J. Nonlinear Var. Anal. 2 (2018), No. 1, pp. 63-90
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.2.2018.1.06

EUCLIDEAN SPACE CONTROLLABILITY CONDITIONS AND MINIMUM ENERGY PROBLEM FOR TIME DELAY SYSTEMS WITH A HIGH GAIN CONTROL

VALERY Y. GLIZER

Department of Applied Mathematics, ORT Braude College of Engineering, 51 Snunit Str., P.O.B. 78, Karmiel 2161002, Israel

Abstract. In this paper, a non-autonomous differential system with state delays and with a large coefficient for the control (high gain control) is considered. By a proper control transformation, this system is converted into equivalent time delay system singularly perturbed by a small positive parameter. The latter system, is decomposed asymptotically into two much simpler parameter-free subsystems, slow and fast ones. The slow subsystem has delays in the state and control variables, while the fast subsystem is delay-free. Based on the assumption of the Euclidean space controllability of the slow subsystem, the Euclidean space controllability of the transformed and original systems is established for all sufficiently small values of the parameter of singular perturbation. Also, the minimum energy control problem for the transformed system is considered. Asymptotic behaviour with respect to the small parameter of the solution to this problem is studied. An illustrative example is presented.

Keywords. Time delay system; High gain control; Euclidean space controllability; Singular perturbation; Minimum energy control.

2010 Mathematics Subject Classification. 34K26, 34K35, 49N10, 93B05, 93C23.

1. INTRODUCTION

The system under the consideration is

$$\frac{dx(t)}{dt} = \sum_{i=0}^{1}\left[A_{1i}(t)x(t-h_i) + A_{2i}(t)y(t-h_i)\right]$$
$$+\int_{-h}^{0}\left[G_1(t,\eta)x(t+\eta) + G_2(t,\eta)y(t+\eta)\right]d\eta, \quad t \geq 0, \tag{1.1}$$

$$\frac{dy(t)}{dt} = \sum_{i=0}^{1}\left[A_{3i}(t)x(t-h_i) + A_{4i}(t)y(t-h_i)\right]$$
$$+\int_{-h}^{0}\left[G_3(t,\eta)x(t+\eta) + G_4(t,\eta)y(t+\eta)\right]d\eta + \frac{1}{\varepsilon}u(t), \quad t \geq 0, \tag{1.2}$$

where for any $t \geq 0$, $x(t) \in E^n$, $y(t) \in E^m$, $u(t) \in E^m$ (u is a control); $\varepsilon > 0$ is a small parameter; $h_0 = 0$, $h_1 = h$ is a given constant independent of ε; $A_{ki}(t)$, $G_k(t,\eta)$, $(i = 0,1;\ k = 1,...,4)$ are matrices of corresponding dimensions; for any given $\bar{t} > 0$, the matrix-valued functions $A_{ki}(t)$, $(i = 0,1;\ k = 1,...,4)$ are bounded in the interval $[0,\bar{t}]$, while the matrix-valued functions $G_k(t,\eta)$, $(k = 1,...,4)$ are bounded

E-mail addresses: valery48@braude.ac.il, valgl120@gmail.com.
Received December 29, 2017; Accepted January 11, 2018.

in the domain $\Omega(\bar{t}) \stackrel{\triangle}{=} \{(t,\eta) : t \in [0,\bar{t}], \eta \in [-h,0]\}$; for any given $\bar{t} > 0$, the control function $u(\cdot) \in L^2[0,\bar{t};E^m]$; E^q denotes the real q-dimensional Euclidean space.

The system (1.1)-(1.2) is functional-differential. Therefore, it is infinite dimensional. The state variables of this system have the form $\big(x(t),x(t+\eta)\big)$, $\big(y(t),y(t+\eta)\big)$, $\eta \in [-h,0]$. In what follows, for any given $t \geq 0$, we consider the pairs $\big(x(t),x(t+\eta)\big)$ and $\big(y(t),y(t+\eta)\big)$ in the spaces $\mathscr{M}[-h,0;n]$ and $\mathscr{M}[-h,0;m]$, respectively, where $\mathscr{M}[a,b;q] \stackrel{\triangle}{=} E^q \times L^2[a,b;E^q]$. Therefore, the components $x(t)$ and $y(t)$ of the state variables are called their Euclidean parts, while the components $x(t+\eta)$ and $y(t+\eta)$ are called the functional parts of the corresponding state variables.

Due to the smallness of the parameter ε, the system (1.1)-(1.2) is a high gain control system. High gain control systems appear in various applications. For example, such systems appear in a model matching problem ([31]), in a trajectory/output tracking problem ([28]), in a sliding mode control ([59]), in a chattering suppression ([10]), in automotive systems ([9, 52]), in an adaptive control ([54, 55]), in a control of unmanned aerial vehicles ([4]), in cheap control problems ([44]), in a solution of singular control problems by regularization ([22, 24, 25, 27]), and in some other applications (see [34] and references therein). A high gain control has been mostly studied for finite-dimensional differential systems (see e.g. [44, 48, 53, 58] and references therein). However, infinite-dimensional differential systems with a high gain control were analyzed much less (see [18, 20, 21, 32, 33]).

Multiplying the equation (1.2) by ε, one can convert the system (1.1)-(1.2) to a singularly perturbed system.

Singularly perturbed controlled systems are extensively studied in the literature, due to their appearance in various real-life problems and the considerable theoretical meaning. Study of such systems without delays can be found, for instance, in [8, 12, 44, 45, 60] and references therein, while analysis of singularly perturbed controlled systems with delays can be found in [8, 49, 60] and references therein.

In the present paper, we study the Euclidean space controllability of the system (1.1)-(1.2). Controllability of a system is its important property, which means the ability to transfer the system from any position of a given set of initial positions to any position of a given set of terminal positions in a finite time by a proper choice of the control function. Different types of controllability for various finite-dimensional and infinite-dimensional systems were extensively studied in the literature (see e.g. [3, 6, 35, 38, 39, 42] and references therein). In order to check a proper type of controllability for a given singularly perturbed system, corresponding controllability conditions can be directly applied for any specified value of a small parameter $\varepsilon > 0$ of singular perturbation. However, a possible stiffness and a high dimension of the singularly perturbed system, can considerably complicate this application. Moreover, such an application depends, in general, on the value of ε, i.e., it is not robust with respect to this parameter, while in most of real-life problems this value is unknown. Therefore, controllability analysis of singularly perturbed systems requires another approach, independent of the parameter of singular perturbation. One of such approaches is the approach based on the asymptotic decomposition of a singularly perturbed system into two simpler parameter-free subsystems (see e.g. [44]). Using such a decomposition, the complete controllability of some linear and nonlinear systems without delays was analyzed in the works [43, 44, 56, 57]. In [50], a linear time-invariant undelayed system with multipliers for the derivatives, which are different positive integer powers of a small positive parameter, was considered. A generalization of the asymptotic decomposition was proposed for the complete controllability analysis of this system. In [46], a linear standard singularly perturbed time-invariant system with a single point-wise delay in the state variables was considered. For this system, based on the system's decomposition, the robust complete Euclidean space controllability was studied in the case of nonsmall delay. In [14, 15], using the decomposition approach, parameter-free conditions of complete Euclidean space controllability, robust with respect to ε, were obtained for linear standard singularly perturbed systems with point-wise and distributed small delays (of order of ε) in the state variables. In [16], such a result was obtained for nonstandard singularly perturbed systems with multiple point-wise and distributed small delays in the state variables. In [26], a singularly perturbed linear time-dependent controlled system with a small point-wise delay in state and control variables was considered. Applying the asymptotic decomposition approach, parameter-free conditions of the complete Euclidean space controllability were established for both, standard and nonstandard, types of this system. In [19], a linear singularly perturbed system with small state delays (multiple point-wise and distributed) was considered. The asymptotic decomposition approach is not applicable for its controllability analysis. The parameter-free complete Euclidean space controllability conditions for this system were derived using the first-order asymptotic expansion of the controllability matrix. In [47], the defining equations approach (see e.g. [11]) was applied for analysis of the complete Euclidean space controllability of a linear singularly perturbed neutral type system with a single nonsmall point-wise delay.

In the present paper, the Euclidean space controllability of the system (1.1)-(1.2) is studied by application of its slow-fast asymptotic decomposition. However, due to the specific type of singular perturbation in this system, (the high gain of the control), the slow and fast subsystems differ considerably from such subsystems in the above mentioned works. Namely, the slow subsystem has delays in both, state and control, variables, while the fast subsystem is delay-free and completely controllable. Based on this feature, it is established in the paper, that the Euclidean space controllability of the slow subsystem yields the Euclidean space controllability of the original high gain control system. This controllability is based on parameter-free conditions, while it is valid for all sufficiently small values of the parameter $\varepsilon > 0$ in the control's gain, i.e., robustly with respect to this parameter. To the best of our knowledge, controllability properties of time delay systems with high gain controls have not yet been studied in the literature.

Along with the controllability of the system (1.1)-(1.2), we study an asymptotic behaviour for $\varepsilon \to +0$ of the solution to the minimum energy control problem for a system, obtained from (1.1)-(1.2) by a linear control transformation. This new system also is a time delay high gain control system. The minimum energy control problem was studied extensively in the literature for systems without and with delays (see e.g. [5, 36, 40, 41] and references therein). However, an asymptotic behaviour of the solution to the minimum energy control problem for a singularly perturbed system was studied much less. Thus, in [37] the minimum energy control problem was studied for a singularly perturbed system without delays. In the present paper, we study an asymptotic behaviour of the solution to the minimum energy control problem for a class of singularly perturbed systems with delays in the state variable. To the best of our knowledge, such a problem has not yet been studied in the literature.

The paper is organized as follows. In the next section, a high dimension Euclidean space controllability condition for the system (1.1)-(1.2) is derived. In Section 3, an asymptotic decomposition of the system (1.1)-(1.2) is carried out. Main definitions are presented. Auxiliary results are obtained in Section 4. These results include: (a) a linear control transformation in the system (1.1)-(1.2), invariant with respect to the Euclidean space controllability; (b) estimates of solutions to some singularly perturbed matrix time delay differential equations. In Section 5, a lower dimension Euclidean space controllability condition for the system (1.1)-(1.2) is derived. Asymptotic properties of the solution to the minimum energy control problem for the system, obtained from (1.1)-(1.2) by the control transformation, are studied in Section 6. In Section 7, an illustrative example is presented. Section 8 is devoted to conclusions.

2. High dimension controllability condition for the system (1.1)-(1.2)

First of all let us note that, due to the results of [7], for any given $\varepsilon > 0$, $t_c > 0$, $u(t) \in L^2[0,t_c;E^m]$, $f_{x0}(\eta) = \big(x_0, \varphi_x(\eta)\big) \in \mathscr{M}[-h,0;E^n]$ and $f_{y0}(\eta) = \big(y_0, \varphi_y(\eta)\big) \in \mathscr{M}[-h,0;E^m]$, the system (1.1)-(1.2) subject to the initial conditions

$$x(\eta) = \varphi_x(\eta), \quad y(\eta) = \varphi_y(\eta), \quad \eta \in [-h,0); \quad x(0) = x_0, \ \ y(0) = y_0, \tag{2.1}$$

has the unique absolutely continuous solution $\big(x(t), y(t)\big)$, $t \in [0,t_c]$ with the derivatives $dx(t)/dt$ and $dy(t)/dt$ belonging to $L^2[0,t_c;E^n]$ and $L^2[0,t_c;E^m]$, respectively.

Let

$$t_c > h \tag{2.2}$$

be a given time instant.

Definition 2.1. For a given $\varepsilon > 0$, the system (1.1)-(1.2) is said to be completely Euclidean space controllable at the time instant t_c if for any $f_{x0}(\eta) = \big(x_0, \varphi_x(\eta)\big) \in \mathscr{M}[-h,0;E^n]$, $f_{y0}(\eta) = \big(y_0, \varphi_y(\eta)\big) \in \mathscr{M}[-h,0;E^m]$ and $x_c \in E^n$, $y_c \in E^m$, there exists a control $u(t) \in L^2[0,t_c;E^m]$ such that the system (1.1)-(1.2) with the initial conditions (2.1) and the terminal conditions

$$x(t_c) = x_c, \quad y(t_c) = y_c, \tag{2.3}$$

has a solution.

