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HADAMARD WELL-POSEDNESS FOR VECTOR PARAMETRIC EQUILIBRIUM PROBLEMS

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Abstract. We consider some extensions of the Hadamard well-posedness notion for vector parametric equilibrium problems. We obtain some necessary and sufficient conditions for Hadamard well-posedness of these problems and also obtain the equivalence between their Hadamard well-posedness and their scalarizations.

Keywords. Extended Hadamard well-posedness; Generalized Hadamard well-posedness; Topological vector space; Vector equilibrium problem.

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1. INTRODUCTION

Well-posedness has played a crucial role in optimization theory. This fact has motivated many authors to study the well-posedness of optimization problems. The first concept of well-posedness is due to Tykhonov [26] for unconstrained optimization problems. Tykhonov well-posedness [26], requires the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Another fundamental generalization of Tykhonov well-posedness for an optimization problem (in the scalar case) is the well-posedness by perturbations due to Dontchev and Zolezzi [7] and Zolezzi [28, 29]. Levitin and Polyak [15] extended the notion to the constrained case. Levitin-Polyak well-posedness requires the existence of the solution and convergence of each Levitin-Polyak minimizing sequence to the solution, where (x_n) is Levitin-Polyak minimizing sequence iff, for all n, x_n to be outside of the feasible set and g is continuous map and K is a closed set and distance between $g(x_n)$ and K tends to zero. There are various notions and generalizations of the concepts of well-posedness, see [4, 6, 12, 13, 14, 27].

The concept of Hadamard well-posedness is due to Hadamard [10]. Another fundamental generalization of Hadamard well-posed was given in [18, 21, 24]. Motivated and inspired by the above works, in this paper, we are investigated some new versions of Hadamard well-posedness for vector parametric

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equilibrium problems. The outline of the paper is as follows: In the first Section, we introduce a generalized parametric vector equilibrium problem and some preliminary results which are used in the sequel. Section 2 deals with notions of Hadamard well-posedness under perturbations for vector equilibrium problem and obtain a sufficient condition from Hadamard well-posedness. In Section 3, by introducing a gap function of our problem, we deduce the equivalent between its Hadamard well-posedness of our problem with the Hadamard well-posedness of its gap function.

Let X be a metric topological vector space, Y be a Hausdorff topological vector space, W and Z be topological vector spaces and P be a metric space. We denote the family of neighborhood of $x \in X$ by $\mathfrak{U}(x)$. Let A, B and D be nonempty sets of X, W and Z, respectively and $C : X \times P \longrightarrow 2^Y$ be a set-valued mapping such that for any $x \in X$ and for any $p \in P$, $C(x,p)$ is a closed, convex and pointed cone in Y such that $intC(x,p) \neq \emptyset$. Assume that $e : X \times P \longrightarrow Y$ is a continuous vector valued mapping satisfying $e(x,p) \in intC(x,p)$. Suppose that $K_1 : A \times P \longrightarrow 2^A$, $K_2 : A \times P \longrightarrow 2^B$ and $K_3 : A \times P \longrightarrow 2^D$ are defined. Let the machinery of the problems be expressed by $F : A \times B \times D \times P \longrightarrow 2^Y$.

For any subsets A and B, we adopt the following notations

$$(u,v)\ r_1\ A \times B \text{ means } \forall u \in A,\ \forall v \in B,$$

$$(u,v)\ r_2\ A \times B \text{ means } \forall u \in A,\ \exists v \in B,$$

$$(u,v)\ r_3\ A \times B \text{ means } \exists u \in A,\ \forall v \in B.$$

For $r \in \{r_1, r_2, r_3\}$, we consider the following vector parametric quasi-equilibrium problem, for given $\varepsilon \in \mathbb{R}^+$ and $p \in P$:

$$(P_r(F,p,\varepsilon)) \qquad \exists \bar{x} \in clK_1(\bar{x},p):\ (y,z)\ r\ K_2(\bar{x},p) \times K_3(\bar{x},p), \quad F(\bar{x},y,z,p) + \varepsilon e(\bar{x},p) \subseteq C(\bar{x},p).$$

We denote the solution set of the above problem by $S_r(F,p,\varepsilon)$.

Remark 1.1. (a) In fact, we deduce those problems that were considered in [3, 24].
(b) If $\varepsilon = 0$, then the solution set of Problem $(P_r(F,p,\varepsilon))$, is the efficient solution of equilibrium problem for set-valued map F, that we denote its efficient solution set with $S_r(F,p)$.
(c) If $D : X \times P \longrightarrow 2^Y$ is a set-valued mapping such that for any $x \in X$ and for any $p \in P$, $D(x,p)$ is a closed, convex and pointed cone in Y such that $\mathrm{int}D(x,p) \neq \emptyset$, then solutions of Problem $(P_r(F,p,\varepsilon))$, with assumption $C(x,p) = Y \setminus \mathrm{int}D(x,p)$ is weakly ε-efficient solutions of F that were considered in works [8, 18]. Furthermore, if $\varepsilon = 0$, then weakly ε-efficient solutions are weakly efficient solutions, see [1, 11, 17, 19] and references therein.

Definition 1.1. Let $T : X \longrightarrow 2^Y$ be a set-valued map. Then,
(a) T is said to be upper semi continuous (u.s.c.), iff for each closed set $B \subset Y$,
$T^-(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$ is closed in X.
(b) T is said to be lower semi continuous (l.s.c.), iff for each open set $B \subset Y$,
$T^-(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$ is open in X.
(c) T is said to be closed iff $Gr(T) = \{(x,y) \in X \times Y :\ y \in T(x), x \in X\}$ is closed in $X \times Y$.

It is often convenient to characterize the upper and lower semicontinuity in terms of nets, as in the following lemma (see, for example, [[2], Theorems 17.16 and 17.19]).

Lemma 1.1. [2] *Let X and Y be topological spaces and $T : X \longrightarrow 2^Y$ be a set-valued map.*
(i) If T has compact values, then T is u.s.c. iff for every net $x_\alpha$in X converging to $x \in X$ and for any net y_α with $y_\alpha \in T(x_\alpha)$, there exist $y \in T(x)$ and a subnet y_{α_i} of y_α converging to y.
(ii) T is l.s.c. iff for any net $x_\alpha$in X converging to $x \in X$ and each $y \in T(x)$, there exists a net y_α converging to y, with $y_\alpha \in T(x_\alpha)$, for all α. □

Motivated by Definition 4.1 in [22], here we define outer converge continuously, inner converge continuously and converge continuously for a sequence of set-valued mas.

