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ON NORM ONE LINEAR MAPS BETWEEN C^*-ALGEBRAS

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Dedicated to Professor Anthony To-Ming Lau in the occasion of his 75 birthday, with thanks for his influences on my career

Abstract. We give a description of norm one linear mappings between two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ that map a C^*-subalgebra of $\mathscr{A}$ isometrically onto one of $\mathscr{B}$. This generalises previous results of Paterson and Sinclar, and of Kadison.

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1. Introduction

The well-known theorem of Kadison [1] states that a surjective linear isometry $T : \mathscr{A} \to \mathscr{B}$ between two unital C^*-algebras must be a Jordan $*$-isomorphism of the given C^*-algebras follows by a multiplication by a unitary element of the range. In the case that $\mathscr{A}$ and/or $\mathscr{B}$ are not unital, the theorem could be applied to $T^{**} : \mathscr{A}^{**} \to \mathscr{B}^{**}$. But the issue is to have the obtained Jordan $*$-isomorphism maps $\mathscr{A}$ into $\mathscr{B}$. This was later resolved by Paterson and Sinclair in [2], where the unitary is now required to be in the multiplier algebra of the range.

In this paper, we shall generalise these to the situation where isometricity is only assumed on a C^*-subalgebra (Theorem 2.1). This will be carried out through first extending Tomiyama's theorem [5] to mappings that satisfy weaker condition than being idempotent.

Recall that Tomiyama's theorem on norm one idempotent says that if $\mathscr{A}$ is a C^*-algebra and $\mathscr{A}_0$ is a C^*-subalgebra of $\mathscr{A}$, and if $\pi : \mathscr{A} \to \mathscr{A}_0$ is a norm one linear idempotent, i.e. a norm one linear mapping whose restriction to $\mathscr{A}_0$ is the identity, then π must be a conditional expectation, i.e. a postive linear map such that

$$\pi(ax) = a\pi(x) \quad \text{and} \quad \pi(xa) = \pi(x)a \qquad (a \in \mathscr{A}_0,\ x \in \mathscr{A}).$$

In Proposition 2.1, we shall relax the requirement of being idempotent to merely that π preserves the unitary property of the elements of $\mathscr{A}_0$, assuming that $\mathscr{A}_0$ is unital, (and this naturally allows the range of π to be in a different algebra). This relaxation comes at a cost that we shall no longer be able to conclude that π is an $\mathscr{A}_0$-module mapping, but only that it preserve the Jordan product of an element in $\mathscr{A}$ and one in $\mathscr{A}_0$.

Our method is to first prove rather modestly that the given mapping is positive, and hence self-adjoint, and then use an unitary element to transform the given mapping to a new one that shares the same property. This is then translated back to show that the given mapping is Jordan multiplicative.

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Kadison's proof of his theorem in [1] had two parts:

(1) He showed that a surjective linear isometry $T : \mathscr{A} \to \mathscr{B}$ between two unital C^*-algebras must map unitaries of $\mathscr{A}$ to those of $\mathscr{B}$.
(2) Kadison then showed that if in addition T maps $1_{\mathscr{A}}$ to $1_{\mathscr{B}}$, then it is a Jordan $*$-isomorphism.

We shall make use of (i) in order to apply the extension of Tomiyama's theorem mentioned above. But our argument as laid out in the previous paragraph will incidentally give us a new, simpler proof of (ii).

Let us briefly review the notation that we shall use; our reference for the general theory of C^*-algebras is [4]. Let $\mathscr{A}$ be a C^*-algebra. The Jordan product on $\mathscr{A}$ is defined as

$$a \circ b := \frac{ab+ba}{2} \quad (a, b \in \mathscr{A}).$$

We shall denote by $\mathscr{A}_h$ the sets of self-adjoint elements of $\mathscr{A}$. By the Sherman–Takeda theorem (see [3] or [4, Theorem III.2.4] for a proof), the multiplication on $\mathscr{A}$ can be extended to the whole of $\mathscr{A}^{**}$ making it a W^*-algebra with the canonical predual $\mathscr{A}^*$.