Consider the following block matrices:

$$A_i(t) = \begin{pmatrix} A_{1i}(t) & A_{2i}(t) \\ A_{3i}(t) & A_{4i}(t) \end{pmatrix}, \ \ i = 0,1, \quad G(t,\eta) = \begin{pmatrix} G_1(t,\eta) & G_2(t,\eta) \\ G_3(t,\eta) & G_4(t,\eta) \end{pmatrix}, \\ B(\varepsilon) = \begin{pmatrix} 0 \\ (1/\varepsilon) I_m \end{pmatrix}, \tag{2.4}$$

and the vector

$$z(t) = \mathrm{col}\big(x(t), y(t)\big). \tag{2.5}$$

Using (2.4)-(2.5), the system (1.1)-(1.2) can be rewritten equivalently in the form:

$$\frac{dz(t)}{dt} = \sum_{i=0}^{1} A_i(t) z(t-h_i) + \int_{-h}^{0} G(t,\eta) z(t+\eta) d\eta + B(\varepsilon) u(t), \quad t \geq 0. \tag{2.6}$$

Let the $(n+m)\times(n+m)$-matrix-valued function $\Phi(t,\sigma)$, $0 \leq \sigma \leq t \leq t_c$, be the fundamental solution of the homogeneous system corresponding to (2.6). This means that, for any $\sigma \in [0,t_c]$, the matrix

$\Phi(t,\sigma)$ is the unique solution of the following initial-value problem:

$$\frac{d\Phi(t,\sigma)}{dt} = \sum_{i=0}^{1} A_i(t)\Phi(t-h_i,\sigma) + \int_{-h}^{0} G(t,\eta)\Phi(t+\eta,\sigma)d\eta, \quad t\in[\sigma,t_c],$$
$$\Phi(t,\sigma)=0, \quad t<\sigma; \quad \Phi(\sigma,\sigma)=I_{n+m}. \tag{2.7}$$

Let us partition the matrix $\Phi(t,\sigma)$ into blocks as:

$$\Phi(t,\sigma) = \begin{pmatrix} \Phi_1(t,\sigma) & \Phi_2(t,\sigma) \\ \Phi_3(t,\sigma) & \Phi_4(t,\sigma) \end{pmatrix}, \tag{2.8}$$

where the blocks $\Phi_1(t,\sigma)$, $\Phi_2(t,\sigma)$, $\Phi_3(t,\sigma)$ and $\Phi_4(t,\sigma)$ are of the dimensions $n\times n$, $n\times m$, $m\times n$ and $m\times m$, respectively.

Lemma 2.1. *For any $\varepsilon>0$, the system (1.1)-(1.2) is completely Euclidean space controllable at the time instant t_c if and only if the $(n+m)\times(n+m)$-matrix*

$$W(t_c) = \int_0^{t_c} \begin{pmatrix} \Phi_2(t_c,\sigma)\Phi_2^T(t_c,\sigma) & \Phi_2(t_c,\sigma)\Phi_4^T(t_c,\sigma) \\ \Phi_4(t_c,\sigma)\Phi_2^T(t_c,\sigma) & \Phi_4(t_c,\sigma)\Phi_4^T(t_c,\sigma) \end{pmatrix} d\sigma \tag{2.9}$$

is invertible.

Proof. By virtue of the results of [7], for a given $\varepsilon>0$ the system (1.1)-(1.2) is completely Euclidean space controllable at the time instant t_c if and only if the matrix

$$\widetilde{W}(t_c,\varepsilon) = \int_0^{t_c} \Phi(t_c,\sigma)B(\varepsilon)B^T(\varepsilon)\Phi^T(t_c,\sigma)d\sigma \tag{2.10}$$

is invertible. Substitution of the block representation for the matrices $B(\varepsilon)$ and $\Phi(t,\sigma)$ (see equations (2.4) and (2.8)) into (2.10), and use of (2.9) yield after a routine algebra:

$$\widetilde{W}(t_c,\varepsilon) = \frac{1}{\varepsilon^2}W(t_c). \tag{2.11}$$

The latter equation directly implies the statement of the lemma. □

Remark 2.1. The condition of the complete Euclidean space controllability for the system (1.1)-(1.2), obtained in Lemma 2.1, is independent of ε. However, this condition requires to deal with the high-dimension ($(n+m)\times(n+m)$-dimension) controllability matrix $W(t_c)$, including obtaining the solution of the initial-value problem for the time delay differential equation with respect to the matrix $\begin{pmatrix} \Phi_2(t,\sigma) \\ \Phi_4(t,\sigma) \end{pmatrix}$. All this means that the use of the controllability condition, obtained in Lemma 2.1, is a very complicated task. In subsequent sections of the paper, we derive another controllability condition for the system (1.1)-(1.2). This condition deals with a lower-dimension ($n\times n$-dimension) controllability matrix, which requares to solve an initial-value problem for a time delay differential equation with respect to an $n\times n$-dimensional unknown matrix-valued function. Moreover, this lower dimension controllability condition is independent of $\varepsilon>0$, while provides the complete Euclidean space controllability of the system (1.1)-(1.2) for all sufficiently small values of this parameter. The derivation of the lower dimension controllability condition is based on an asymptotic decomposition of (1.1)-(1.2).

3. Asymptotic decomposition of system (1.1)-(1.2) and main definitions

Multiplying the equation (1.2) by ε, we obtain

$$\varepsilon\frac{dy(t)}{dt}=\varepsilon\Bigg(\sum_{i=0}^{1}\big[A_{3i}(t)x(t-h_i)+A_{4i}(t)y(t-h_i)\big]$$
$$+\int_{-h}^{0}\big[G_3(t,\eta)x(t+\eta)+G_4(t,\eta)y(t+\eta)\big]d\eta\Bigg)+u(t),\quad t\geq 0. \tag{3.1}$$

It is clear that, for any $\varepsilon>0$, the auxiliary system (1.1), (3.1) is equivalent to the original system (1.1)-(1.2). Moreover, the system (1.1), (3.1) has the explicit singular perturbation form. For the sake of the further analysis, we decompose asymptotically this system (and, therefore, the system (1.1)-(1.2)) into two much simpler ε-free subsystems, the slow and fast ones.

3.1. **Slow subsystem.** Let us set formally $\varepsilon=0$ in the system (1.1), (3.1) and re-denote the variables $x(\cdot)$, $y(\cdot)$ and $u(\cdot)$ as $x_s(\cdot)$, $y_s(\cdot)$ and $u_s(\cdot)$, respectively. Thus, we have

$$\frac{dx_s(t)}{dt}=\sum_{i=0}^{1}\big[A_{1i}(t)x_s(t-h_i)+A_{2i}(t)y_s(t-h_i)\big]$$
$$+\int_{-h}^{0}\big[G_1(t,\eta)x_s(t+\eta)+G_2(t,\eta)y_s(t+\eta)\big]d\eta,\quad t\geq 0, \tag{3.2}$$

$$u_s(t)=0,\quad t\geq 0. \tag{3.3}$$

Thus, in the system (3.2)-(3.3), the control $u_s(t)$ is identically zero vector-valued function, while the state variable $\big(y_s(t),y_s(t+\eta)\big)$, $\eta\in[-h,0]$ does not satisfy any equation. Therefore, we can choose this variable to satisfy a desirable property of the system (3.2). The latter means that the variable $\big(y_s(t),y_s(t+\eta)\big)$, $\eta\in[-h,0]$ can be considered as a control in (3.2). The system (3.2), where $\big(x_s(t),x_s(t+\eta)\big)$, $\eta\in[-h,0]$ is a state variable, while $\big(y_s(t),y_s(t+\eta)\big)$, $\eta\in[-h,0]$ is a control, is called the slow subsystem associated with the system (1.1), (3.1) (and, therefore, with the original system (1.1)-(1.2)).

Definition 3.1. The slow subsystem (3.2) is said to be completely Euclidean space controllable at the time instant t_c if for any $f_{x0}(\eta)=\Big(x_0,\varphi_x(\eta)\Big)\in\mathcal{M}[-h,0;E^n]$, $\varphi_y(\eta)\in L^2[-h,0;E^m]$ and $x_c\in E^n$, there exists a control $y_s(t)\in L^2[0,t_c;E^m]$ such that the system (3.2) with the initial and terminal conditions

$$x_s(\eta)=\varphi_x(\eta),\quad \eta\in[-h,0);\qquad x_s(0)=x_0, \tag{3.4}$$

$$y_s(\eta)=\varphi_y(\eta),\quad \eta\in[-h,0), \tag{3.5}$$

$$x_s(t_c)=x_c, \tag{3.6}$$

has a solution.

Let the $n\times n$-matrix-valued function $\Phi_s(t,\sigma)$, $0\leq\sigma\leq t\leq t_c$, be the fundamental solution of the homogeneous system corresponding to (3.2). This means that, for any $\sigma\in[0,t_c]$, the matrix $\Phi_s(t,\sigma)$ is the unique solution of the following initial-value problem:

$$\frac{d\Phi_s(t,\sigma)}{dt}=\sum_{i=0}^{1}A_{1i}(t)\Phi_s(t-h_i,\sigma)+\int_{-h}^{0}G_1(t,\eta)\Phi_s(t+\eta,\sigma)d\eta,\quad t\in[\sigma,t_c],$$
$$\Phi_s(t,\sigma)=0,\quad t<\sigma;\qquad \Phi_s(\sigma,\sigma)=I_n. \tag{3.7}$$

Consider the following $n \times m$-matrix-valued function:

$$D_s(t,\sigma) = \sum_{i=0}^{1} \Phi_s(t,\sigma+h_i)A_{2i}(\sigma+h_i) + \int_{-h}^{0} \Phi_s(t,\sigma-\eta)G_2(\sigma-\eta,\eta)d\eta, \quad 0 \le \sigma \le t \le t_c. \tag{3.8}$$

In (3.8), it is assumed that

$$A_{21}(t) = A_{21}(t_c), \quad G_2(t,\eta) = G_2(t_c,\eta), \quad t > t_c, \quad \eta \in [-h,0]. \tag{3.9}$$

By virtue of the results of [2, 23], we have the assertion.

Lemma 3.1. *The slow subsystem (3.2) is completely Euclidean space controllable at the time instant t_c if and only if the matrix*

$$W_s(t_c) = \int_0^{t_c} D_s(t_c,\sigma)D_s^T(t_c,\sigma)d\sigma \tag{3.10}$$

is invertible.

Based on Lemma 3.1, let us obtain another criterion of the complete Euclidean space controllability of the system (3.2), which is more convenient for the further analysis. To do this, we consider the following terminal-value problem with respect to the $n \times n$-matrix-valued function $\Lambda_s(\sigma)$:

$$\frac{d\Lambda_s(\sigma)}{d\sigma} = -\sum_{i=0}^{1} A_{1i}^T(\sigma+h_i)\Lambda_s(\sigma+h_i) - \int_{-h}^{0} G_1^T(t-\eta,\eta)\Lambda_s(\sigma-\eta)d\eta, \quad \sigma \in [0,t_c],$$
$$\Lambda_s(t_c) = I_n; \quad \Lambda_s(\sigma) = 0, \quad \sigma > t_c. \tag{3.11}$$

In this problem, we assume

$$A_{11}(t) = A_{11}(t_c), \quad G_1(t,\eta) = G_1(t_c,\eta), \quad t > t_c, \quad \eta \in [-h,0]. \tag{3.12}$$

Applying the results of [29] (Section 4.3) to the problems (3.7) and (3.11), we obtain that the latter has the unique solution $\Lambda_s(\sigma)$, and

$$\Phi_s(t_c,\sigma) = \Lambda_s^T(\sigma), \quad \sigma \in [0,t_c]. \tag{3.13}$$

Consider the following $n \times m$-matrix-valued function:

$$C_s(\sigma) = \sum_{i=0}^{1} \Lambda_s^T(\sigma+h_i)A_{2i}(\sigma+h_i) + \int_{-h}^{0} \Lambda_s^T(\sigma-\eta)G_2(\sigma-\eta,\eta)d\eta, \quad 0 \le \sigma \le t_c. \tag{3.14}$$

Using the matrix-valued function $C_s(\sigma)$, the equation (3.13) and Lemma 3.1, we directly have the assertion.

Corollary 3.1. *The slow subsystem (3.2) is completely Euclidean space controllable at the time instant t_c if and only if the matrix*

$$M_s(t_c) = \int_0^{t_c} C_s(\sigma)C_s^T(\sigma)d\sigma \tag{3.15}$$

is invertible.

3.2. **Fast subsystem.** The fast subsystem is derived from the equation (3.1) in the following formal way: (i) the expression, multiplied by ε, is removed from the right-hand side of (3.1); (ii) the transformation of the variables $t = \varepsilon\xi$, $y(\varepsilon\xi) = y_f(\xi)$, $u(\varepsilon\xi) = u_f(\xi)$ is made in the resulting system, where ξ is a new independent variable.

Thus, we obtain the fast subsystem associated with the system (1.1), (3.1) (and, therefore, with the original system (1.1)-(1.2)):

$$\frac{dy_f(\xi)}{d\xi} = u_f(\xi), \quad \xi \geq 0, \tag{3.16}$$

where, for any $\xi \geq 0$, $y_f(\xi) \in E^m$, $u_f(\xi) \in E^m$, ($u_f(\xi)$ is a control).

Definition 3.2. The fast subsystem (3.16) is said to be completely controllable if for any $y_0 \in E^m$ and $y_c \in E^m$, there exists a number $\xi_c > 0$, independent of y_0, y_c, and a control $u_f(\xi) \in L^2[0,\xi_c;E^m]$ such that the system (3.16) with the initial and terminal conditions

$$y_f(0) = y_0, \quad y_f(\xi_c) = y_c, \tag{3.17}$$

has a solution.