Definition 1.2. Let X be a metric topological vector space, Y be a Hausdorff topological vector space, $G_n : X \longrightarrow 2^Y$ be a sequence of set-valued maps and $G : X \longrightarrow 2^Y$ be a set-valued map. The sequence G_n is said to be outer converge continuously (resp. inner converge continuously) to G at x_0 if

$$\limsup_n G_n(x_n) \subseteq G(x_0), (\text{resp. } G(x_0) \subseteq \liminf_n G_n(x_n)) \text{ for all } x_n \longrightarrow x_0,$$

where

$$\liminf_n G_n(x_n) := \{y \in Y : y = \lim_n y_n, y_n \in G_n(x_n) \text{ for sufficiently large } n\},$$

$$\limsup_n G_n(x_n) := \{y \in Y : y = \lim_k y_{n_k}, y_{n_k} \in G_{n_k}(x_{n_k}), \{n_k\} \text{ is a subsequence of } \{n\}\}.$$

The sequence G_n is said to be converge continuously to G at x_0 if $\limsup_n G_n(x_n) \subseteq G(x_0) \subseteq \liminf_n G_n(x_n)$, for all $x_n \longrightarrow x_0$. If G_n converges continuously to G at every point $x \in X$, then it is said that G_n converges continuously to G on X.

2. Hadamard Well-posedness

Here we define the notion of Γ_{Crp}-convergence for a sequence of set-valued functions similar to the Definition 2.1 in [20].

Definition 2.1. (Definition 2.1 in [20]) Let X be a topological space, Y be a topological vector space and $C \subseteq Y$ be a closed, convex and pointed cone such that $intC \neq \emptyset$ and $\mathfrak{U}(x)$ be the family of neighborhoods of x. Let $f_n, f : X \longrightarrow \bar{Y}$, $n \in \mathbb{N}$ be given functions, We say that (f_n) is Γ_C-convergence to f and we shall write $f_n \xrightarrow{\Gamma_C} f$, if for every $x \in X$, the following statements are true:
(i) for all $U \in \mathfrak{U}(x)$ and for all $\varepsilon \in \mathbb{R}^+$, there exists $n_{\varepsilon,U} \in \mathbb{N}$ such that

$$\forall\, n \geq n_{\varepsilon,U}\ \exists x_n \in U\ f_n(x_n) \leq f(x) + \varepsilon e,$$

(ii) for all $\varepsilon \in \mathbb{R}^+$ there exist $U_\varepsilon \in \mathfrak{U}(x)$ and n_ε such that

$$\forall\, \bar{x} \in U_\varepsilon,\ \forall\, n \geq n_\varepsilon\ f_n(\bar{x}) \geq f(x) - \varepsilon e.$$

Definition 2.2. Suppose for all $n \in \mathbb{N}$, $F_n, F : A \times B \times D \times P \longrightarrow 2^Y$ are defined. Then, the sequence (F_n) is said to be Γ_{Crp}- convergence to F and we denote by $F_n \overset{\Gamma_{Crp}}{\longrightarrow} F$, iff for every $x \in X$ the following statements hold,
(i) for all $U \in \mathfrak{U}(x)$ and for all $\varepsilon \in \mathbb{R}^+$, there exists $n_{\varepsilon,U} \in \mathbb{N}$ such that for all $n \geq n_{\varepsilon,U}$ there exists $x_n \in U$ such that

$$(y,z)rK_2(x,p) \times K_3(x,p), \quad F_n(x_n,y,z,p) - F(x,y,z,p) - \varepsilon e(x,p) \subseteq -C(x,p),$$

(ii) for all $\varepsilon \in \mathbb{R}^+$ there exist $U_\varepsilon \in \mathfrak{U}(x)$ and $n_\varepsilon \in \mathbb{N}$ such that for all $\bar{x} \in U_\varepsilon$ and for all $n \geq n_\varepsilon$, we obtain

$$(y,z)rK_2(x,p) \times K_3(x,p), \quad F(x,y,z,p) - F_n(\bar{x},y,z,p) - \varepsilon e(x,p) \subseteq -C(x,p).$$

Now, we can define some new notions of Hadamard well-posedness for vector parametric equilibrium problem that include Definitions 2.5 and 2.6 in [16], Definition 4 in [24] and Definitions 4.1 and 4.2 in [9].

Definition 2.3. Let $(\lambda_n, p_n) \subseteq \Lambda \times P$ be a sequence converging to (λ_0, p_0) and $F_{\lambda_n} \overset{\Gamma_{Crp}}{\longrightarrow} F_{\lambda_0}$. The Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$, is said to be
(a) Hadamard well-posed corresponding to (F_{λ_n}), iff:
(i) there exists only one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) for all sequence $(\varepsilon_n) \subseteq \mathbb{R}^+$ that $\varepsilon_n \longrightarrow \varepsilon$,

$$\limsup_n [S_r(F_{\lambda_n}, p_n, \varepsilon_n)] \subseteq S_r(F_{\lambda_0}, p_0, \varepsilon).$$

(b) Generalized Hadamard well-posed corresponding to (F_{λ_n}), iff:
(i) there exists one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) for all sequence $(\varepsilon_n) \subseteq \mathbb{R}^+$ that $\varepsilon_n \longrightarrow \varepsilon$, condition (ii) of part (a) holds.
(c) Extended Hadamard well-posed corresponding to (F_{λ_n}), iff
(i) there exists one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) there exists $\varepsilon_0 \in \mathbb{R}^+$ such that for all $0 \leq \varepsilon' \leq \varepsilon_0$,

$$\limsup_n [S_r(F_{\lambda_n}, p_\alpha, \varepsilon')] \subseteq S_r(F_{\lambda_0}, p_0, \varepsilon').$$

Example 2.1. Let $X = P = \mathbb{R}$, $Y = \mathbb{R}^2$, $K_1(x,p) = K_2(x,p) = K_3(x,p) = [-1,0]$, for all $x \in X$ and $p \in P$, $r = r_1$, $C(x,p) = \mathbb{R}^2{}_+$, $e(x,p) = (1,1)$ and $F : X \times X \times X \times P \longrightarrow 2^Y$ be defined by $F(x,y,z,p) = \{(x,x)\}$ and for all $n \in \mathbb{N}$, $F_n : X \times X \times X \times P \longrightarrow 2^Y$ be defined by $F_n(x,y,z,p) = \{(x + \frac{1}{n}, t) : \ t \in [x, x + \frac{1}{n}[\}$. We show that $F_n \overset{\Gamma_{Crp}}{\longrightarrow} F$. In fact for all $x \in X$, $U \in \mathfrak{U}(x)$ and $\varepsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$ and for all $n \geq N$, $x - \frac{1}{n} \in U$. Therefore, for $x_n = x - \frac{1}{n}$, we have