2. Main results

Proposition 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras, and let $\mathscr{A}_0$ be a C^*-subalgebra of $\mathscr{A}$. Suppose that $\mathscr{A}_0$ and $\mathscr{B}$ have units, and $\pi : \mathscr{A} \to \mathscr{B}$ is a norm one linear mapping that maps the unitaries of $\mathscr{A}_0$* into *those of $\mathscr{B}$ and maps $1_{\mathscr{A}_0} \mapsto 1_{\mathscr{B}}$. Then $\pi(a \circ x) = \pi(a) \circ \pi(x)$ for every $a \in \mathscr{A}_0$ and $x \in \mathscr{A}$.*

Proof. We start with a standard argument: Take $\varphi \in \mathscr{B}^*_+$, and set $\psi := \varphi \circ \pi \in \mathscr{A}^*$. Then $\|\psi\| \le \|\varphi\| = \varphi(1_{\mathscr{B}}) = \psi(1_{\mathscr{A}_0}) \le \|\psi\|$, and so $\psi \in \mathscr{A}^*_+$. This shows that π is order-preserving. Moreover, we also obtain that $\psi(1_{\mathscr{A}^{**}} - 1_{\mathscr{A}_0}) = \|\psi\| - \|\psi\| = 0$, and so

$$\psi(x - 1_{\mathscr{A}_0}x) = \psi(x - x1_{\mathscr{A}_0}) = 0 \qquad (x \in \mathscr{A}).$$

Since $\varphi \in \mathscr{B}^*_+$ is arbitrary, these equations show $\pi(x) = \pi(1_{\mathscr{A}_0}x) = \pi(x1_{\mathscr{A}_0})$ for every $x \in \mathscr{A}$. This allows us to replace $\mathscr{A}$ by $1_{\mathscr{A}_0}\mathscr{A}1_{\mathscr{A}_0}$ if necessary, and suppose that $\mathscr{A}$ is unital with $1_{\mathscr{A}} = 1_{\mathscr{A}_0}$.

Take a unitary u of $\mathscr{A}_0$, let us now consider $\pi_u(x) := \pi(u)^*\pi(ux)$. Then, as $\pi(u)$ is a unitary of $\mathscr{B}$, $\pi_u : \mathscr{A} \to \mathscr{B}$ is again a norm one linear mapping, that maps unitaries of $\mathscr{A}_0$ into those of $\mathscr{B}$. Since we also have $\pi_u(1_{\mathscr{A}_0}) = 1_{\mathscr{B}}$, we obtain, by the first paragraph, that π_u is order-preserving. In particular, $\pi_u(x^*) = \pi_u(x)^*$ for every $x \in \mathscr{A}$. This implies that $\pi(uxu) = \pi(u)\pi(x)\pi(u)$ for every $x \in \mathscr{A}$ and every unitary u of $\mathscr{A}_0$; note that u must now also be a unitary of $\mathscr{A}$ by the assumption at the end of the preceding paragraph. Take $a \in (\mathscr{A}_0)_h$, then the preceding sentence gives

$$\pi\left(\mathrm{e}^{ita}x\mathrm{e}^{ita}\right) = \pi\left(\mathrm{e}^{ita}\right)\pi(x)\pi\left(\mathrm{e}^{ita}\right) \quad (t \in \mathbb{R}).$$

Differentiating at $t = 0$ then shows $\pi(ax + xa) = \pi(a)\pi(x) + \pi(x)\pi(a)$, which is true for every $x \in \mathscr{A}$ and $a \in \mathscr{A}_0$ by linearity. □

Using this, we shall now prove the following generalisation of [2] (which in turn is a generalisation of [1]) on isometries between C^*-algebras.