Let us observe that the dimension of the state variable in (3.16) equals to the dimension of the control variable, and the matrix of the coefficients for the latter is invertible. Using this observation and the results of [38], we directly obtain the assertion.

Lemma 3.2. *The fast subsystem (3.16) is completely controllable.*

4. Auxiliary results

4.1. **Linear control transformation and its properties.**

4.1.1. *Control transformation in the system (1.1)-(1.2).* Let us transform the control in the system (1.1)-(1.2) as:

$$u(t) = -y(t) + v(t), \tag{4.1}$$

where $v(t)$ is a new control.

Due to this transformation, the system (1.1)-(1.2) becomes the one consisting of the equation (1.1) and the equation

$$\begin{aligned}\frac{dy(t)}{dt} &= \sum_{i=0}^{1} A_{3i}(t)x(t-h_i) + \left(A_{40}(t) - \frac{1}{\varepsilon}I_m\right)y(t) + A_{41}(t)y(t-h) \\ &\quad + \int_{-h}^{0}\left[G_3(t,\eta)x(t+\eta) + G_4(t,\eta)y(t+\eta)\right]d\eta + \frac{1}{\varepsilon}v(t).\end{aligned} \tag{4.2}$$

Lemma 4.1. *For a given $\varepsilon > 0$, the system (1.1)-(1.2) is completely Euclidean space controllable at the time instant t_c if and only if the system (1.1),(4.2) is completely Euclidean space controllable at this time instant.*

Proof. Necessity: Suppose that for some $\varepsilon > 0$, the system (1.1)-(1.2) is Euclidean space controllable at t_c. Let $f_{x0}(\eta) = \big(x_0, \varphi_x(\eta)\big) \in \mathscr{M}[-h,0;E^n]$, $f_{y0}(\eta) = \big(y_0, \varphi_y(\eta)\big) \in \mathscr{M}[-h,0;E^m]$, $x_c \in E^n$ and $y_c \in E^m$ be arbitrary given. Due to Definition 2.1, there exists a control function $u(t) \in L^2[0,t_c;E^m]$

such that the system (1.1)-(1.2) subject to the initial (2.1) and terminal (2.3) conditions has a solution $\big(x(t),y(t)\big)$, $t\in[0,t_c]$. For this control, the equation (1.2) can be rewritten in the equivalent form

$$\frac{dy(t)}{dt}=\sum_{i=0}^{1}A_{3i}(t)x(t-h_i)+\left(A_{40}(t)-\frac{1}{\varepsilon}I_m\right)y(t)+A_{41}(t)y(t-h)$$
$$+\int_{-h}^{0}\big[G_3(t,\eta)x(t+\eta)+G_4(t,\eta)y(t+\eta)\big]d\eta+\frac{1}{\varepsilon}\big[u(t)+y(t)\big]. \tag{4.3}$$

Denote $v(t)\overset{\triangle}{=}u(t)+y(t)$. Since $u(t)\in L^2[0,t_c;E^m]$ and $y(t)$ is an absolutely continuous function in the interval $[0,t_c]$, then $v(t)\in L^2[0,t_c;E^m]$. This observation and the equivalence of the equations (1.2) and (4.3), along with the above assumed Euclidean space controllability of the system (1.1)-(1.2), imply the Euclidean space controllability of the system (1.1), (4.2) at the time instant t_c.
Sufficiency: The sufficiency is proven similarly to the necessity. □

Consider the following block matrix:

$$\mathscr{A}_0(t,\varepsilon)=\begin{pmatrix}A_{10}(t) & A_{20}(t)\\ A_{30}(t) & A_{40}(t)-\frac{1}{\varepsilon}I_m\end{pmatrix}. \tag{4.4}$$

Using the matrices $A_1(t)$, $G(t,\eta)$, $B(\varepsilon)$, $\mathscr{A}_0(t,\varepsilon)$ and vector $z(t)$ (see equations (2.4), (4.4) and (2.5)), we can rewrite the system (1.1), (4.2) in the equivalent form:

$$\frac{dz(t)}{dt}=\mathscr{A}_0(t,\varepsilon)z(t)+A_1(t)z(t-h)+\int_{-h}^{0}G(t,\eta)z(t+\eta)d\eta+B(\varepsilon)v(t),\quad t\geq 0. \tag{4.5}$$

Let for a given $\varepsilon>0$, the $(n+m)\times(n+m)$-matrix-valued function $\Psi(t,\sigma,\varepsilon)$, $0\leq\sigma\leq t\leq t_c$, be the fundamental solution of the homogeneous system corresponding to (4.5). This means that, for any $\sigma\in[0,t_c]$, the matrix $\Psi(t,\sigma,\varepsilon)$ is the unique solution of the following initial-value problem:

$$\frac{d\Psi(t,\sigma,\varepsilon)}{dt}=\mathscr{A}_0(t,\varepsilon)\Psi(t,\sigma,\varepsilon)+A_1(t)\Psi(t-h,\sigma,\varepsilon)+\int_{-h}^{0}G(t,\eta)\Psi(t+\eta,\sigma,\varepsilon)d\eta,\ t\in[\sigma,t_c],$$
$$\Psi(t,\sigma,\varepsilon)=0,\quad t<\sigma;\qquad \Psi(\sigma,\sigma,\varepsilon)=I_{n+m}. \tag{4.6}$$

By virtue of the results of [7], we directly have the assertion.

Lemma 4.2. *For a given $\varepsilon>0$, the system (1.1),(4.2) is completely Euclidean space controllable at the time instant t_c if and only if the $(n+m)\times(n+m)$-matrix*

$$\mathscr{W}(t_c,\varepsilon)=\int_0^{t_c}\Psi(t_c,\sigma,\varepsilon)B(\varepsilon)B^T(\varepsilon)\Psi^T(t_c,\sigma,\varepsilon)d\sigma \tag{4.7}$$

is invertible.

For the sake of further analysis, we rewrite the matrix $\Psi(t_c,\sigma,\varepsilon)$ (and, therefore, the matrix $\mathscr{W}(t_c,\varepsilon)$) in another form. For this purpose, for a given $\varepsilon>0$, we consider the following terminal-value problem with respect to the $(n+m)\times(n+m)$-matrix-valued function $\Lambda(\sigma,\varepsilon)$:

$$\frac{d\Lambda(\sigma,\varepsilon)}{d\sigma}=-\mathscr{A}_0^T(\sigma,\varepsilon)\Lambda(\sigma,\varepsilon)-A_1^T(t)\Lambda(\sigma+h,\varepsilon)$$
$$-\int_{-h}^{0}G^T(\sigma-\eta,\eta)\Lambda(\sigma-\eta,\varepsilon)d\eta,\quad \sigma\in[0,t_c],$$
$$\Lambda(t_c,\varepsilon)=I_{n+m};\quad \Lambda(\sigma,\varepsilon)=0,\quad \sigma>t_c. \tag{4.8}$$

In this problem, it is assumed that:

$$A_1(t) = A_1(t_c), \quad G(t,\eta) = G(t_c,\eta), \quad t > t_c, \quad \eta \in [-h,0]. \tag{4.9}$$

Using the results of [29] (Section 4.3), we obtain that the problem (4.8) has the unique solution $\Lambda(\sigma,\varepsilon)$ for any $\varepsilon > 0$. Moreover,

$$\Psi(t_c,\sigma,\varepsilon) = \Lambda^T(\sigma,\varepsilon), \quad \sigma \in [0,t_c]. \tag{4.10}$$

Let us partition the matrix $\Lambda(\sigma,\varepsilon)$ into block as:

$$\Lambda(\sigma,\varepsilon) = \begin{pmatrix} \Lambda_1(\sigma,\varepsilon) & \Lambda_2(\sigma,\varepsilon) \\ \Lambda_3(\sigma,\varepsilon) & \Lambda_4(\sigma,\varepsilon) \end{pmatrix}, \tag{4.11}$$

where the blocks $\Lambda_1(\sigma,\varepsilon)$, $\Lambda_2(\sigma,\varepsilon)$, $\Lambda_3(\sigma,\varepsilon)$ and $\Lambda_4(\sigma,\varepsilon)$ are of the dimensions $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively.

Corollary 4.1. *For a given $\varepsilon > 0$, the system (1.1),(4.2) is completely Euclidean space controllable at the time instant t_c if and only if the $(n+m)\times(n+m)$-matrix*

$$M(t_c,\varepsilon) = \int_0^{t_c} \begin{pmatrix} \Lambda_3^T(\sigma,\varepsilon)\Lambda_3(\sigma,\varepsilon) & \Lambda_3^T(\sigma,\varepsilon)\Lambda_4(\sigma,\varepsilon) \\ \Lambda_4^T(\sigma,\varepsilon)\Lambda_3(\sigma,\varepsilon) & \Lambda_4^T(\sigma,\varepsilon)\Lambda_4(\sigma,\varepsilon) \end{pmatrix} d\sigma \tag{4.12}$$

is invertible.

Proof. Due to Lemma 4.2 and the equation (4.10), we directly have that the system (1.1),(4.2) is completely Euclidean space controllable at the time instant t_c if and only if the matrix

$$\widetilde{M}(t_c,\varepsilon) = \int_0^{t_c} \Lambda^T(\sigma,\varepsilon)B(\varepsilon)B^T(\varepsilon)\Lambda(\sigma,\varepsilon)d\sigma \tag{4.13}$$

is invertible.

Using the block representations for the matrices $B(\varepsilon)$ and $\Lambda(\sigma,\varepsilon)$ (see equations (2.4) and (4.11)), we obtain after a routine rearrangement that

$$\widetilde{M}(t_c,\varepsilon) = (1/\varepsilon^2)M(t_c,\varepsilon). \tag{4.14}$$

This equality directly yields the statement of the corollary. □

4.1.2. *Asymptotic decomposition of the transformed system (1.1),(4.2).* Similarly to the asymptotic decomposition of the original system (1.1)-(1.2), we can decompose the transformed system (1.1), (4.2). Thus, the slow subsystem, associated with (1.1), (4.2), consists of the equation (3.2) and the equation

$$0 = -y_s(t) + v_s(t), \quad t \geq 0, \tag{4.15}$$

where, for any $t \geq 0$, $y_s(t) \in E^m$, $v_s(t) \in E^m$, (y_s is a state, while v_s is a control). The system (3.2), (4.15) is differential-algebraic. Eliminating the state variable y_s from this system, we obtain the purely differential form of the slow subsystem, associated with (1.1), (4.2):

$$\frac{dx_s(t)}{dt} = \sum_{i=0}^{1} \left[A_{1i}(t)x_s(t-h_i) + A_{2i}(t)v_s(t-h_i)\right]$$
$$+ \int_{-h}^{0} \left[G_1(t,\eta)x_s(t+\eta) + G_2(t,\eta)v_s(t+\eta)\right]d\eta, \quad t \geq 0, \tag{4.16}$$

where $\big(x_s(t), x_s(t+\eta)\big)$, $\eta \in [-h,0]$ is a state variable, while $\big(v_s(t), v_s(t+\eta)\big)$, $\eta \in [-h,0]$ is a control variable.

Remark 4.1. The slow subsystem (4.16), associated with the system (1.1), (4.2), coincides with the slow subsystem (3.2), associated with the system (1.1)-(1.2). Therefore, Definition 3.1, Lemma 3.1 and Corollary 3.1 also are valid for the system (4.16).

The fast subsystem, associated with the system (1.1), (4.2), is the differential equation

$$\frac{dy_f(\xi)}{d\xi} = -y_f(\xi) + v_f(\xi), \quad \xi \geq 0, \tag{4.17}$$

where, for any $\xi \geq 0$, $y_f(\xi) \in E^m$, $v_f(\xi) \in E^m$, (y_f is a state, while v_f is a control).

The complete controllability of the fast subsystem (4.17), associated with the system (1.1), (4.2), is defined similarly to such a controllability of the fast subsystem (3.16), associated with the system (1.1)-(1.2) (see Definition 3.2). Moreover, for the fast subsystem (4.17) the assertion, similar to Lemma 3.2, is valid.

4.2. **Some estimates of solutions to singularly perturbed matrix differential equations with delays.** In what follows, we assume:

A1. The matrix-valued functions $A_{ki}(t)$, $(k = 1,...,4;\ i = 0,1)$ are continuously differentiable in the interval $[0,t_c]$.

A2. The matrix-valued functions $G_k(t,\eta)$, $(k = 1,...,4)$ are piece-wise continuous with respect to $\eta \in [-h,0]$ for each $t \in [0,t_c]$, and they are continuously differentiable with respect to $t \in [0,t_c]$ uniformly in $\eta \in [-h,0]$.

As a particular case of the results of [17] (Lemma 6.2), we have the assertion.