$$\begin{aligned} F_n(x_n,y,z,p) - F(x,y,z,p) - \varepsilon(1,1) &= (x_n + \frac{1}{n}, t) - (x,x) - (\varepsilon,\varepsilon) && (2.1)\\ &= (x - \frac{1}{n} + \frac{1}{n}, t) - (x,x) - (\varepsilon,\varepsilon) && (2.2)\\ &= (0, t-x) - (\varepsilon,\varepsilon) && (2.3)\\ &= (-\varepsilon, t-x-\varepsilon), && (2.4) \end{aligned}$$

that is, $t \in [x_n, x_n + \frac{1}{n}[= [x - \frac{1}{n}, x[$ and

$$t - x - \varepsilon < x - x - \varepsilon < -\varepsilon < 0. \tag{2.5}$$

Therefore, $F_n(x_n, y, z, p) - F(x, y, z, p) - \varepsilon(1,1) \subseteq -C(x, p)$ and condition (i) of Definition 2.2 holds. For condition (ii) of Definition 2.2, for each $x \in X$ and $\varepsilon \in \mathbb{R}^+$, we suppose that $U_\varepsilon = B_\varepsilon(x)$ (where $B_\varepsilon(x)$ is the ball with center x and radius ε). Since there exists $N \in \mathbb{N}$ such that $\frac{2}{N} < \varepsilon$, for all $n \geq N$ and $x^{'} \in U_\varepsilon$, we have

$$F(x, y, z, p) - F_n(x^{'}, y, z, p) - \varepsilon(1,1) = (x, x) - (x^{'} + \frac{1}{n}, t) - (\varepsilon, \varepsilon) \tag{2.6}$$

$$= (x - x^{'} - \frac{1}{n} - \varepsilon, x - t - \varepsilon), \tag{2.7}$$

but $x - x^{'} - \frac{1}{n} - \varepsilon = (x - x^{'} - \varepsilon) - \frac{1}{n} < -\frac{1}{n} < 0$. On the other hand $\frac{1}{n} < \frac{2}{N} < \varepsilon$, we have

$$x - t - \varepsilon < x - x^{'} - \varepsilon < 0,$$

and for all $n \geq N$ and $x^{'} \in U_\varepsilon$, we obtain $F(x, y, z, p) - F_n(x^{'}, y, z, p) - \varepsilon(1,1) \subseteq -C(x, p)$.

Obviously, Problem $(P_r(F, p, \varepsilon))$ is extended Hadamard well-posed corresponding to (F_n). Since, for all $\varepsilon \in \mathbb{R}^+$, $S_{r_1,p}(F, p, \varepsilon) = S_{r_1,p}(F_n, p, \varepsilon) = [-\varepsilon, 0]$. It follows that

$$\limsup_n [S_{r_1,p}(F_n, p, \varepsilon)] \subseteq S_{r_1}(F, p, \varepsilon).$$

Note that, if Problem $(P_r(F, p, \varepsilon))$ is Hadamard well-posed corresponding to (F_{λ_n}), it is also generalized Hadamard well-posed corresponding to (F_{λ_n}). In the following result, we show that if Problem $(P_r(F, p, \varepsilon))$ is extended Hadamard well-posed corresponding to (F_{λ_n}), it is also generalized Hadamard well-posed corresponding to (F_{λ_n}).

Proposition 2.1. *Let $(\lambda_n, p_n) \subseteq \Lambda \times P$ be a sequence converging to (λ_0, p_0) and $F_{\lambda_n} \overset{\Gamma_{Crp_0}}{\longrightarrow} F_{\lambda_0}$. If Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$, is extended Hadamard well-posed corresponding to (F_{λ_n}), then Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$, is generalized Hadamard well-posed corresponding to (F_{λ_n}).*

Proof. The proof with minor modifications is similar to the proof of Proposition 2.2 in [16], therefore, it is omitted. □

Motivated by an idea of Tam and Loan in [25] and by using Proposition 2.1, we can deduce some sufficient conditions for generalized Hadamard well-posedness.

Theorem 2.1. *Suppose that X is a metric vector topological space, Y is a topological vector space and $(\lambda_n, p_n) \subseteq \Lambda \times P$ is a sequence converging to (λ_0, p_0), $F_{\lambda_n}, F_{\lambda_0} : X \longrightarrow 2^Y$ and the following conditions hold:*
(i) $F_{\lambda_n} \overset{\Gamma_{Cr_1p_0}}{\longrightarrow} F_{\lambda_0}$;
(ii) for all $\varepsilon \in \mathbb{R}^+$, the solution set of Problem $(P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))$ is nonempty;
(iii) clK_1 is upper semi continuous and compact valued;
(iv) K_2 and K_3 are lower semi continuous;

(v) F_{λ_n} *is inner converge continuously to* F_{λ_0}.
Then, Problem $(P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))$ *is extended Hadamard well-posed corresponding to* (F_{λ_n}).

Proof. We show for all $\varepsilon \in \mathbb{R}^+$

$$\limsup_n [S_{r_1}(F_{\lambda_n}, p_n, \varepsilon)] \subseteq S_{r_1}(F_{\lambda_0}, p_0, \varepsilon).$$

Suppose that $w \in \limsup_n [S_{r_1}(F_{\lambda_n}, p_n, \varepsilon)]$ and $w \notin S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$. Then, there exist a subsequence of $(S_{r_1}(F_{\lambda_n}, p_n, \varepsilon))$ and sequence w_{n_k} such that for all n_k, $w_{n_k} \in S_{r_1}(F_{\lambda_{n_k}}, p_{n_k}, \varepsilon)$ and $w_{n_k} \longrightarrow w$. Since $w_{n_k} \in S_{r_1}(F_{\lambda_{n_k}}, p_{n_k}, \varepsilon)$, one sees that $w_{n_k} \in clK_1(w_{n_k}, p_{n_k})$ and for all $(y,z) \in K_2(w_{n_k}, p_{n_k}) \times K_3(w_{n_k}, p_{n_k})$

$$F_{\lambda_{n_k}}(w_{n_k}, y, z, p_{n_k}) + \varepsilon e(w_{n_k}, p_{n_k}) \subseteq C(w_{n_k}, p_{n_k}). \tag{2.8}$$

But $w_{n_k} \in clK_1(w_{n_k}, p_{n_k})$ and clK_1 is upper semi continuous and compact valued. Then $w \in clK_1(w, p)$. Since $w \notin S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$, one sees that

$$\exists (y_0, z_0) \in K_2(w, p_0) \times K_3(w, p_0) \; : \; F_{\lambda_0}(w, y_0, z_0, p_0) + \varepsilon e(w, p_0) \nsubseteq C(w, p_0). \tag{2.9}$$

