y, z, p_0) - F_{\lambda_n}(x, y, z, p_0) - e e \subset -C. \quad (3.4)$$
If \( t \) belongs to
\[
\{ t \in \mathbb{R}^+ : F_{\lambda_0}(x, y, z, p_0) + \varepsilon \cdot e + te \subseteq C \forall (y, z) \in B \times D \},
\]
by summing (3.4) and (3.5), we deduce \( t + \varepsilon \) belonging to
\[
\{ t \in \mathbb{R}^+ : F_{\lambda_0}(x, y, z, p_0) + \varepsilon' \cdot e + te \subseteq C \forall (y, z) \in B \times D \}.
\]
So, \( \varphi_{\lambda_0}(x, p_0, \varepsilon') \leq \varphi_{\lambda_0}(x, p_0, \varepsilon') + \varepsilon \), and condition (ii) of Definition 2.1 holds and \( -\varphi_{\lambda_0} \xrightarrow{\Gamma_{t^+}} -\varphi_{\lambda_0} \). \( \square \)

In the following theorem, we obtain an equivalence relation between generalized Hadamard well-posedness of Problem \((P_t(F_{\lambda_0}, p_0, \varepsilon))\) and the generalized Hadamard well-posedness of scalar optimization problems.

**Theorem 3.1.** Suppose that \((\lambda_n, p_n)\) is a sequence converging to \((\lambda_0, p_0)\) and \(F_{\lambda_0}\) and \(F_{\lambda_n}\) have compact values and \(F_{\lambda_n} \xrightarrow{\Gamma_{t^+}} F_{\lambda_0}\), such that \(F_{\lambda_0}\) is inner converge continuously to \(F_{\lambda_0}\). Then Problem \((P_t(F_{\lambda_0}, p_0, \varepsilon))\) is generalized Hadamard wellposed corresponding to \((F_{\lambda_0})\) if the following scalar optimization problem
\[
(\text{OP}(\varphi_{F_{\lambda_0}}, p_0, \varepsilon)) \max_{x \in S} -\varphi_{F_{\lambda_0}}(x, p_0, \varepsilon)
\]
is generalized Hadamard wellposed corresponding to \((-\varphi_{F_{\lambda_0}})\) defined in Remark 3.2.

**Proof.** We denote the solution set of Problem \((\text{OP}(\varphi_{F_{\lambda_0}}, p_0, \varepsilon))\) by \(\text{eff}(-\varphi_{F_{\lambda_0}}, p_0, \varepsilon)\). Suppose that \((e_n) \subseteq \mathbb{R}^+, e_n \rightarrow \varepsilon'\) and Problem \((\text{OP}(\varphi_{F_{\lambda_0}}, p_0, \varepsilon))\) is generalized Hadamard wellposed corresponding to \((-\varphi_{F_{\lambda_0}})\). Then
\[
\limsup_n [\text{eff}(-\varphi_{F_{\lambda_0}}, p_0, e_n)] \subseteq \text{eff}(-\varphi_{F_{\lambda_0}}, p_0, e').
\]
If \(x' \in \limsup_n [S_t(F_{\lambda_0}, p_0, e_n)]\), then there exists \((x'_n, k)\) such that for all \(n_k\), \(x'_n \in S_t(F_{\lambda_{n_k}}, p_{n_k}, e_{n_k})\) and \(x'_n \rightarrow x'\). Since \(x'_n \in S_t(F_{\lambda_{n_k}}, p_{n_k}, e_{n_k})\), we have \(\varphi_{F_{\lambda_{n_k}}}(x'_n, p_{n_k}, e_{n_k}) = 0\) and \(x'_n \in \text{eff}(-\varphi_{F_{\lambda_{n_k}}}, p_{n_k}, e_{n_k})\).

Hence, we deduce \(x' \in \limsup_n [\text{eff}(-\varphi_{F_{\lambda_0}}, p_0, e_n)]\) and \(x' \in \text{eff}(-\varphi_{F_{\lambda_0}}, p_0, e')\). For completing the proof, it is enough to show that \(\varphi_{F_{\lambda_0}}(x', p_0, e') = 0\). Suppose on the contrary, there exists \(t_0 > 0\) such that \(\varphi_{F_{\lambda_0}}(x', p_0, e') = t_0\) and for all \(t < t_0\), there exists \(u_t \in F_{\lambda_0}(x', p_0, e')\) that \(u_t + \varepsilon' + te \not\subseteq C\). Now for a fix \(t > t_0\), Since \(F_{\lambda_0}\) is inner converge continuously to \(F_{\lambda_0}\), we see that there exists \(u_{n_k} \in F_{\lambda_{n_k}}(x'_n, p_{n_k}, e_{n_k})\) such that \(\lim_k u_{n_k} = u_t\), since \(\varphi_{F_{\lambda_{n_k}}}(x'_n, p_{n_k}, e_{n_k}) = 0\) and \(C\) is closed cone. Hence, 
\[
u_t + \varepsilon' = \lim_k u_{n_k} + \varepsilon_{n_k} \in C,
\]
since \(e \in \text{int} C\). Then we deduce \(u_t + \varepsilon' + te \subseteq C\). This is a contradiction. \(\square\)

The following definition is a generalization of Definition 2.3 in [16].