Lemma 4.3. *Let the assumptions A1-A2 be valid. Let $\Psi_1(t,\sigma,\varepsilon)$, $\Psi_2(t,\sigma,\varepsilon)$, $\Psi_3(t,\sigma,\varepsilon)$, and $\Psi_4(t,\sigma,\varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand and lower right-hand blocks of the matrix $\Psi(t,\sigma,\varepsilon)$ of the dimensions $n\times n$, $n\times m$, $m\times n$ and $m\times m$, respectively. Then, there exists a positive number ε_1 such that, for all $\varepsilon \in (0,\varepsilon_1]$, the following inequalities are satisfied*

$$\left\|\Psi_l(t,\sigma,\varepsilon)\right\| \leq a, \quad l = 1,3, \qquad \left\|\Psi_2(t,\sigma,\varepsilon)\right\| \leq a\varepsilon, \quad 0 \leq \sigma \leq t \leq t_c, \tag{4.18}$$

$$\left\|\Psi_4(t,\sigma,\varepsilon)\right\| \leq a\left[\varepsilon + \exp\left(-\frac{\beta(t-\sigma)}{\varepsilon}\right)\right], \quad 0 \leq \sigma \leq t < \min\{\sigma + h_1, t_c\}, \tag{4.19}$$

and

$$\left\|\Psi_4(t,\sigma,\varepsilon)\right\| \leq a\varepsilon, \quad \sigma + h_1 \leq t \leq t_c, \tag{4.20}$$

where $\|\cdot\|$ is the Euclidean norm of a matrix; $a > 0$ and $\beta > 0$ are some constants independent of ε.

Let us denote

$$\Lambda_{1s}(\sigma) \stackrel{\triangle}{=} \Lambda_s(\sigma), \tag{4.21}$$

$$\Lambda_{3s}(\sigma) \stackrel{\triangle}{=} \begin{cases} -\sum_{i=0}^{1} A_{2i}^T(\sigma+h_i)\Lambda_{1s}(\sigma+h_i) - \int_{-h}^{0} G_2^T(\sigma-\eta,\eta)\Lambda_{1s}(\sigma-\eta)d\eta, & \sigma \in [0,t_c], \\ 0, & \sigma > t_c, \end{cases} \tag{4.22}$$

where $\Lambda_s(\sigma)$ is given by the equation (3.11).

Lemma 4.4. *Let the assumptions A1-A2 be valid. Then, for all $\varepsilon \in (0,\varepsilon_1]$, the following inequalities are satisfied:*

$$\left\|\Lambda_1(\sigma,\varepsilon) - \Lambda_{1s}(\sigma)\right\| \leq a\varepsilon, \quad \sigma \in [0,t_c], \tag{4.23}$$

$$\left\|\Lambda_3(\sigma,\varepsilon)-\varepsilon\Lambda_{3s}(\sigma)\right\| \le a\varepsilon \begin{cases} \left[\varepsilon+\exp\left(-\beta(t_c-\sigma)/\varepsilon\right)\right], & \text{if } \ t_c-h<\sigma\le t_c, \\ \left[\varepsilon+\exp\left(-\beta(t_c-h-\sigma)/\varepsilon\right)\right], & \text{if } \ 0\le\sigma\le t_c-h, \end{cases} \tag{4.24}$$

where $\Lambda_k(\sigma,\varepsilon)$, ($k=1,3$) are the corresponding blocks of the solution to the terminal-value problem (4.8) (see equation (4.11)); $a>0$ and $\beta>0$ are some constants independent of ε.

Proof. Let us denote

$$\Lambda_{13}(\sigma,\varepsilon) \stackrel{\triangle}{=} \begin{pmatrix} \Lambda_1(\sigma,\varepsilon) \\ \Lambda_3(\sigma,\varepsilon) \end{pmatrix}, \quad \Lambda_{13,s}(\sigma,\varepsilon) \stackrel{\triangle}{=} \begin{pmatrix} \Lambda_{1s}(\sigma) \\ \varepsilon\Lambda_{3s}(\sigma) \end{pmatrix}, \tag{4.25}$$

$$\Delta(\sigma,\varepsilon) \stackrel{\triangle}{=} \Lambda_{13}(\sigma,\varepsilon)-\Lambda_{13,s}(\sigma,\varepsilon). \tag{4.26}$$

Using the notations (4.21)-(4.22) and (4.25)-(4.26), the terminal-value problems (3.11) and (4.8), we obtain the terminal-value problem for the matrix-valued function $\Delta(\sigma,\varepsilon)$

$$\begin{aligned} \frac{d\Delta(\sigma,\varepsilon)}{d\sigma} &= -\mathscr{A}_0^T(\sigma,\varepsilon)\Delta(\sigma,\varepsilon)-A_1^T(\sigma+h)\Delta(\sigma+h,\varepsilon) \\ -\int_{-h}^{0} & G^T(\sigma-\eta,\eta)\Delta(\sigma-\eta,\varepsilon)d\eta+\Gamma(\sigma,\varepsilon), \quad \sigma\in[0,t_c], \\ \Delta(t_c,\varepsilon) &= \begin{pmatrix} 0 \\ -\varepsilon\Lambda_{3s}(t_c) \end{pmatrix}, \quad \Delta(\sigma,\varepsilon)=0, \quad \sigma>t_c, \end{aligned} \tag{4.27}$$

where the $(n+m)\times n$-matrix-valued function $\Gamma(\sigma,\varepsilon)$ has the block form

$$\Gamma(\sigma,\varepsilon)=\begin{pmatrix} \Gamma_1(\sigma,\varepsilon) \\ \Gamma_2(\sigma,\varepsilon) \end{pmatrix}, \tag{4.28}$$

and the $n\times n$-block $\Gamma_1(\sigma,\varepsilon)$ and $m\times n$-block $\Gamma_2(\sigma,\varepsilon)$ are, respectively,

$$\Gamma_1(\sigma,\varepsilon)=-\varepsilon\left[\sum_{i=0}^{1}A_{3i}^T(\sigma+h_i)\Lambda_{3s}(\sigma+h_i)+\int_{-h}^{0}G_3^T(\sigma-\eta,\eta)\Lambda_{3s}(\sigma-\eta)d\eta\right], \tag{4.29}$$

$$\Gamma_2(\sigma,\varepsilon)=-\varepsilon\left[\frac{d\Lambda_{3s}(\sigma)}{d\sigma}+\sum_{i=0}^{1}A_{4i}^T(\sigma+h_i)\Lambda_{3s}(\sigma+h_i)+\int_{-h}^{0}G_4^T(\sigma-\eta,\eta)\Lambda_{3s}(\sigma-\eta)d\eta\right)\right]. \tag{4.30}$$

Let us estimate $\Gamma_1(\sigma,\varepsilon)$ and $\Gamma_2(\sigma,\varepsilon)$. We start with the first matrix. Note, that $\Lambda_{1s}(\sigma)$ is bounded in the interval $[0,t_c]$. Therefore, $\Lambda_{3s}(\sigma)$ is bounded in this interval. This observation, the assumptions A1-A2 and the fact that $\Lambda_{3s}(\sigma)=0$ for $\sigma>t_c$ yield the estimate:

$$\left\|\Gamma_1(\sigma,\varepsilon)\right\| \le a\varepsilon, \quad \sigma\in[0,t_c], \quad \varepsilon\in(0,\varepsilon_1], \tag{4.31}$$

where $a>0$ is some constant independent of ε.

Proceed to $\Gamma_2(\sigma,\varepsilon)$. We can rewrite this matrix as:

$$\Gamma_2(\sigma,\varepsilon)=-\varepsilon\frac{d\Lambda_{3s}(\sigma)}{d\sigma}+\widetilde{\Gamma}(\sigma,\varepsilon), \tag{4.32}$$

where

$$\widetilde{\Gamma}(\sigma,\varepsilon)=-\varepsilon\left[\sum_{i=0}^{1}A_{4i}^T(\sigma+h_i)\Lambda_{3s}(\sigma+h_i)+\int_{-h}^{0}G_4^T(\sigma-\eta,\eta)\Lambda_{3s}(\sigma-\eta)d\eta\right]. \tag{4.33}$$

For $\widetilde{\Gamma}(\sigma,\varepsilon)$, similarly to (4.31), we have the estimate

$$\left\|\widetilde{\Gamma}(\sigma,\varepsilon)\right\| \le a\varepsilon, \quad \sigma \in [0,t_c], \quad \varepsilon \in (0,\varepsilon_1], \tag{4.34}$$

where $a > 0$ is some constant independent of ε.

Now, let us deal with the derivative $d\Lambda_{3s}(\sigma)/d\sigma$, $\sigma \in [0,t_c]$. Due to (4.22), we can represent $\Lambda_{3s}(\sigma)$ in the interval $[0,t_c]$ as:

$$\Lambda_{3s}(\sigma) = \Lambda_{3s}^1(\sigma) + \Lambda_{3s}^2(\sigma) + \Lambda_{3s}^3(\sigma), \tag{4.35}$$

where

$$\Lambda_{3s}^1(\sigma) = -A_{20}(\sigma)\Lambda_{1s}(\sigma), \quad \sigma \in [0,t_c], \tag{4.36}$$

$$\Lambda_{3s}^2(\sigma) = -A_{21}(\sigma+h)\Lambda_{1s}(\sigma+h), \quad \sigma \in [0,t_c], \tag{4.37}$$

$$\Lambda_{3s}^3(\sigma) = -\int_{\max\{-h,\sigma-t_c\}}^{0} G_2^T(\sigma-\eta,\eta)\Lambda_{1s}(\sigma-\eta)d\eta, \quad \sigma \in [0,t_c]. \tag{4.38}$$

Due to (3.11), (3.12) and (4.21), the derivatives $d\Lambda_{3s}^1(\sigma)/d\sigma$ and $d\Lambda_{3s}^3(\sigma)/d\sigma$ are piecewise continuous functions in the interval $[0,t_c]$ with a single finite break at $\sigma = t_c - h$. Therefore,

$$\left\|\frac{d\Lambda_{3s}^1(\sigma)}{d\sigma}\right\| \le a, \qquad \left\|\frac{d\Lambda_{3s}^3(\sigma)}{d\sigma}\right\| \le a, \quad \sigma \in [0,t_c]. \tag{4.39}$$

In contrast with $d\Lambda_{3s}^1(\sigma)/d\sigma$ and $d\Lambda_{3s}^3(\sigma)/d\sigma$, the derivative $d\Lambda_{3s}^2(\sigma)/d\sigma$ is a generalized function because the function $\Lambda_{3s}^2(\sigma)$ has a break at $\sigma = t_c - h$. Let us represent this function and its derivative in a form convenient for the further analysis. Let $F(\sigma)$ be an $m \times n$-matrix-valued function, differentiable in the interval $(-\infty,+\infty)$, and such that $F(\sigma) = \Lambda_{3s}^2(\sigma)$ in the interval $[0,t_c-h]$. Thus, we can represent the matrix $\Lambda_{3s}^2(\sigma)$ in the form

$$\Lambda_{3s}^2(\sigma) = F(\sigma)\big(1-\theta(\sigma-t_c+h)\big), \quad \sigma \in [0,t_c]. \tag{4.40}$$

where $\theta(\sigma-t_c+h)$ is the Heaviside step function with the break point $\sigma = t_c - h$. Using this representation, we obtain the representation for the generalized derivative $d\Lambda_{3s}^2(\sigma)/d\sigma$

$$\frac{d\Lambda_{3s}^2(\sigma)}{d\sigma} = \frac{dF(\sigma)}{d\sigma}\theta(\sigma-t_c+h) - F(\sigma)\delta(\sigma-t_c+h), \quad \sigma \in [0,t_c], \tag{4.41}$$

where $\delta(\sigma-t_c+h)$ is the Dirac delta-function with the impulse at $\sigma = t_c - h$.

By application of the results of [29] (Section 4.3), we can rewrite the problem (4.27) in the equivalent integral form

$$\Delta(\sigma,\varepsilon) = \Psi^T(t_c,\sigma,\varepsilon)\Delta(t_c,\varepsilon) + \int_{t_c}^{\sigma} \Psi^T(t,\sigma,\varepsilon)\Gamma(t,\varepsilon)dt, \quad \sigma \in [0,t_c], \tag{4.42}$$

where $\Psi(t,\sigma,\varepsilon)$, $0 \le \sigma \le t \le t_c$ is given by (2.7).

The equation (4.42), along with Lemma 4.3, the expression for $\Delta(t_c,\varepsilon)$ (see the equation (4.27)), the block representation of $\Gamma(\sigma,\varepsilon)$ (see the equation (4.28)), the estimate for $\Gamma_1(\sigma,\varepsilon)$ (see the equation (4.31)), directly yields the following inequality:

$$\left\|\Delta_1(\sigma,\varepsilon)\right\| \le a\varepsilon, \quad \sigma \in [0,t_c], \quad \varepsilon \in (0,\varepsilon_1], \tag{4.43}$$

where $\Delta_1(\sigma,\varepsilon)$ is the upper block of the matrix $\Delta(\sigma,\varepsilon)$ of the dimension $n \times n$; $a > 0$ is some constant independent of ε.