Therefore, there exists $u_0 \in F_{\lambda_0}(w, y_0, z_0, p_0)$ such that $u_0 + \varepsilon e(w, p_0) \notin C(w, p_0)$. Since K_2 and K_3 are lower semi continuous, one finds that there exists a sequence

$$(y_{n_k}, z_{n_k}) \in K_2(w_{n_k}, p_{n_k}) \times K_3(w_{n_k}, p_{n_k})$$

such that $(y_{n_k}, z_{n_k}) \longrightarrow (y_0, z_0)$. From (2.8),

$$F_{\lambda_{n_k}}(w_{n_k}, y_{n_k}, z_{n_k}, p_{n_k}) + \varepsilon e(w_{n_k}, p_{n_k}) \subseteq C(w_{n_k}, p_{n_k}), \tag{2.10}$$

as $(F_{\lambda_{n_k}})$ is inner converge continuously to F_{λ_0} at (w, y_0, z_0, p_o), therefore

$$u_0 \in F_{\lambda_0}(w, y_0, z_0, p_0) \subseteq \liminf_{n_k} F_{\lambda_{n_k}}(w_{n_k}, y_{n_k}, z_{n_k}, p_{n_k}).$$

So, there exists (u_{n_k}) such that $u_{n_k} \in F_{\lambda_{n_k}}(w_{n_k}, y_{n_k}, z_{n_k}, p_{n_k})$ and $u_{n_k} \longrightarrow u_0$. By using (2.10), $u_{n_k} + \varepsilon e(w_{n_k}, p_{n_k}) \subseteq C(w_{n_k}, p_{n_k})$, and since e is continuous and C is closed, one find that $u_0 + \varepsilon e(w, p_0) \in C(w, p_0)$, which is a contradiction. This completes the proof. □

Remark 2.1. (a) In the previous theorem, if we replace condition (iv) by the following condition:
$(iv)'$ K_2 is lower semi continuous on $A \times P$ and K_3 is upper semi continuous and compact valued on $A \times P$. Then, with minor modifications in the proof, one can conclude the extended Hadamard well-posedness of the Problem $(P_{r_2}(F_{\lambda_0}, p_0, \varepsilon))$.
(b) In the previous theorem, if we replace condition (iv) by the following condition:
$(iv)''$ K_2 is upper semi continuous and compact valued on $A \times P$ and K_3 is lower semi continuous on $A \times P$.
Then, with minor modifications in the proof, one can deduce the extended Hadamard well-posedness of the Problem $(P_{r_3}(F_{\lambda_0}, p_0, \varepsilon))$.

The next example shows that our assumptions in Theorem 2.3 is weaker than Salamon's assumptions in [24].

Example 2.2. For $r=1$ and fixed p, let $K_1, K_2 : [-1,1]\times\{p\} \longrightarrow [-1,1]$, $K_1(x,p)=K_2(x,p)=[-1,1]$ and $K_3 : [-1,1]\times\{p\} \longrightarrow \{1\}$ and let $F : [-1,1]\times[-1,1]\times\{1\}\times\{p\} \longrightarrow \{0,1\}$ and for all $n\in\mathbb{N}$, $F_n : [-1,1]\times[-1,1]\times\{1\}\times\{p\} \longrightarrow \{0,1\}$ be defined by

$$F(x,y,1,p) = \begin{cases} 0 & x=0, \\ \\ 1 & \text{o.w}, \end{cases}$$

$$F_n(x,y,1,p) = \begin{cases} 0 & x\in]-\frac{1}{n},\frac{1}{n}[, \\ \\ 1 & \text{o.w}\,, \end{cases}$$

For all $x,p\in[-1,1]$, let $C(x,p)=[0,+\infty[$. Since condition (iii) of Theorem 1 in [24] doesn't hold, therefore we can not achieve any results. But the conditions of Theorem 2.3 hold, therefore this problem is extended Hadamard well-posed corresponding by (F_n).

The next theorem and its corollary show some alternative characterization for extended Hadamard well-posedness and Hadamard well-posedness of Problem $(P_r(F,p,\varepsilon))$. In fact, we obtain a relation between Hadamard well-posed of Problem $(P_r(F,p,\varepsilon))$ and its approximate solutions. For this idea, we need to define approximate solutions of Problem $(P_r(F,p,\varepsilon))$, denoted by:

$$\begin{aligned}&\Pi_r(F,p,\varepsilon,\delta)\\ &=\{x\in clK_l(x,p):\ F(x,y,z,p)+\varepsilon e(x,p)+\delta e(x,p)\subseteq C(x,p),\ (y,z)rK_2(x,p)\times K_3(x,p)\}.\end{aligned}$$

Obviously, if $\delta_1\leq\delta_2$, then $\Pi_r(F,p,\delta_1)\subseteq\Pi_r(F,p,\delta_2)$ and

$$S_r(F,p,\varepsilon)=\cap_{\delta>0}\Pi_r(F,p,\varepsilon,\delta)=\Pi_r(F,p,\varepsilon,0).$$

The following theorem and its corollary improve Theorems 4.8 and 4.10 in [9].

Theorem 2.2. *Let X be a metric topological vector space, Y be a Hausdorff topological vector space and (λ_n,p_n) be a sequence converges to (λ_0,p_0), $\bar{E}$ (where, $\bar{E}(p_0)=\{x\in X:\ x\in clK_1(x,p_0)\}$) be a compact-valued and upper semi continuous set-valued map and $F_{\lambda_n}\overset{\Gamma_{Cr_1p_0}}{\longrightarrow}F_{\lambda_0}$. If Problem $(P_{r_1}(F_{\lambda_0},p_0,\varepsilon))$ is extended Hadamard well-posed corresponding to (F_{λ_n}), then there exists a nonempty and compact subset M of $S_{r_1}(F_{\lambda_0},p_0,\varepsilon)$ such that for every neighborhood U of 0_X, there exists $\delta>0$ such that*

$$x\in\Pi_{r_1}(F_{\lambda_n},p_n,\varepsilon,\delta)\implies x\in M+V.$$

Conversely, if
(i) there exists a nonempty compact subset M of $S_{r_1}(F_{\lambda_0},p_0,\varepsilon)$ such that for every neighborhood V of 0, there exists $\delta>0$ such that

$$x\in\Pi_{r_1}(F_{\lambda_n},p_n,\varepsilon,\delta)+V\implies x\in M+V;$$

(ii) Π_{r_1}, is upper semi continuous in its third argument.
Then, Problem $(P_{r_1}(F_{\lambda_0},p_0,\varepsilon))$, is extended Hadamard well-posed corresponding to (F_{λ_n}).