**Definition 3.2.** Let \(F : X \times B \times D \times P \rightarrow 2^Y\) be a set-valued map. \(F\) is said to be strongly upper \(C_{(e)}\)-semicontinuous at the point \(x_0 \in X\), if for all \(e \in \mathbb{R}^+, \) there exists \(U_{x_0, e} \in \Delta(x_0)\) such that
\[
\forall x \in U_{x_0, e}, F(x, y, z, p) - F(x_0, y, z, p) - \varepsilon e \subseteq -\text{int} C, \forall (y, z, p) \in B \times D \times P.
\]

**Lemma 3.3.** Let \(F\) be a strongly upper \(C_{(e)}\)-semicontinuous map at the first argument and lower semi continuous at the forth argument. Then, function \(-\varphi_F\) that defined in Remark 3.2 is a strongly \(\mathbb{R}^{(1)}_+\)-upper semi-continuous map at the first argument.
Proof. By Remark 2.1 in [16], we have to show that $-\varphi_F$ is upper semi continuous, i.e., for all $a \in \mathbb{R}$, \[ \{(x, p, \varepsilon) : -\varphi_F(x, p, \varepsilon) \geq a\} \] is a closed set. Suppose there exists sequence $(x_n, p_n, \varepsilon_n) \to (x_0, p_0, \varepsilon_0)$ such that $-\varphi_F(x_n, p_n, \varepsilon_n) \geq a$. We show $-\varphi_F(x_0, p_0, \varepsilon_0) \geq a$.

Since $F$ is a strongly upper $C_{(e)}$-semicontinuous map at the point $x_0 \in X$, then for all $\delta \in \mathbb{R}^+$, there exists $U_{x_0, \delta} \in \mathcal{U}(x_0)$ such that

\[ \forall x \in U_{x_0, \delta}, F(x, y, z, p) - F(x, y, z, p) - \delta e \subseteq \text{int} C, \quad \forall (y, z, p) \in B \times D \times P. \]

Since $x_n \to x_0$, we have

\[ F(x_n, y, z, p) - F(x_0, y, z, p) - \delta e \subseteq \text{int} C, \quad \forall (y, z, p) \in B \times D \times P. \quad (3.6) \]

On the other hand, $-\varphi_F(x_n, p_n, \varepsilon_n) \geq a$. Then there exists $t_0 \leq -a$ belonging to

\[ \{t \in \mathbb{R}^+ : F(x_n, y, z, p_n) + \varepsilon_n e + te \subseteq C, \quad \forall (y, z) \in B \times D\}. \]

Therefore, one has

\[ F(x_n, y, z, p_n) + \varepsilon_n e + t_0 e \subseteq C \quad \forall (y, z) \in B \times D. \quad (3.7) \]

Putting $p = p_n$ in (3.6) and summing (3.6) and (3.7), we obtain

\[ F(x_0, y, z, p_n) + \varepsilon_n e + t_0 e + \delta e \subseteq \text{int} C. \]

Since $F$ is lower semi continuous on the forth argument and $C$ is closed cone, one has

\[ F(x_0, y, z, p) + \varepsilon_0 e + t_0 e + \delta e \subseteq C. \]

So, $t_0 + \delta$ belongs to

\[ \{t \in \mathbb{R}^+ : F(x_0, y, z, p_0) + \varepsilon_0 e + te \subseteq C, \quad \forall (y, z) \in B \times D\}. \]

Then $\varphi_F(x_0, p_0, \varepsilon_0) \leq t_0 + \delta$. If $\delta \to 0$, then $\varphi_F(x_0, p_0, \varepsilon_0) \leq t_0$. So, $-\varphi_F(x_0, p_0, \varepsilon) \geq -t_0 \geq a$. It follows that $-\varphi_F$ is upper semicontinuous at the point $x_0$. \[ \square \]

**Theorem 3.2.** (Theorem 4.1 in [16]) Assume that $\varphi : S \to Y$, $\varphi \overset{\Gamma_{\mathbb{R}^+}}{\longrightarrow} \varphi$ and $\varphi$ is strongly upper $\mathbb{R}^+$-semi continuous. Then Problem $\min_{x \in S} \varphi(x)$ is extended Hadamard wellposed with respect to $(\varphi_n)$.

**Theorem 3.3.** Suppose that $F_n, F : A \times B \times D \times P \to \mathcal{Y}$ are compact valued, $F_n \overset{\Gamma_{\mathbb{R}^+}}{\longrightarrow} F$ and $F$ is strongly upper $C_{(e)}$-semicontinuous and inner converge continuously. Then Problem $(P_r(F_{x_0}, p_0, \varepsilon))$ is extended Hadamard wellposed with respect to $F_{x_0}$.

**Proof.** By using of Lemma 3.3, Theorem 3.1 and the above theorem, we can obtain the desired conclusion immediately. \[ \square \]
4. CONCLUSION

Well-posedness plays a crucial role in the theory and numerical methods of optimization problems. There are three concepts of well-posedness, namely: Tykhonov well-posedness, Levitin-Polyak well-posedness and Hadamard well-posedness. The two first concepts of well-posedness deal with the behavior of a prescribed class of approximating solution sequences. While the Hadamard well-posedness of a problem means the continuous behavior of the solution with respect to the perturbations of the data. In this article, we introduce two kinds of Hadamard well-posedness for vector parametric quasi-equilibrium problems which include some of the main results in this area. Furthermore, by introducing a gap function, we establish a scalarization for our problem. We obtain a sufficient condition for Hadamard well-posedness of vector parametric quasi-equilibrium problem in terms of the Hadamard well-posedness of the gap function.

REFERENCES