Similarly, using the equations (4.32), (4.35)-(4.38), (4.41) and the inequalities (4.34), (4.39), we obtain

$$\|\Delta_2(\sigma,\varepsilon)\| \le a\varepsilon \begin{cases} \left[\varepsilon + \exp\left(-\beta(t_c-\sigma)/\varepsilon\right)\right], & \text{if } \ t_c - h < \sigma \le t_c, \\ \left[\varepsilon + \exp\left(-\beta(t_c-h-\sigma)/\varepsilon\right)\right], & \text{if } \ 0 \le \sigma \le t_c - h, \end{cases} \tag{4.44}$$

where $\Delta_2(\sigma,\varepsilon)$ is the lower block of the matrix $\Delta(\sigma,\varepsilon)$ of the dimension $m \times n$; $a > 0$ and $\beta > 0$ are some constants independent of ε.

The inequalities (4.43) and (4.44), along with the notations (4.25) and (4.26), imply the validity of the inequalities stated in the lemma. □

Let the $m \times m$-matrix-valued function $\Lambda_{4f}(\xi)$ be the solution of the problem

$$\frac{d\Lambda_{4f}(\xi)}{d\xi} = -\Lambda_{4f}(\xi), \quad \xi > 0,$$
$$\Lambda_{4f}(0) = I_m; \quad \Lambda_{4f}(\xi) = 0, \quad \xi < 0. \tag{4.45}$$

We directly have

$$\Lambda_{4f}(\xi) = \exp(-\xi) I_m, \quad \|\Lambda_{4f}(\xi)\| = \exp(-\xi), \quad \xi \ge 0. \tag{4.46}$$

Lemma 4.5. *Let the assumptions A1-A2 be valid. Then, there exists a positive number ε_2 such that, for all $\varepsilon \in (0,\varepsilon_2]$, the following inequalities are satisfied:*

$$\|\Lambda_2(\sigma,\varepsilon)\| \le a\varepsilon, \quad \left\|\Lambda_4(\sigma,\varepsilon) - \Lambda_{4f}\left(\frac{t_c-\sigma}{\varepsilon}\right)\right\| \le a\varepsilon, \quad \sigma \in [0,t_c], \tag{4.47}$$

where $\Lambda_l(\sigma,\varepsilon)$, ($l = 2,4$) are the corresponding blocks of the solution to the terminal-value problem (4.8) (see equation (4.11)); $a > 0$ is some constant independent of ε.

Proof. The lemma is proven similarly to Lemma 4.4. □

5. Lower dimension controllability condition for system (1.1)-(1.2)

Theorem 5.1. *Let the assumptions A1-A2 be valid. Let the slow subsystem (3.2) be completely Euclidean space controllable at the time instant t_c. Then, there exists a positive number $\varepsilon^* \le \min\{\varepsilon_1,\varepsilon_2\}$ such that, for all $\varepsilon \in (0,\varepsilon^*]$, the system (1.1)-(1.2) is completely Euclidean space controllable at the time instant t_c.*

Proof. First, let us show that, for all sufficiently small $\varepsilon > 0$, the system (1.1), (4.2) is completely Euclidean space controllable at the time instant t_c.

Denote

$$M_1(t_c,\varepsilon) \triangleq \int_0^{t_c} \Lambda_3^T(\sigma,\varepsilon)\Lambda_3(\sigma,\varepsilon)d\sigma, \tag{5.1}$$

$$M_2(t_c,\varepsilon) \triangleq \int_0^{t_c} \Lambda_3^T(\sigma,\varepsilon)\Lambda_4(\sigma,\varepsilon)d\sigma, \tag{5.2}$$

$$M_3(t_c,\varepsilon) \triangleq \int_0^{t_c} \Lambda_4^T(\sigma,\varepsilon)\Lambda_4(\sigma,\varepsilon)d\sigma. \tag{5.3}$$

Using these notations, the matrix $M(t_c,\varepsilon)$, given by (4.12), can be rewritten as:

$$M(t_c,\varepsilon) = \begin{pmatrix} M_1(t_c,\varepsilon) & M_2(t_c,\varepsilon) \\ M_2^T(t_c,\varepsilon) & M_3(t_c,\varepsilon) \end{pmatrix}. \tag{5.4}$$

Applying Lemmas 4.4-4.5 and the equation (4.46) to the equations (5.1)-(5.3), we obtain the existence of a positive number $\bar{\varepsilon}_1 \leq \min\{\varepsilon_1, \varepsilon_2\}$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_1]$, the following inequalities are satisfied:

$$\left\| M_1(t_c,\varepsilon) - \varepsilon^2 \bar{M}_1(t_c) \right\| \leq a\varepsilon^3, \tag{5.5}$$

$$\left\| M_2(t_c,\varepsilon) - \varepsilon^2 \bar{M}_2(t_c) \right\| \leq a\varepsilon^2, \tag{5.6}$$

$$\left\| M_3(t_c,\varepsilon) - \varepsilon \bar{M}_3 \right\| \leq a\varepsilon^2, \tag{5.7}$$

where

$$\bar{M}_1(t_c) = \int_0^{t_c} \Lambda_{3s}^T(\sigma)\Lambda_{3s}(\sigma)d\sigma, \quad \bar{M}_2(t_c) = \Lambda_{3s}^T(t_c), \quad \bar{M}_3 = \frac{1}{2} I_m, \tag{5.8}$$

$a > 0$ is some constant independent of ε.

Using the assumption of the theorem on the complete Euclidean space controllability of the slow subsystem (3.2) at the time instant t_c, as well as the equation (4.22), we obtain that the matrix $\bar{M}_1(t_c)$ is invertible, i.e.,

$$\det \bar{M}_1(t_c) \neq 0. \tag{5.9}$$

Denote

$$L(\varepsilon) \triangleq \begin{pmatrix} (1/\varepsilon)I_n & 0 \\ 0 & (1/\sqrt{\varepsilon})I_m \end{pmatrix}. \tag{5.10}$$

It is clear that for all $\varepsilon > 0$

$$\det\big(L(\varepsilon)\big) \neq 0. \tag{5.11}$$

Using this matrix, the inequalities (5.5)-(5.7) and the equation (5.8), we directly obtain the existence of a positive number $\bar{\varepsilon}_2 \leq \bar{\varepsilon}_1$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_2]$, the following inequality is valid:

$$\left| \det\big(L(\varepsilon)M(t_c,\varepsilon)L(\varepsilon)\big) - (1/2^m)\det\big(\bar{M}_1(t_c)\big) \right| \leq a\varepsilon, \tag{5.12}$$

where $a > 0$ is some constant independent of ε.

The inequalities (5.9), (5.11) and (5.12) imply the existence of a positive number $\varepsilon^* \leq \bar{\varepsilon}_2$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, the following inequality is valid:

$$\det\big(M(t_c,\varepsilon)\big) \neq 0. \tag{5.13}$$

Thus, by virtue of Corollary 4.1, the system (1.1), (4.2) is completely Euclidean space controllable at the time instant t_c for all $\varepsilon \in (0, \varepsilon^*]$. The latter, along with Lemma 4.1, directly yields the complete Euclidean space controllability of the original system (1.1)-(1.2) at the time instant t_c for all $\varepsilon \in (0, \varepsilon^*]$. This completes the proof of the theorem. □

6. Minimum energy control problem

6.1. **Minimum energy control of the system (1.1), (4.2).** Along with the system (1.1), (4.2), let us consider the functional

$$J(v) = \int_0^{t_c} v^T(t)R(t)v(t)dt, \tag{6.1}$$

where $R(t)$ is a given $m \times m$-matrix-valued function.

In what follows, we assume:

A3. The matrix-valued function $R(t)$ is continuously differentiable in the interval $[0, t_c]$, and for any $t \in [0, t_c]$ the matrix $R(t)$ is symmetric and positive definite.

Let $f_{x0}(\eta) = \big(x_0, \varphi_x(\eta)\big) \in \mathscr{M}[-h,0;E^n]$, $f_{y0}(\eta) = \big(y_0, \varphi_y(\eta)\big) \in \mathscr{M}[-h,0;E^m]$, $x_c \in E^n$ and $y_c \in E^m$ be arbitrary given. For a given $\varepsilon > 0$, consider the set $V(\varepsilon)$ of all controls $v(t,\varepsilon) \in L^2[0,t_c;E^m]$ such that the system (1.1), (4.2) with $v(t) = v(t,\varepsilon)$, and with the initial conditions (2.1) and terminal conditions (2.3) has a solution.

Let $\varepsilon > 0$ by any given. The minimum energy control problem for the system (1.1), (4.2) is to find a control $v^*(t,\varepsilon) \in V(\varepsilon)$ such that

$$J\big(v^*(t,\varepsilon)\big) \le J\big(v(t,\varepsilon)\big) \quad \forall v(t,\varepsilon) \in V(\varepsilon). \tag{6.2}$$

Based on the Halanay Transformation for a linear system with state delays (see [30]), let us make the following change of the state variable in the system (1.1), (4.2):

$$w(t) = \Lambda^T(t,\varepsilon)z(t) + \int_t^{t+h} \Lambda^T(\rho,\varepsilon)A_1(\rho)z(\rho-h)d\rho$$
$$+ \int_t^{t+h}\left(\int_t^{\rho} \Lambda^T(\chi,\varepsilon)G\big(\chi,\rho-\chi-h\big)d\chi\right)z(\rho-h)d\rho, \tag{6.3}$$

where $w(\cdot)$ is a new state variable; $z(\cdot)$ is the original state variable given by (2.5); the matrix-valued function $\Lambda(t,\varepsilon)$ is given by (4.8).

Denote

$$\mathscr{B}(t,\varepsilon) \triangleq \Lambda^T(t,\varepsilon)B(\varepsilon), \quad t \in [0,t_c], \tag{6.4}$$

$$w_0(\varepsilon) \triangleq \Lambda^T(0,\varepsilon)z_0 + \int_{-h}^{0} \Lambda^T(\eta+h,\varepsilon)A_1(\eta+h)\varphi_z(\eta)d\eta$$
$$+ \int_{-h}^{0}\left(\int_0^{\eta+h} \Lambda^T(\sigma,\varepsilon)G\big(\sigma,\eta-\sigma\big)d\sigma\right)\varphi_z(\eta)d\eta, \tag{6.5}$$

where

$$z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \varphi_z(\eta) = \begin{pmatrix} \varphi_x(\eta) \\ \varphi_y(\eta) \end{pmatrix}. \tag{6.6}$$

By virtue of the results of [30, 23], we directly obtain the following assertion.

Proposition 6.1. *Let for any given $\varepsilon > 0$ and control $v(t,\varepsilon) \in V(\varepsilon)$, the vector-valued function $z(t,\varepsilon) = \mathrm{col}\big(x(t,\varepsilon), y(t,\varepsilon)\big)$, $t \in [0,t_c]$ be the absolutely continuous solution of the system (1.1), (4.2) subject to the conditions (2.1) and (2.3). Then the vector-valued function $w(t,\varepsilon)$, $t \in [0,t_c]$ is the unique absolutely continuous solution of the differential equation*

$$\frac{dw(t)}{dt} = \mathscr{B}(t,\varepsilon)v(t), \tag{6.7}$$

subject to the initial

$$w(0) = w_0(\varepsilon) \tag{6.8}$$

and terminal

$$w(t_c) = z(t_c,\varepsilon) = z_c \tag{6.9}$$

conditions, where

$$z_c = \begin{pmatrix} x_c \\ y_c \end{pmatrix}. \tag{6.10}$$

For a given $\varepsilon > 0$, consider the set $V_w(\varepsilon)$ of all controls $v(t,\varepsilon) \in L^2[0,t_c;E^m]$ such that the differential equation (6.7) with $v(t) = v(t,\varepsilon)$, and with the initial (6.8) and terminal (6.9) conditions has a solution. For a given $\varepsilon > 0$, the minimum energy control problem for the differential equation (6.7) is to find a control $v_w^*(t,\varepsilon) \in V_w(\varepsilon)$ such that

$$J\big(v_w^*(t,\varepsilon)\big) \le J\big(v(t,\varepsilon)\big) \quad \forall v(t,\varepsilon) \in V_w(\varepsilon), \tag{6.11}$$

where the functional $J(v)$ is given by (6.1).