Proof. Suppose that Problem $(P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))$ is extended Hadamard well-posed corresponding to (F_{λ_n}). Let $M = S_{r_1}(F_{\lambda_0}, p_0, \varepsilon) \neq \emptyset$. We will show that $S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$ is a compact set. If (x_n) is a sequence in $S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$, then

$$(y,z)r_1K_2(x_n, p_0) \times K_3(x_n, p_0), \quad F_{\lambda_0}(x_n, y, z, p_0) + \varepsilon e(x_n, p_0) \subseteq C(x_n, p_0). \tag{2.11}$$

Since $F_{\lambda_n} \overset{\Gamma_{Cr_1p_0}}{\longrightarrow} F_{\lambda_0}$, one sees that if $(\delta_n) \subseteq \mathbb{R}^+$ and $\delta_n \longrightarrow 0$. For any δ_n, $(y,z)r_1K_2(x_n, p_0) \times K_3(x_n, p_0)$

$$F_{\lambda_0}(x_n, y, z, p_0) - F_{\lambda_n}(x_n, y, z, p_0) - \delta_n e(x_\alpha, p_0) \subseteq -C(x_n, p_0). \tag{2.12}$$

By summing (2.11) and (2.12), we obtain

$$(y,z)r_1K_2(x_n, p_0) \times K_3(x_n, p_0), \;\; F_{\lambda_n}(x_n, y, z, p_0) + \varepsilon e(x_n, p_0) + \delta_n e(x_n, p_0) \subseteq C(x_n, p_0). \tag{2.13}$$

Therefore $x_n \in S_{r_1}(F_{\lambda_n}, p_0, \varepsilon + \delta_n)$. Since $x_n \in \bar{E}(p_0)$, one sees that (x_n) has a subsequence converges to some x_0 of $\bar{E}(p_0)$. Let for all n, $p_n = p_0$ and $\varepsilon_n = \varepsilon + \delta_n$. Since Problem $(P_{r_1}(F_{\lambda_0}, p_0, \varepsilon))$ is generalized Hadamard well-posed corresponding to (F_{λ_n}), one finds that

$$\limsup_n [S_{r_1}(F_{\lambda_n}, p_0, \varepsilon_n)] \subseteq S_{r_1}(F_{\lambda_0}, p_0, \varepsilon),$$

and $x_0 \in S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$. Therefore $S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$ is a compact set.

Now, we show the existence of δ for any neighborhood of 0_X. On the contrary, there existence a neighborhood U of 0_X and a sequence $(\delta_n) \subseteq \mathbb{R}^+$ such that $\delta_n \longrightarrow 0$ and $(x_n) \subseteq X$ such that for all n, $x_n \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta_n)$, but $x_n \notin S_{r_1}(F_{\lambda_0}, p_0, \varepsilon) + U$. Since $x_n \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta_n)$, one obtains that $x_n \in S_{r_1}(F_{\lambda_n}, p_n, \varepsilon + \delta_n)$. On the other hand, $x_n \in \bar{E}(p_n)$. Using our assumption, one obtains that there exists $x_0 \in \bar{E}(p_0)$ and subsequence x_{n_k} such that $x_{n_k} \longrightarrow x_0$. But Problem $(P_{r_1}(F_{\lambda_0}, p, \varepsilon))$ is generalized Hadamard well-posed corresponding to (F_{λ_n}). Then

$$\limsup_n [S_{r_1}(F_{\lambda_{n_k}}, p_{n_k}, \varepsilon + \delta_{n_k})] \subseteq S_{r_1}(F_{\lambda_0}, p_0, \varepsilon),$$

and $x_0 \in S_{r_1}(F_{\lambda_0}, p_0, \varepsilon)$, which is a contradiction.

Conversely, we show that for any sequence $(\lambda_n, p_n) \subseteq \Lambda \times P$ converges to (λ_0, p_0), $\varepsilon_n \longrightarrow \varepsilon$ and $F_{\lambda_n} \overset{\Gamma_{Cr_1p_0}}{\longrightarrow} F_{\lambda_0}$. Note that

$$\limsup_n [S_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n)] \subseteq S_{r_1}(F_{\lambda_0}, p_0, \varepsilon).$$

Let $\bar{x} \in \limsup_n [S_{r_1}(F_{\lambda_n}, p_0, \varepsilon_n)]$. Hence, there exists a sequence (x_n) such that $x_n \longrightarrow \bar{x}$ and $x_n \in S_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n)$. For completing the proof, we show that (x_n) contains a subsequence converges to a point $x_0 \in M$. Suppose on the contrary that for all $x \in M$ there exists a neighborhood U_x of 0 such that $\{x\} + U_x$ does not contain any subsequence of x_n. Also, for all $x \in M$, there exists a neighborhood V_x of 0 such that $V_x + V_x \subseteq U_x$. Since $M \subseteq \bigcup_{x \in M}(\{x\} + V_x)$ and M is compact, we see that there exists $n \in \mathbb{N}$ such that

$$M \subseteq \bigcup_{i=1}^{n}(\{x_i\} + V_{x_i}).$$

Letting $V = \bigcap_{i=1}^{n} V_{x_i}$, we find from the assumption that there exists $\delta > 0$ such that

$$x \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta) + V \;\Rightarrow\; x \in M + V.$$

Obviously, there exists n_1 such that for all $n \geq n_1$,

$$\Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n, \delta_n) \subseteq \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n, \delta).$$

On the other hand, $x_n \in S_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n)$. Then $x_n \in \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n, 0) \subseteq \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n, \delta_n)$. Since Π_{r_1} is upper semi continuous in its third argument and $\varepsilon_n \longrightarrow \varepsilon$, we see that there exists n_2 such that for all $n \geq n_2$, $\Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n, \delta) \subseteq \Pi_{r_1}(F_{\lambda_n}, p_n, \varepsilon, \delta) + V$. Now, if $n \geq \max\{n_1, n_2\}$, then $x_n \in \Pi_r(F_{\lambda_n}, p_0, \delta) + V$. Therefore, $x_n \in M + V$. But

$$\begin{aligned} M + V \subseteq \bigcup_{i=1}^{n}(\{x_i\} + V_{x_i}) + V \quad &\subseteq \quad \bigcup_{i=1}^{n}(\{x_i\} + V_{x_i} + V) \\ \subseteq \bigcup_{i=1}^{n}(\{x_i\} + V_{x_i} + V_{x_i}) \quad &\subseteq \quad \bigcup_{i=1}^{n}(\{x_i\} + U_{x_i}). \end{aligned}$$

Hence, for those $n \geq \max\{n_1, n_2\}$, $x_n \in \bigcup_{i=1}^{n}(\{x_i\} + U_{x_i})$, which is a contradiction, since for all $x \in M$, $\{x\} + U_x$ does not contain any subsequence of x_n, but as the limit is unique. Therefore, $x_0 = \bar{x}$. □

Now, we obtain the Tykhonov well-posedness and the Levitin-Polayk well-posedness from the Hadamard well-posedness.