Theorem 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras, and let $\mathscr{A}_0$ and $\mathscr{B}_0$ be their respective C^*-subalgebras with $\mathscr{B}_0$ contains a bounded approximate unit for $\mathscr{B}$. Let $T : \mathscr{A} \to \mathscr{B}$ be a norm one linear mapping that* *maps $\mathscr{A}_0$ isometrically onto $\mathscr{B}_0$. Then there are a norm one linear mapping $\pi : \mathscr{A} \to \mathscr{B}$ and a unitary $\tilde{u} \in \mathscr{B}^{**}$ such that*

$$T(x) = \tilde{u}\pi(x) \quad \textit{and} \quad \pi(x) = \tilde{u}^* T(x) \qquad (x \in \mathscr{A}), \tag{2.1}$$

with the following additional properties:

(1) π is order preserving, $\pi(\mathscr{A}_0) = \mathscr{B}_0$, $\tilde{u}\mathscr{B}_0 = \mathscr{B}_0$, and
(2) $\pi(a \circ x) = \pi(a) \circ \pi(x)$ for every $a \in \mathscr{A}_0$ and $x \in \mathscr{A}$.

Proof. Consider $T^{**} : \mathscr{A}^{**} \to \mathscr{B}^{**}$. Then $\mathscr{A}_0^{**}$ is a W^*-subalgebra of $\mathscr{A}^{**}$, and $T^{**}(\mathscr{A}_0^{**}) = \mathscr{B}_0^{**}$ is a unital W^*-subalgebra of $\mathscr{B}^{**}$ since $\mathscr{B}_0$ contains a bounded approximate unit for $\mathscr{B}$. Since $T^{**} : \mathscr{A}_0^{**} \to \mathscr{B}_0^{**}$ is an isometric linear surjection, it maps unitaries of $\mathscr{A}_0^{**}$ onto those of $\mathscr{B}_0^{**}$, as proved by Kadison in [1]. In particular, if we set

$$\tilde{u} := T^{**}(1_{\mathscr{A}_0^{**}}) \quad \text{and} \quad \widetilde{\pi} := \tilde{u}^* T^{**},$$

then $\tilde{u}$ is a unitary of $\mathscr{B}_0^{**}$, which is also a unitary of $\mathscr{B}^{**}$, and $\widetilde{\pi} : \mathscr{A}^{**} \to \mathscr{B}^{**}$ is a norm one linear mapping whose restriction $\widetilde{\pi}|_{\mathscr{A}_0^{**}}$ is a unital surjective isometry onto $\mathscr{B}_0^{**}$. Thus, by [1, Theorem 7], the latter must be a Jordan $*$-isomorphism; in fact, more generally, as $\widetilde{\pi}$ satisfies the conditions of Proposition 2.1, we see that it is order-preserving and

$$\widetilde{\pi}(\tilde{a} \circ \tilde{x}) = \widetilde{\pi}(\tilde{a}) \circ \widetilde{\pi}(\tilde{x}) \quad \text{for every } \tilde{a} \in \mathscr{A}_0^{**} \text{ and } \tilde{x} \in \mathscr{A}^{**}. \tag{2.2}$$

Set $\pi := \widetilde{\pi}|_{\mathscr{A}}$. Then π is a norm one linear mapping $\mathscr{A} \to \mathscr{B}^{**}$ that is order preserving and satisfies (2.1) and (ii). Since for each $x \in \mathscr{A}$

$$\pi(x)^* \pi(x) = T(x)^* \tilde{u}\tilde{u}^* T(x) = T(x)^* T(x) \in \mathscr{B}, \tag{2.3}$$

we see that $\pi(x) \in \mathscr{B}$ whenever $x \in \mathscr{A}_+$ as $\pi(x) \in \mathscr{B}_+^{**}$ in that case. Thus $\pi(\mathscr{A}) \subseteq \mathscr{B}$. Similarly, $\pi(\mathscr{A}_0) \subseteq \mathscr{B}_0$. Equations (2.3) and (2.2) also show that $\mathscr{B}_0 \subseteq \pi(\mathscr{A}_0)$. Thus

$$\mathscr{B}_0 = \pi(\mathscr{A}_0) = \tilde{u}^* T(\mathscr{A}_0) = \tilde{u}^* \mathscr{B}_0.$$

This completes the proof. □

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