Proposition 6.2. *Let, for a given $\varepsilon > 0$, $v(t,\varepsilon) \in V_w(\varepsilon)$. Then, $v(t,\varepsilon) \in V(\varepsilon)$.*

Proof. Let the statement of the proposition be wrong. Then, the solution $z(t,\varepsilon) = \mathrm{col}\big((x(t,\varepsilon), y(t,\varepsilon)\big)$ of the initial-value problem (1.1), (4.2), (2.1) with $v(t) = v(t,\varepsilon)$ satisfies the inequality $z(t_c,\varepsilon) \ne z_c$. By virtue of Proposition 6.1 (the first equality in (6.9)), the solution $w(t,\varepsilon)$ of the initial-value problem (6.7)-(6.8) with $v(t) = v(t,\varepsilon)$ satisfies the equality $w(t_c,\varepsilon) = z(t_c,\varepsilon)$, and therefore, the inequality $w(t_c,\varepsilon) \ne z_c$. The latter contradicts the inclusion $v(t,\varepsilon) \in V_w(\varepsilon)$, meaning that the statement of the proposition is true. □

Propositions 6.1 and 6.2 directly imply that for any $\varepsilon > 0$:

$$V_w(\varepsilon) = V(\varepsilon). \tag{6.12}$$

Proposition 6.3. *If for a given $\varepsilon > 0$, $v^*(t,\varepsilon)$ is a solution of the minimum energy control problem for the system (1.1), (4.2), then it is a solution of the minimum energy control problem for the differential equation (6.7). Vice versa: if for a given $\varepsilon > 0$, $v_w^*(t,\varepsilon)$ is a solution of the minimum energy control problem for the differential equation (6.7), then it is a solution of the minimum energy control problem for the system (1.1), (4.2).*

Proof. Based on the equation (6.12), the proposition is proven similarly to [23] (Proposition 2). □

Denote

$$W_{\mathscr{B}}(t_c,\varepsilon) \triangleq \int_0^{t_c} \mathscr{B}(t,\varepsilon) R^{-1}(t) \mathscr{B}^T(t,\varepsilon)dt. \tag{6.13}$$

Using the expression (6.4) for the matrix $\mathscr{B}(t,\varepsilon)$, the equation (6.13) can be rewritten as:

$$W_{\mathscr{B}}(t_c,\varepsilon) = \int_0^{t_c} \Lambda^T(t,\varepsilon) B(\varepsilon) R^{-1}(t) B^T(\varepsilon) \Lambda(t,\varepsilon)dt. \tag{6.14}$$

Theorem 6.1. *Let the assumptions A1-A3 be valid. Let the slow subsystem (4.16), associated with the system (1.1), (4.2), be completely Euclidean space controllable at the time instant t_c. Then, for all $\varepsilon \in (0,\varepsilon^*]$ (ε^* is introduced in Theorem 5.1), the solution of the minimum energy control problem for the system (1.1), (4.2) exists and has the form:*

$$v^*(t,\varepsilon) = R^{-1}(t)\mathscr{B}^T(t,\varepsilon) W_{\mathscr{B}}^{-1}(t_c,\varepsilon)\big(z_c - w_0(\varepsilon)\big), \quad t \in [0,t_c], \tag{6.15}$$

where $w_0(\varepsilon)$ is given by (6.5).

Moreover,

$$J_\varepsilon^* \triangleq J\big(v^*(t,\varepsilon)\big) = \big(z_c - w_0(\varepsilon)\big)^T W_{\mathscr{B}}^{-1}(t_c,\varepsilon)\big(z_c - w_0(\varepsilon)\big). \tag{6.16}$$

Proof. First, let us consider the minimum energy control problem for the differential equation (6.7). This equation is undelayed. Applying the well known result on the solution of the minimum energy control problem for undelayed systems (see e.g. [5, 40] and references therein) to the equation (6.7), we obtain the following. If for a given $\varepsilon > 0$, the matrix $W_{\mathscr{B}}(t_c,\varepsilon)$ is invertible, then the control (6.15) solves the minimum energy problem for this equation. Moreover, the optimal value of the functional in this problem has the form (6.16). Comparing the equations (4.13) and (6.14), and taking into account the positive definiteness of the matrix $R(t)$, one can conclude that both matrices $\widetilde{M}(t_c,\varepsilon)$ and $W_{\mathscr{B}}(t_c,\varepsilon)$ either are invertible or are noninvertible simultaneously. Due to Remark 4.1, as well as the proof of Theorem 5.1 and the equation (4.14), the matrix $\widetilde{M}(t_c,\varepsilon)$ is invertible for all $\varepsilon \in (0,\varepsilon^*]$. Therefore, the matrix $W_{\mathscr{B}}(t_c,\varepsilon)$ is invertible for these values of ε. Now, the statement of the theorem follows immediately from Proposition 6.3. □

6.2. **Minimum energy control of system (4.16).** Along with system (4.16), we consider the functional

$$J_s(v_s) = \int_0^{t_c} v_s^T(t)R(t)v_s(t)dt, \tag{6.17}$$

where $R(t)$ is the same $m\times m$-matrix-valued function as in (6.1).

Let $f_{x0}(\eta) = \big(x_0, \varphi_x(\eta)\big) \in \mathscr{M}[-h,0;E^n]$, $\varphi_y(\eta) \in L^2[-h,0;E^m]$ and $x_c \in E^n$ be arbitrary given. Let us consider the set V_s of all controls $v_s(t) \in L^2[0,t_c;E^m]$ such that system (4.16) with the initial (3.4) and terminal (3.6) conditions for the state variable, and with the initial condition for the control variable

$$v_s(\eta) = \varphi_y(\eta), \quad \eta \in [-h,0), \tag{6.18}$$

has a solution.

The minimum energy control problem for the system (4.16) is to find a control $v_s^*(t) \in V_s$ such that

$$J_s\big(v_s^*(t)\big) \le J_s\big(v_s(t)\big) \quad \forall v_s(t) \in V_s. \tag{6.19}$$

Based on the Halanay Transformation for a linear system with state delays (see [30]) and on the Kwon-Pearson-Artstein transformation for a linear system with control delays (see [51, 1]), we make the following change of the state variable in system (4.16):

$$w_s(t) = p_s(t) + \int_{t-h}^{t} \Lambda_s^T(\zeta+h)A_{21}(\zeta+h)v_s(\zeta)d\zeta$$
$$+\int_{-h}^{0} d\eta\left(\int_{t+\eta}^{t} \Lambda_s^T(\chi-\eta)G_2(\chi-\eta,\eta)v_s(\chi)d\chi\right), \tag{6.20}$$

where

$$p_s(t) = \Lambda_s^T(t)x_s(t) + \int_t^{t+h} \Lambda_s^T(\rho)A_{11}(\rho)x_s(\rho-h)d\rho$$
$$+\int_t^{t+h}\left(\int_t^{\rho} \Lambda_s^T(\chi)G_1\big(\chi,\rho-\chi-h\big)d\chi\right)x_s(\rho-h)d\rho; \tag{6.21}$$

$w_s(\cdot)$ is a new state variable; $x_s(\cdot)$ is the original state variable; the matrix-valued function $\Lambda_s(t)$ is given by (3.11).

Denote

$$p_{s,0} \triangleq \Lambda_s^T(0)x_0 + \int_{-h}^{0} \Lambda_s^T(\eta+h)A_{11}(\eta+h)\varphi_x(\eta)d\eta$$
$$+ \int_{-h}^{0} \left(\int_{0}^{\eta+h} \Lambda_s^T(\sigma)G_1\Big(\sigma,\eta-\sigma\Big)d\sigma \right) \varphi_x(\eta)d\eta, \tag{6.22}$$

$$w_{s,0} \triangleq p_{s,0} + \int_{-h}^{0} \Lambda_s^T(\eta+h)A_{21}(\eta+h)\varphi_y(\eta)d\eta$$
$$+ \int_{-h}^{0} d\eta \left(\int_{\eta}^{0} \Lambda_s^T(\chi-\eta)G_2(\chi-\eta,\eta)\varphi_y(\chi)d\chi \right). \tag{6.23}$$

The routine change of the integration order and a transformation of the integration variable in the resulting integral yield

$$w_{s,0} = p_{s,0} + \int_{-h}^{0} \Lambda_s^T(\eta+h)A_{21}(\eta+h)\varphi_y(\eta)d\eta$$
$$+ \int_{-h}^{0} \left(\int_{0}^{\eta+h} \Lambda_s^T(\sigma)G_2(\sigma,\eta-\sigma)d\sigma \right) \varphi_y(\eta)d\eta. \tag{6.24}$$

By virtue of the results of [23], we directly obtain the following assertion.

Proposition 6.4. *Let for any given control $v_s(t) \in V_s$, the vector-valued function $x_s(t)$, $t \in [0,t_c]$ be the absolutely continuous solution of the system (4.16) subject to the conditions (3.4), (3.6) and (6.18). Then the vector-valued function $w_s(t)$, $t \in [0,t_c]$ is the unique absolutely continuous solution of the differential equation*

$$\frac{dw_s(t)}{dt} = C_s(t)v_s(t), \tag{6.25}$$

subject to the initial

$$w(0) = w_{s,0} \tag{6.26}$$

and terminal

$$w_s(t_c) = x_s(t_c) = x_c \tag{6.27}$$

conditions, where the matrix-valued function $C_s(t)$ is given by (3.14).

Consider the set $V_{s,w}$ of all controls $v_s(t) \in L^2[0,t_c;E^m]$ such that the differential equation (6.25) with the initial (6.26) and terminal (6.27) conditions has a solution. The minimum energy control problem for the differential equation (6.25) is to find a control $v_{s,w}^*(t) \in V_{s,w}$ such that

$$J_s\big(v_{s,w}^*(t)\big) \le J_s\big(v_s(t)\big) \quad \forall v_s(t) \in V_{s,w}, \tag{6.28}$$

where the functional $J_s(v)$ is given by (6.17).

Similarly to the equation (6.12), we obtain the following equation

$$V_{s,w} = V_s. \tag{6.29}$$

Moreover, similarly to Proposition 6.3, we have the assertion.

Proposition 6.5. *If $v_s^*(t)$ is a solution of the minimum energy control problem for the system (4.16), then it is a solution of the minimum energy control problem for the differential equation (6.25). Vice versa: if $v_{s,w}^*(t)$ is a solution of the minimum energy control problem for the differential equation (6.25), then it is a solution of the minimum energy control problem for the system (4.16).*

Denote

$$W_C(t_c) \stackrel{\triangle}{=} \int_0^{t_c} C_s(t)R^{-1}(t)C_s^T(t)dt. \tag{6.30}$$

Based on Proposition 6.5, we obtain (similarly to Theorem 6.1) the following theorem.

Theorem 6.2. *Let the assumptions A1-A3 be valid. Let the system (4.16) be completely Euclidean space controllable at the time instant t_c. Then, the solution of the minimum energy control problem for this system exists and has the form:*

$$v_s^*(t) = R^{-1}(t)C_s^T(t)W_C^{-1}(t_c)\big(x_c - w_{s,0}\big), \quad t \in [0,t_c]. \tag{6.31}$$

Moreover,

$$J_s^* \stackrel{\triangle}{=} J_s\big(v_s^*(t)\big) = \big(x_c - w_{s,0}\big)^T W_C^{-1}(t_c)\big(x_c - w_{s,0}\big). \tag{6.32}$$

6.3. **Relation between solutions of the minimum energy control problems for the systems (1.1), (4.2) and (4.16).** Denote

$$N_{\mathscr{B}}(t_c,\varepsilon) \stackrel{\triangle}{=} W_{\mathscr{B}}^{-1}(t_c,\varepsilon) = \begin{pmatrix} N_{\mathscr{B},1}(t_c,\varepsilon) & N_{\mathscr{B},2}(t_c,\varepsilon) \\ N_{\mathscr{B},2}^T(t_c,\varepsilon) & N_{\mathscr{B},3}(t_c,\varepsilon) \end{pmatrix}, \tag{6.33}$$

where the blocks $N_{\mathscr{B},1}(t_c,\varepsilon)$, $N_{\mathscr{B},2}(t_c,\varepsilon)$ and $N_{\mathscr{B},3}(t_c,\varepsilon)$ are of the dimensions $n \times n$, $n \times m$ and $m \times m$, respectively.

Lemma 6.1. *Let the assumptions A1-A3 be valid. Let the system (4.16) be completely Euclidean space controllable at the time instant t_c. Then, there exists a positive number $\tilde{\varepsilon}_1 \le \varepsilon^*$ such that, for all $\varepsilon \in (0,\tilde{\varepsilon}_1]$, the following inequalities are satisfied:*

$$\big\|N_{\mathscr{B},1}(t_c,\varepsilon) - W_C^{-1}(t_c\big\| \le a\varepsilon, \quad \big\|N_{\mathscr{B},l}(t_c,\varepsilon)\big\| \le a\varepsilon, \quad l = 2,3, \tag{6.34}$$

where $a > 0$ is some constant independent of ε.

Proof. Let us partition the matrix $W_{\mathscr{B}}(t_c,\varepsilon)$ into blocks as:

$$W_{\mathscr{B}}(t_c,\varepsilon) = \begin{pmatrix} W_{\mathscr{B},1}(t_c,\varepsilon) & W_{\mathscr{B},2}(t_c,\varepsilon) \\ W_{\mathscr{B},2}^T(t_c,\varepsilon) & W_{\mathscr{B},3}(t_c,\varepsilon) \end{pmatrix}, \tag{6.35}$$

where the blocks $W_{\mathscr{B},1}(t_c,\varepsilon)$, $W_{\mathscr{B},2}(t_c,\varepsilon)$ and $W_{\mathscr{B},3}(t_c,\varepsilon)$ are of the dimensions $n \times n$, $n \times m$ and $m \times m$, respectively.