Definition 2.4. [5] The Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$, is said to be
(a) Tykhonov wellposed iff
(i) there exists only one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) for any sequence $(\lambda_n, p_n) \subseteq \Lambda \times P$ converging to (λ_0, p_0), every asymptotically solving sequence for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ corresponding to (λ_n, p_n), converges to $S_r(F_{\lambda_0}, p_0, \varepsilon)$.
(b) Tykhonov well-posed in the general sense iff
(i) there exists one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) for any sequence $(\lambda_n, p_n) \subseteq \Lambda \times P$ converging to (λ_0, p_0), every asymptotically solving sequence for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ corresponding to (λ_n, p_n), contains a susequence converges to some point of $S_r(F_{\lambda_0}, p_0, \varepsilon)$.

Definition 2.5. [4] Let X and Y be two metric spaces, $(\lambda_n, p_n) \subseteq \Lambda \times P$ be a sequence converging to (λ_0, p_0). A sequence $\{x_n\} \subset A$ is said to be
(a) type I LP asymptotically solving sequence corresponding to (λ_n, p_n), for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$, if $x_n \in clK_1(x_n, p_n)$ and there exists a sequence $(\varepsilon_n) \subseteq \mathbb{R}^+$ with $\varepsilon_n \longrightarrow 0$ such that

$$(y,z)\ r\ K_2(x_n, p_n) \times K_3(x_n, p_n), \quad F(x_n, y, z, p_n) + \varepsilon_n e(x_n, \lambda_n, p_n) \subseteq C(x_n, p_n). \tag{2.14}$$

(b) type II LP asymptotically solving sequence corresponding to (λ_n, p_n), for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$, if there exists a sequence $(\varepsilon_n) \subseteq \mathbb{R}^+$ with $\varepsilon_n \longrightarrow 0$ such that

$$d(x_n, K_1(x_n, p_n)) \leq \varepsilon_n \tag{2.15}$$

and (2.14) holds.
(c) The Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ is LP well-posed of type I (resp. type II) if and only if

(i) there is only one solution for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$;
(ii) for any sequence $(\lambda_n, p_n) \subseteq \Lambda \times P$, which converges to (λ_0, p_o), every LP of type I (resp. type II) asymptotically solving sequence for Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ corresponding to (λ_n, p_n), converges to $S_r(F_{\lambda_0}, p_0, \varepsilon)$.

In the following result, we obtain a generalization of Theorem 2.2 in [21]. As a matter of fact, Revalski in [21], have shown that if X is a real Banach space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex lower semi-continuous extended real-valued function, then the Tykhonov well-posedness and the Levitin-Polayk well-posedness are deduced from Hadamard well-posedness. Here, we obtain those results for set-valued maps in topological spaces.

Theorem 2.3. *Let $(\lambda_n, p_n) \subseteq \Lambda \times P$ be a sequence converging to (λ_0, p_0) for all n, $F_{\lambda_n} = F_{\lambda_0}$. If Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ is Hadamard well-posed corresponding to (F_{λ_n}) and one of the following conditions holds:*
(i) clK_1 is compact valued and upper semi-continuous in the first argument;
(ii) clK_1 is closed map.
Then, Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ is Tykhonov well-posed, type I Levitin-Polayk well-posed and type II Levitin-Polayk well-posed.

Proof. (i) It suffices to prove for generalized Hadamard well-posed. Other case obtains from Proposition 2.1. Suppose Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ is generalized Hadamard well-posed and clK_1 is compact valued and upper semi-continuous. If (x_n) is an asymptotically solving sequence, (resp. type I LP asymptotically solving sequence and type II LP asymptotically solving sequence), since for all n, $x_n \in clK_1(x_n, p_0)$, then there exists $x_0 \in clK_1(x_0, p_0)$ such that $x_n \longrightarrow x_0$. On the other hand, (x_n) is an asymptotically solving sequence (resp. type I LP asymptotically solving sequence and type II LP asymptotically solving sequence) then for all n, $x_n \in S_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n)$. Since for all n, $F_{\lambda_n} = F_{\lambda_0}$ and $x_0 \in \limsup_n [S_{r_1}(F_{\lambda_n}, p_n, \varepsilon_n))] \subseteq S_{r_1}(F_{\lambda_0}, p_0)$, so the proof is complete. □

If condition (ii) holds, then (x_n) has a subsequence converges to x_0. The remaining proof is similar to the proof of part (i).

3. A SCALARIZATION FOR HADAMARD WELL-POSEDNESS

In this section, let $(\lambda_n, p_n) \subseteq \Lambda \times P$ be a sequence converging to (λ_0, p_0). We shall show that the Hadamard well-posedness of Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ reaches from the Hadamard wellposed of scalar optimization problem. In this Section, we suppose that maps C, e, K_2, K_3 are constant maps and therefore, for all $x \in X$ and $p \in P$, $C(x, p) = C$ and $e(x, p) = e \in \text{int}C$, $K_2(x, p) = B$ and $K_3(x, p) = D$.

Here, we use a modified version of a result of Sach [23] for obtaining a nonlinear scalarization function to define a gap function for problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$.

Following the idea in [23], we need to define the following notations.

Definition 3.1. Let $Q \subset Y$, C be a closed convex cone in a topological vector space Y. Then
(i) Q is called C-bounded if for each neighborhood U of the origin of Y there exists a positive t, such that

$Q \subset C + tU$.

(ii) Q is called $-C$-closed if $Q - C$ is closed.

Remark 3.1. One can show that when Q is C-compact, then Q is $-C$-closed and C-bounded. If the set-valued function F satisfies condition(i) (resp. (ii)) of Definition 3.1 at each point of $A \times B \times D \times P$, then we say that F is C-bounded (resp. $-C$-closed). It is evident that if F has bounded values in Y, then it is C-bounded and furthermore, if F has compact values in Y, then F is simultaneously C-bounded and $-C$-closed .

The proof of the following results are similar to the corresponding ones in [23], with replacing C by $-C$, therefore it is omitted.

Lemma 3.1. *Let $Q \subset Y$, C be a closed convex cone in Y with $\mathrm{int}C \neq \emptyset$ and $e \in \mathrm{int}\, C$. For a subset Q of Y, $e \in \mathrm{int}C$ and $\varepsilon > 0$, we have*

(i) If Q is C-bounded, then $s_Q := \min\{t \geq 0 :\ Q + te \subseteq C\}$ is well-defined.

(ii) If Q is C-bounded, then $s_Q = 0$ iff $Q \subseteq C$ and $s_Q \leq \varepsilon$ iff $Q \subseteq C - \varepsilon e$.

Remark 3.2. Let F be a set-valued map with compact values. Using the Remark 3.1, all of the consequences of Lemma 3.1 is valid for

$\varphi_F :\ X \times P \times \mathbb{R} \longrightarrow \mathbb{R}$ which is defined by

$$\varphi_F(x, p, \varepsilon) := \min\{t \in \mathbb{R}^+ :\ F(x, y, z, p) + \varepsilon e + te \subseteq C\ \forall\ (y, z) \in B \times D\}. \tag{3.1}$$

Obviously, $\bar{x} \in S_{r_1}(F, p, \varepsilon)$ iff, $\varphi_F(\bar{x}, p, \varepsilon) = 0$.