Due to the equations (2.4), (4.11) and (6.14),

$$W_{\mathscr{B},1}(t_c,\varepsilon) = \frac{1}{\varepsilon^2}\int_0^{t_c} \Lambda_3^T(t,\varepsilon)R^{-1}(t)\Lambda_3(t,\varepsilon)dt, \tag{6.36}$$

$$W_{\mathscr{B},2}(t_c,\varepsilon) = \frac{1}{\varepsilon^2}\int_0^{t_c} \Lambda_3^T(t,\varepsilon)R^{-1}(t)\Lambda_4(t,\varepsilon)dt, \tag{6.37}$$

$$W_{\mathscr{B},3}(t_c,\varepsilon) = \frac{1}{\varepsilon^2}\int_0^{t_c} \Lambda_4^T(t,\varepsilon)R^{-1}(t)\Lambda_4(t,\varepsilon)dt. \tag{6.38}$$

Using Lemmas 4.4-4.5 and the equations (3.14), (4.21)-(4.22), (4.46), (6.30), we obtain the existence of a positive number $\hat{\varepsilon}_1 \le \varepsilon^*$ such that the following inequalities are satisfied:

$$\big\|W_{\mathscr{B},1}(t_c,\varepsilon) - W_C(t_c)\big\| \le a\varepsilon, \quad \varepsilon \in (0,\hat{\varepsilon}_1], \tag{6.39}$$

$$\big\|W_{\mathscr{B},2}(t_c,\varepsilon) - \Lambda_{3s}^T(t_c)R^{-1}(t_c)\big\| \le a, \quad \varepsilon \in (0,\hat{\varepsilon}_1], \tag{6.40}$$

$$\big\|W_{\mathscr{B},3}(t_c,\varepsilon) - (1/2\varepsilon)R^{-1}(t_c)\big\| \le a, \quad \varepsilon \in (0,\hat{\varepsilon}_1]. \tag{6.41}$$

where $a > 0$ is some constant independent of ε.

Remember that, due to the complete Euclidean space controllability of the system (4.16), the matrix $W_C(t_c)$ is invertible. Therefore, by virtue of the inequality (6.39), there exists a positive number $\hat{\varepsilon}_2 \le \hat{\varepsilon}_1$ such that the matrix $W_{\mathscr{B},1}(t_c,\varepsilon)$ is invertible for all $\varepsilon \in (0,\hat{\varepsilon}_2]$, and

$$\left\| W_{\mathscr{B},1}^{-1}(t_c,\varepsilon) - W_C^{-1}(t_c) \right\| \le a\varepsilon, \quad \varepsilon \in (0,\hat{\varepsilon}_2]. \tag{6.42}$$

Moreover, by virtue of Theorem 5.1, the matrix $W_{\mathscr{B}}(t_c,\varepsilon)$ is invertible for all these ε.

Applying the Frobenius formula (see e.g. [13]) to the calculation of the inverse matrix for $W_{\mathscr{B}}(t_c,\varepsilon)$, given by (6.35), we obtain the matrix $N_{\mathscr{B}}(t_c,\varepsilon)$ (see (6.33)), where

$$N_{\mathscr{B},1}(t_c,\varepsilon) = W_{\mathscr{B},1}^{-1}(t_c,\varepsilon) + W_{\mathscr{B},1}^{-1}(t_c,\varepsilon)W_{\mathscr{B},2}(t_c,\varepsilon)H(t_c,\varepsilon)W_{\mathscr{B},2}^{T}(t_c,\varepsilon)W_{\mathscr{B},1}^{-1}(t_c,\varepsilon), \tag{6.43}$$

$$N_{\mathscr{B},2}(t_c,\varepsilon) = -W_{\mathscr{B},1}^{-1}(t_c,\varepsilon)W_{\mathscr{B},2}(t_c,\varepsilon)H(t_c,\varepsilon), \tag{6.44}$$

$$N_{\mathscr{B},3}(t_c,\varepsilon) = H(t_c,\varepsilon), \tag{6.45}$$

and

$$H(t_c,\varepsilon) = \Big(W_{\mathscr{B},3}(t_c,\varepsilon) - W_{\mathscr{B},2}^{T}(t_c,\varepsilon)W_{\mathscr{B},1}^{-1}(t_c,\varepsilon)W_{\mathscr{B},2}(t_c,\varepsilon) \Big)^{-1}. \tag{6.46}$$

Using the inequalities (6.40)-(6.41) and (6.42), we obtain the existence of a positive number $\hat{\varepsilon}_3 \le \hat{\varepsilon}_2$ such that

$$\left\| H(t_c,\varepsilon) \right\| \le a\varepsilon, \quad \varepsilon \in (0,\hat{\varepsilon}_3], \tag{6.47}$$

where $a > 0$ is some constant independent of ε.

Now, the equations (6.43)-(6.45) and the inequalities (6.40)-(6.41), (6.42), (6.47) directly yield the inequalities in (6.34), which completes the proof of the lemma. □

Let us partition the vector $w_0(\varepsilon)$, given by (6.5), into blocks as:

$$w_0(\varepsilon) = \begin{pmatrix} w_{0,1}(\varepsilon) \\ w_{0,2}(\varepsilon) \end{pmatrix}, \quad w_{0,1}(\varepsilon) \in E^n, \quad w_{0,2}(\varepsilon) \in E^m. \tag{6.48}$$

Lemma 6.2. *Let the assumptions A1-A2 be valid. Then, there exists a positive number $\tilde{\varepsilon}_2 \le \varepsilon^*$ such that, for all $\varepsilon \in (0,\tilde{\varepsilon}_2]$, the following inequalities are satisfied:*

$$\left\| w_{0,1}(\varepsilon) - w_{s,0} \right\| \le a\big(z_0,\varphi_z(\eta)\big)\varepsilon, \quad \left\| w_{0,2}(\varepsilon) \right\| \le a\big(z_0,\varphi_z(\eta)\big)\varepsilon^{1/2}, \tag{6.49}$$

where the vector $w_{s,0}$ is given by (6.24); $a\big(z_0,\varphi_z(\eta)\big) > 0$ is some constant independent of ε, while dependent on z_0 and $\varphi_z(\eta)$.

Proof. Let us represent the vector $w_0(\varepsilon)$ as follows:

$$w_0(\varepsilon) = b_1(\varepsilon) + b_2(\varepsilon) + b_3(\varepsilon), \tag{6.50}$$

where

$$b_1(\varepsilon) = \Lambda^T(0,\varepsilon)z_0, \tag{6.51}$$

$$b_2(\varepsilon) = \int_{-h}^{0} \Lambda^T(\eta+h,\varepsilon)A_1(\eta+h)\varphi_z(\eta)d\eta, \tag{6.52}$$

$$b_2(\varepsilon) = \int_{-h}^{0} \left(\int_{0}^{\eta+h} \Lambda^T(\sigma,\varepsilon)G\big(\sigma,\eta-\sigma\big)d\sigma \right) \varphi_z(\eta)d\eta. \tag{6.53}$$

Let us partition the vectors $b_k(\varepsilon)$, $(k=1,2,3)$ into blocks as:

$$b_k(\varepsilon)=\begin{pmatrix} b_{k,1}(\varepsilon)\\ b_{k,2}(\varepsilon)\end{pmatrix},\quad b_{k,1}(\varepsilon)\in E^n,\quad b_{k,2}(\varepsilon)\in E^m,\quad k=1,2,3. \tag{6.54}$$

Using the equations (2.4), (4.11), (4.21)-(4.22), (6.6), Lemmas 4.4-4.5, and the Cauchy-Bunyakovsky-Schwarz inequality directly yields the existence of a positive number $\tilde{\varepsilon}_2\le\varepsilon^*$ such that, for all $\varepsilon\in(0,\tilde{\varepsilon}_2]$, the following inequalities are satisfied:

$$\left\|b_{1,1}(\varepsilon)-\Lambda_{1s}^T(0)x_0\right\|\le a_1(z_0)\varepsilon,\qquad \left\|b_{1,2}(\varepsilon)\right\|\le a_1(z_0)\varepsilon, \tag{6.55}$$

$$\Bigg\|b_{2,1}(\varepsilon)-\int_{-h}^{0}\Lambda_{1s}^T(\eta+h)A_{11}(\eta+h)\varphi_x(\eta)d\eta$$
$$-\int_{-h}^{0}\Lambda_{1s}^T(\eta+h)A_{21}(\eta+h)\varphi_y(\eta)d\eta\Bigg\|\le a_2\big(\varphi_z(\eta)\big)\varepsilon,\quad \left\|b_{2,2}(\varepsilon)\right\|\le a_2\big(\varphi_z(\eta)\big)\varepsilon^{1/2}, \tag{6.56}$$

$$\Bigg\|b_{3,1}(\varepsilon)-\int_{-h}^{0}\left(\int_{0}^{\eta+h}\Lambda_{1s}^T(\sigma)G_1(\sigma,\eta-\sigma)d\sigma\right)\varphi_x(\eta)d\eta$$
$$-\int_{-h}^{0}\left(\int_{0}^{\eta+h}\Lambda_{1s}^T(\sigma)G_2(\sigma,\eta-\sigma)d\sigma\right)\varphi_y(\eta)d\eta\Bigg\|\le a_3\big(\varphi_z(\eta)\big)\varepsilon,\ \ \left\|b_{3,2}(\varepsilon)\right\|\le a_3\big(\varphi_z(\eta)\big)\varepsilon, \tag{6.57}$$

where $a_1\big(z_0\big)>0$, $a_2\big(\varphi_z(\eta)\big)>0$, $a_3\big(\varphi_z(\eta)\big)>0$ are some constants independent of ε, while dependent on z_0 and $\varphi_z(\eta)$.

Now, the statement of the lemma follows immediately from the equations (6.22), (6.24), (6.48), (6.50)-(6.54) and the inequalities (6.55)-(6.57). □

Based on Theorems 6.1-6.2, Lemmas 4.4-4.5 and 6.1-6.2, and using the equations (4.11), (6.4), (6.10), we directly obtain the theorem.

Theorem 6.3. *Let the assumptions A1-A3 be valid. Let the system (4.16) be completely Euclidean space controllable at the time instant t_c. Then, for any $t\in[0,t_c-h)\cup(t_c-h,t_c)$, the following limit equality is fulfilled:*

$$\lim_{\varepsilon\to+0}v^*(t,\varepsilon)=v_s^*(t). \tag{6.58}$$

Moreover,

$$\lim_{\varepsilon\to+0}J_\varepsilon^*=J_s^*. \tag{6.59}$$

7. Example

Consider the following system, a particular case of the system (1.1)-(1.2):

$$\frac{dx_1(t)}{dt}=x_1(t)-x_2(t)+\exp\big(2t-1)x_1(t-1)-\exp(-2t+1)x_2(t-1)+y(t-1), \tag{7.1}$$

$$\frac{dx_2(t)}{dt}=2x_1(t)+4x_2(t)-\exp\big(-3t+1.5)x_1(t-1)-\exp(3t-1.5)x_2(t-1), \tag{7.2}$$

$$\frac{dy(t)}{dt}=tx_1(t)-x_2(t)-(\cos t)x_1(t-1)+(\sin t)x_2(t-1)+(t-4)y(t)-y(t-1)+\frac{1}{\varepsilon}u(t), \tag{7.3}$$

where $t\ge0$; $x_1(\cdot)$, $x_2(\cdot)$, $y(\cdot)$, $u(\cdot)$ are scalar functions (u is a control); $\varepsilon>0$ is a small parameter.