Lemma 3.2. *Let $(\lambda_n, p_n) \subseteq \Lambda \times P$ be a sequence converging to (λ_0, p_0) and $F_{\lambda_n} \xrightarrow{\Gamma_{Crp_0}} F_{\lambda_0}$. Then $-\varphi_{F_{\lambda_n}} \xrightarrow{\Gamma_{\mathbb{R}^+}} -\varphi_{F_{\lambda_0}}$.*

Proof. Suppose that $F_{\lambda_n} \xrightarrow{\Gamma_{Crp_0}} F_{\lambda_0}$. For every $\bar{x} \in X$, conditions (i) and (ii) of Definition 2.2 hold. Since $F_{\lambda_n} \xrightarrow{\Gamma_{Crp_0}} F_{\lambda_0}$, of condition (i) of Definition 2.2, for all $U \in \mathfrak{U}(\bar{x})$ and for all $\varepsilon \in \mathbb{R}^+$, there exists $\alpha_{\varepsilon,U}$ such that for all $n \geq n_{\varepsilon,U}$ and there exists $x_n \in U$ such that for all $y, z \in B$

$$F_{\lambda_n}(x_n, y, z, p_0) - F_{\lambda_0}(\bar{x}, y, z, p_0) - \varepsilon e \subseteq -C. \tag{3.2}$$

If t belongs to

$$\{t \in \mathbb{R}^+ :\ F_{\lambda_n}(x_n, y, z, p_0) + \varepsilon' e + te \subseteq C\ \forall\ (y, z) \in B \times D\}, \tag{3.3}$$

then by summing (3.2) and (3.3), we obtain $t + \varepsilon$ belongs to

$$\{t \in \mathbb{R}^+ :\ F_{\lambda_0}(\bar{x}, y, z, p_0) + \varepsilon' e + te \subseteq C\ \forall\ (y, z) \in B \times D\}.$$

Therefore, $\varphi_{\lambda_0}(\bar{x}, p_0, \varepsilon') \leq t + \varepsilon$ and $\varphi_{\lambda_0}(\bar{x}, p_0, \varepsilon') \leq \varphi_{\lambda_n}(x_\alpha, p_0, \varepsilon') + \varepsilon$. Then condition (i) of Definition 2.1 holds. From condition (ii) of Definition 2.2, we obtain for all $\varepsilon \in \mathbb{R}^+$ there exist $U_\varepsilon \in \mathfrak{U}(\bar{x})$ and n_ε such that for all $x \in U_\varepsilon$, for all $n \geq n_\varepsilon$, and for all $y, z \in S$, we have

$$F_{\lambda_0}(\bar{x}, y, z, p_0) - F_{\lambda_n}(x, y, z, p_0) - \varepsilon e \subseteq -C. \tag{3.4}$$

If t belongs to

$$\{t \in \mathbb{R}^+ : \ F_{\lambda_0}(\bar{x}, y, z, p_0) + \varepsilon' e + te \subseteq C \ \forall \ (y,z) \in B \times D\}, \tag{3.5}$$

by summing (3.4) and (3.5), we deduce $t+\varepsilon$ belonging to

$$\{t \in \mathbb{R}^+ : \ F_{\lambda_n}(x, y, z, p_0) + \varepsilon' e + te \subseteq C \ \forall \ (y,z) \in B \times D\}.$$

So, $\varphi_{\lambda_n}(x, p_0, \varepsilon') \leq \varphi_{\lambda_0}(\bar{x}, p_0, \varepsilon') + \varepsilon$, and condition (ii) of Definition 2.1 holds and $-\varphi_{\lambda_\alpha} \overset{\Gamma_{\mathbb{R}^+}}{\longrightarrow} -\varphi_{\lambda_0}$. □

In the following theorem, we obtain an equivalence relation between generalized Hadamard well-posedness of Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ and the generalized Hadamard well-posedness of scalar optimization problems.

Theorem 3.1. *Suppose that (λ_n, p_n) is a sequence converging to (λ_0, p_0) and F_{λ_0} and F_{λ_n} have compact values and $F_{\lambda_n} \overset{\Gamma_{Crp_0}}{\longrightarrow} F_{\lambda_0}$, such that F_{λ_n} is inner converge continuously to F_{λ_0}. Then Problem $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ is generalized Hadamard wellposed corresponding to (F_{λ_n}) if the following scalar optimization problem*

$$(OP(\varphi_{F_{\lambda_0}}, p_0, \varepsilon)) \ \max_{x \in S} -\varphi_{F_{\lambda_0}}(x, p_0, \varepsilon)$$

is generalized Hadamard wellposed corresponding to $(-\varphi_{F_{\lambda_n}})$ defined in Remark 3.2.

Proof. We denote the solution set of Problem $(OP(\varphi_{F_{\lambda_0}}, p_0, \varepsilon))$ by $eff(-\varphi_{F_{\lambda_0}}, p, \varepsilon))$. Suppose that $(\varepsilon_n) \subseteq \mathbb{R}^+$, $\varepsilon_n \longrightarrow \varepsilon'$ and Problem $(OP(\varphi_{F_{\lambda_0}}, p_0, \varepsilon))$ is generalized Hadamard wellposed corresponding to $(-\varphi_{F_{\lambda_n}})$. Then

$$\limsup_n [eff(-\varphi_{F_{\lambda_n}}, p_n, \varepsilon_n)] \subseteq eff(-\varphi_{F_{\lambda_0}}, p_0, \varepsilon').$$

If $x' \in \limsup_n [S_r(F_{\lambda_n}, p_n, \varepsilon_n)]$, then there exists (x'_{n_k}) such that for all n_k, $x'_{n_k} \in S_r(F_{\lambda_{n_k}}, p_{n_k}, \varepsilon_{n_k})$ and $x'_{n_k} \longrightarrow x'$. Since $x'_{n_k} \in S_{r_1}(F_{\lambda_{n_k}}, p_{n_k}, \varepsilon_{n_k})$, we have $\varphi_{F_{\lambda_{n_k}}}(x'_{n_k}, p_{n_k}, \varepsilon_{n_k}) = 0$ and $x'_{n_k} \in eff(-\varphi_{F_{\lambda_{n_k}}}, p_{n_k}, \varepsilon_{n_k})$. Hence, we deduce $x' \in \limsup_n [eff(\varphi_{F_{\lambda_n}}, p_n, \varepsilon_n)]$ and $x' \in eff(-\varphi_{F_{\lambda_0}}, p_0, \varepsilon')$. For completing the proof, it is enough to show that $\varphi_{F_{\lambda_0}}(x', p_0, \varepsilon') = 0$. Suppose on the contrary, there exists $t_0 > 0$ such that $\varphi_{F_{\lambda_0}}(x', p_o, \varepsilon') = t_0$ and for all $t < t_0$, there exists $u_t \in F_{\lambda_0}(x', p_o, \varepsilon')$ that $u_t + \varepsilon' + te \notin C$. Now for a fix $t > t_0$, Since F_{λ_n}, is inner converge continuously to F_{λ_0}, we see that there exists $u_{n_k} \in F_{\lambda_{n_k}}(x'_{n_k}, p_{n_k}, \varepsilon_{n_k})$ such that $\lim_k u_{n_k} = u_t$, since $\varphi_{F_{\lambda_{n_k}}}(x'_{n_k}, p_{n_k}, \varepsilon_{n_k}) = 0$ and C is closed cone. Hence,

$$u_t + \varepsilon' = \lim_k u_{n_k} + \varepsilon_{n_k} \in C,$$

since $e \in intC$. Then we deduce $u_t + \varepsilon' + te \in C$. This is a contradiction. □

The following definition is a generalization of Definition 2.3 in [16].

Definition 3.2. Let $F : X \times B \times D \times P \to 2^Y$ be a set-valued map. F is said to be strongly upper $C_{(e)}$-semicontinuous at the point $x_0 \in X$, if for all $\varepsilon \in \mathbb{R}^+$, there exists $U_{x_0,\varepsilon} \in \mathfrak{U}(x_0)$ such that

$$\forall x \in U_{x_0,\varepsilon}, \ F(x,y,z,p) - F(x_0,y,z,p) - \varepsilon e \subseteq -\text{int}C, \ \forall (y,z,p) \in B \times D \times P.$$

Lemma 3.3. *Let F be a strongly upper $C_{(e)}$-semicontinuous map at the first argument and lower semi continuous at the forth argument. Then, function $-\varphi_F$ that defined in Remark 3.2 is a strongly $\mathbb{R}^+_{(1)}$-upper semi-continuous map at the first argument.*

Proof. By Remark 2.1 in [16], we have to show that $-\varphi_F$ is upper semi continuous, i.e., for all $a \in \mathbb{R}$, $\{(x,p,\varepsilon) : -\varphi_F(x,p,\varepsilon) \geq a\}$ is a closed set. Suppose there exists sequence $(x_n, p_n, \varepsilon_n) \longrightarrow (x_0, p_0, \varepsilon_0)$ such that $-\varphi_F(x_n, p_n, \varepsilon_n) \geq a$. We show $-\varphi_F(x_0, p_0, \varepsilon_0) \geq a$.

Since F is a strongly upper $C_{(e)}$-semicontinuous map at the point $x_0 \in X$, then for all $\delta \in \mathbb{R}^+$, there exists $U_{x_0,\delta} \in \mathfrak{U}(x_0)$ such that

$$\forall x \in U_{x_0,\delta},\ F(x,y,z,p) - F(x_0,y,z,p) - \delta e \subseteq -intC,\ \forall (y,z,p) \in B \times D \times P.$$

Since $x_n \longrightarrow x_0$, we have

$$F(x_n,y,z,p) - F(x_0,y,z,p) - \delta e \subseteq -intC,\ \forall (y,z,p) \in B \times D \times P. \tag{3.6}$$

On the other hand, $-\varphi_F(x_n, p_n, \varepsilon_n) \geq a$. Then there exists $t_0 \leq -a$ belonging to

$$\{t \in \mathbb{R}^+ :\ F(x_n,y,z,p_n) + \varepsilon_n e + te \subseteq C,\ \forall\ (y,z) \in B \times D\}.$$

Therefore, one has

$$F(x_n,y,z,p_n) + \varepsilon_n e + t_0 e \subseteq C\ \forall\ (y,z) \in B \times D. \tag{3.7}$$

Putting $p = p_n$ in (3.6) and summing (3.6) and (3.7), we obtain

$$F(x_0,y,z,p_n) + \varepsilon_n e + t_0 e + \delta e \subseteq intC.$$

Since F is lower semi continuous on the forth argument and C is closed cone, one has

$$F(x_0,y,z,p_o) + \varepsilon_o e + t_0 e + \delta e \subseteq C.$$

So, $t_0 + \delta$ belongs to

$$\{t \in \mathbb{R}^+ :\ F(x_0,y,z,p_0) + \varepsilon_0 e + te \subseteq C,\ \forall\ (y,z) \in B \times D\}.$$

Then $\varphi_F(x_0,p_0,\varepsilon_0) \leq t_0 + \delta$. If $\delta \longrightarrow 0$, then $\varphi_F(x_0,p_0,\varepsilon_0) \leq t_0$. So, $-\varphi_F(x_0,p_0,\varepsilon_)) \geq -t_0 \geq a$. It follows that $-\varphi_F$ is upper semicontinuous at the point x_0. □

Theorem 3.2. *(Theorem 4.1 in [16]) Assume that* $\varphi_n, \varphi : S \longrightarrow Y$, $\varphi_n \xrightarrow{\Gamma_{\mathbb{R}_+}} \varphi$ *and* φ *is strongly upper* $\mathbb{R}_+$*-semi continuous. Then Problem* $\min_{x\in S} \varphi(x)$ *is extended Hadamard wellposed with respect to* (φ_n).

Theorem 3.3. *Suppose that* $F_n, F : A \times B \times D \times P \to 2^Y$ *are compact valued,* $F_n \xrightarrow{\Gamma_{Crp}} F$ *and* F *is strongly upper* $C_{(e)}$*-semicontinuous and inner converge continuously. Then Problem* $(P_r(F_{\lambda_0}, p_0, \varepsilon))$ *is extended Hadamard wellposed with respect to* F_{λ_n}.

Proof. By using of Lemma 3.3, Theorem 3.1 and the above theorem, we can obtain the desired conclusion immediately. □

4. Conclusion

Well-posedness plays a crucial role in the theory and numerical methods of optimization problems. There are three concepts of well-posedness, namely: Tykhonov well-poedness, Levitin-Polyak well-posedness and Hadamard well-posedness. The two first concepts of well-posedness deal with the behavior of a prescribed class of approximating solution sequences. While the Hadamard well-posedness of a problem means the continuous behavior of the solution with respect to the perturbations of the data. In this article, we introduce two kinds of Hadamard well-posedness for vector parametric quasi-equilibrium problems which include some of the main results in this area. Furthermore, by introducing a gap function, we establish a scalarization for our problem. We obtain a sufficient condition for Hadamard well-posedness of vector parametric quasi-equilibrium problem in terms of the Hadamard well-posedness of the gap function.

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