Thus, comparing this system with the system (1.1)-(1.2), we obtain that $n = 2$, $m = 1$, $h = 1$, and

$$A_{10}(t) = A_{10} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}, \quad A_{11}(t) = \begin{pmatrix} \exp(2t-1) & -\exp(-2t+1) \\ -\exp(-3t+1.5) & -\exp(3t-1.5) \end{pmatrix}, \tag{7.4}$$

$$A_{20}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A_{21}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7.5}$$

$$A_{30}(t) = (t, -1), \quad A_{31}(t) = (-\cos t, \sin t), \tag{7.6}$$

$$A_{40}(t) = t - 4, \quad A_{41}(t) = -1, \tag{7.7}$$

$$G_1(t,\eta) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; G_2(t,\eta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; G_3(t,\eta) = (0,0), \; G_4(t,\eta) = 0, \;\; t \geq 0, \;\; \eta \in [-1,0]. \tag{7.8}$$

In this example, we study the complete Euclidean space controllability of the system (7.1)-(7.3) at the time instant $t_c = 1.5$. First, let us note that the assumptions A1 and A2 are valid for this system. Thus, by virtue of Theorem 5.1 and Corollary 3.1, the invertibility of the matrix $M_s(t_c)$ (see equation (3.15)) yields the complete Euclidean space controllability of (7.1)-(7.3) for all sufficiently small $\varepsilon > 0$. To calculate $M_s(t_c)$, we need to calculate the matrix-valued function $C_s(\sigma)$. In the present example, this matrix-valued function becomes a two-dimensional vector-valued function. It is seen from the equation (3.14), that $C_s(\sigma)$ depends on the solution $\Lambda_s(\sigma)$ of the terminal-value problem (3.11). To derive this solution, first, we consider the problem (3.11) in the interval $(t_c - h, t_c] = (0.5, 1.5]$:

$$\frac{d\Lambda_s^1(\sigma)}{d\sigma} = -A_{10}^T \Lambda_s^1(\sigma), \quad \sigma \in (0.5, 1.5]; \qquad \Lambda_s^1(1.5) = I_2, \tag{7.9}$$

where the 2×2-matrix-valued function $\Lambda_s^1(\sigma)$ has the block form

$$\Lambda_s^1(\sigma) = \begin{pmatrix} \Lambda_{s,1}^1(\sigma) & \Lambda_{s,2}^1(\sigma) \\ \Lambda_{s,3}^1(\sigma) & \Lambda_{s,4}^1(\sigma) \end{pmatrix}. \tag{7.10}$$

Solving problem (7.9) and taking into account the form of the matrix A_{10} (see equation (7.4)), we obtain

$$\Lambda_{s,1}^1(\sigma) = -\exp(-3\sigma + 4.5) + 2\exp(-2\sigma + 3), \quad \sigma \in (0.5, 1.5], \tag{7.11}$$

$$\Lambda_{s,2}^1(\sigma) = 2\exp(-3\sigma + 4.5) - 2\exp(-2\sigma + 3), \quad \sigma \in (0.5, 1.5], \tag{7.12}$$

$$\Lambda_{s,3}^1(\sigma) = -\exp(-3\sigma + 4.5) + \exp(-2\sigma + 3), \quad \sigma \in (0.5, 1.5], \tag{7.13}$$

$$\Lambda_{s,4}^1(\sigma) = 2\exp(-3\sigma + 4.5) - \exp(-2\sigma + 3), \quad \sigma \in (0.5, 1.5]. \tag{7.14}$$

Now, let us consider problem (3.11) in the interval $[0, t_c - h] = [0, 0.5]$:

$$\frac{d\Lambda_s^2(\sigma)}{d\sigma} = -A_{10}^T \Lambda_s^2(\sigma) - A_{11}^T(\sigma+1)\Lambda_s^1(\sigma+1), \quad \sigma \in (0, 0.5]; \qquad \Lambda_s^2(0.5) = \Lambda_s^1(0.5+0), \tag{7.15}$$

where the 2×2-matrix-valued function $\Lambda_s^2(\sigma)$ has the block form

$$\Lambda_s^2(\sigma) = \begin{pmatrix} \Lambda_{s,1}^2(\sigma) & \Lambda_{s,2}^2(\sigma) \\ \Lambda_{s,3}^2(\sigma) & \Lambda_{s,4}^2(\sigma) \end{pmatrix}. \tag{7.16}$$

Solving the problem (7.15), and taking into account the form of the matrices A_{10}, $A_{11}(t)$ (see equation (7.4)) and the form of the matrix $\Lambda_s^1(\sigma)$ (see equations (7.10), (7.11)-(7.14)), we obtain

$$\Lambda_{s,1}^2(\sigma) = \left(\frac{11}{6} - \exp(3)\right)\exp(-3\sigma+1.5) + \big(2.5+2\exp(2)\big)\exp(-2\sigma+1)$$
$$-1+\frac{1}{6}\exp(-6\sigma+3)+\frac{1}{6}\exp(-5\sigma+2.5)-2\exp(-4\sigma+2)$$
$$-1.5\exp(-\sigma+0.5)-\frac{1}{6}\exp(\sigma-0.5), \qquad \sigma\in[0,0.5], \tag{7.17}$$

$$\Lambda_{s,2}^2(\sigma) = \big(2\exp(3)-4\big)\exp(-3\sigma+1.5) - \big(3-2\exp(2)\big)\exp(-2\sigma+1)$$
$$-\frac{2}{3}-\frac{1}{3}\exp(-6\sigma+3)-\frac{1}{6}\exp(-5\sigma+2.5)+2\exp(-4\sigma+2)$$
$$+\frac{1}{6}\exp(\sigma-0.5), \qquad \sigma\in[0,0.5], \tag{7.18}$$

$$\Lambda_{s,3}^2(\sigma) = \left(\frac{11}{6} - \exp(3)\right)\exp(-3\sigma+1.5) + \big(1.25+\exp(2)\big)\exp(-2\sigma+1)$$
$$-0.5-\frac{1}{12}\exp(-6\sigma+3)+\frac{5}{6}\exp(-5\sigma+2.5)-3\exp(-4\sigma+2)$$
$$-0.5\exp(-\sigma+0.5)+\frac{1}{6}\exp(\sigma-0.5), \qquad \sigma\in[0,0.5], \tag{7.19}$$

$$\Lambda_{s,4}^2(\sigma) = \big(2\exp(3)-4\big)\exp(-3\sigma+1.5) + \big(1.5-\exp(2)\big)\exp(-2\sigma+1)$$
$$+\frac{1}{3}+\frac{1}{6}\exp(-6\sigma+3)-\frac{11}{6}\exp(-5\sigma+2.5)+4\exp(-4\sigma+2)$$
$$-\frac{1}{6}\exp(\sigma-0.5), \qquad \sigma\in[0,0.5]. \tag{7.20}$$

Now, using the equations (3.14), (7.5), (7.11)-(7.14) and (7.17)-(7.20), we obtain the expression for the vector-valued function $C_s(\sigma)$:

$$C_s(\sigma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \sigma\in(0.5,1.5], \tag{7.21}$$

$$C_s(\sigma) = \begin{pmatrix} -\exp(-3\sigma+1.5)+2\exp(-2\sigma+1) \\ 2\exp(-3\sigma+1.5)-2\exp(-2\sigma+1) \end{pmatrix}, \qquad \sigma\in[0,0.5]. \tag{7.22}$$

The equations (3.15), (7.21)-(7.22) directly yield

$$M_s(1.5) = \begin{pmatrix} 0.6240 & 0.6681 \\ 0.6681 & 1.2208 \end{pmatrix}, \tag{7.23}$$

and $\det M_s(1.5) = 0.3154 \neq 0$. Thus, the matrix $M_s(1.5)$ is invertible, meaning the complete Euclidean space controllability of the system (7.1)-(7.3) for all sufficiently small values of $\varepsilon > 0$.

Now, let us consider the particular case of the system (1.1), (4.2), corresponding to the system (7.1)-(7.3). This particular case consists of the equations (7.1), (7.2) and the equation

$$\frac{dy(t)}{dt} = tx_1(t) - x_2(t) - (\cos t)x_1(t-1) + (\sin t)x_2(t-1) + \left((t-4)-\frac{1}{\varepsilon}\right)y(t) - y(t-1) + \frac{1}{\varepsilon}v(t), \tag{7.24}$$

where $v(t)$ is a new scalar control.

The slow subsystem, associated with (7.1)-(7.2), (7.24), has the form

$$\frac{dx_{1s}(t)}{dt} = x_{1s}(t) - x_{2s}(t) + \exp\big(2t-1\big)x_{1s}(t-1) - \exp(-2t+1)x_{2s}(t-1) + v_s(t-1), \tag{7.25}$$

$$\frac{dx_{2s}(t)}{dt} = 2x_{1s}(t) + 4x_{2s}(t) - \exp\big(-3t+1.5\big)x_{1s}(t-1) - \exp(3t-1.5)x_{2s}(t-1). \tag{7.26}$$

This system is a particular case of (4.16).

For the system (7.1)-(7.2), (7.24), we consider the minimum energy control problem with the following initial and terminal conditions, and the functional:

$$x(\eta) = \varphi_x(\eta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad y(\eta) = \varphi_y(\eta) = -1, \quad \eta \in [-1,0), \tag{7.27}$$

$$x(0) = x_0 = \begin{pmatrix} 1.25 \\ -1 \end{pmatrix}, \quad y(0) = y_0 = 1, \tag{7.28}$$

$$x(t_c) = x_c = \begin{pmatrix} 25 \\ 35 \end{pmatrix}, \quad y(t_c) = y_c = 10, \tag{7.29}$$

$$J(v) = \int_0^{1.5} v^2(t)dt. \tag{7.30}$$

The latter equation means that in this example $R(t) = 1$, $t \in [0, 1.5]$.

Along with this minimum energy control problem, we consider the minimum energy control problem for the system (7.25)-(7.26) with the following initial and terminal conditions, and the functional:

$$x_s(\eta) = \varphi_x(\eta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v_s(\eta) = \varphi_y(\eta) = -1, \quad \eta \in [-1,0), \tag{7.31}$$

$$x_s(0) = x_0 = \begin{pmatrix} 1.25 \\ -1 \end{pmatrix}, \tag{7.32}$$

$$x_s(t_c) = x_c = \begin{pmatrix} 25 \\ 35 \end{pmatrix}, \tag{7.33}$$

$$J_s(v_s) = \int_0^{1.5} v_s^2(t)dt. \tag{7.34}$$

Let us solve the minimum energy control problem (7.25)-(7.26), (7.31)-(7.33), (7.34). Using the equations (6.22) and (6.24), as well as the equations (7.10)-(7.14), (7.16)-(7.20) and the data of the example (7.4)-(7.4), (7.8), (7.31)-(7.32), we obtain

$$p_{s,0} = \begin{pmatrix} 13.4693 \\ 92.6124 \end{pmatrix}, \quad w_{s,0} = \begin{pmatrix} 24.1686 \\ 33.2890 \end{pmatrix}. \tag{7.35}$$

Based on the equation (6.30) and Theorem 6.2, let us calculate the optimal control $v_s^*(t)$ and the optimal value of the functional J_s^* in the minimum energy problem for the system (7.25)-(7.26). Since in this example $R(t) \equiv 1$, then $W_C(1.5) = M_s(1.5)$, given by (7.23). Using this observation and the equations (7.21)-(7.22), (7.33), (7.35), we obtain

$$v_s^*(t) = \begin{cases} 3.6540\exp(-3t+1.5) - 4.0602\exp(-2t+1), & t \in [0, 0.5], \\ 0, & t \in (0.5, 1.5], \end{cases} \tag{7.36}$$

$$J_s^* = 2.4407. \tag{7.37}$$

Now, by virtue of Theorem 6.3, we have the limit equalities

$$\lim_{\varepsilon\to+0} v^*(t,\varepsilon) = v_s^*(t), \quad t \in [0,0.5)\cup(0.5,1.5), \tag{7.38}$$

$$\lim_{\varepsilon\to+0} J_\varepsilon^* = 2.4407, \tag{7.39}$$

where $v^*(t,\varepsilon)$ and J_ε^* are the optimal control and the optimal value of the functional, respectively, in the minimum energy control problem (7.1)-(7.2), (7.24), (7.27)-(7.29), (7.30).

8. Conclusions

In this paper, a time-dependent controlled system with time delays (pointwise and distributed) in the state variables and with a high gain control was considered. The complete Euclidean space controllability of this system was studied. By the linear invertible control transformation and due to the high gain control, the original system was converted to an equivalent time delay system, singularly perturbed by a small positive parameter ε. The asymptotic decomposition of the latter into two lower dimensions ε-free subsystems, slow and fast ones, was carried out. The slow subsystem is a differential equation with delays in the state and control variables, while the fast subsystem is a differential equation without delays. It was established in the paper that the Euclidean space controllability of the slow subsystem yields the Euclidean space controllability of the transformed singularly perturbed system and the original high gain control system for all sufficiently large values of the gain.

For the transformed singularly perturbed system, the minimum energy control problem in the controllability interval also was considered . It was established that, for ε tending to zero, the optimal control of this problem converges to the optimal control of the minimum energy problem for the slow subsystem. This convergence is pointwise with respect to time almost everywhere in the controllability interval. A-long with the convergence of the optimal control, the convergence of the functional optimal value (the minimum energy) for the singularly perturbed system to such a functional optimal value for the slow subsystem also was established.

References

[1] Z. Artstein, Linear systems with delayed controls: a reduction, IEEE Trans. Autom. Control 27 (1982), 869-879.

[2] K. Balachandran, Controllability of nonlinear systems with delays in both state and control variables, Kybernetika 22 (1986), 340–348.

[3] A. Bensoussan, G. Da Prato, M. C. Delfour, S. K. Mitter, Representation and Control of Infinite Dimensional Systems, Birkhauser, Boston, 2007.

[4] G. Canciello, A. Cavallo, E. D'Amato, M. Mattei, Attitude and velocity high-gain control of a tilt-trirotor UAV, Proceedings of 2017 International Conference on Unmanned Aircraft Systems (2017), 867–876.

[5] C. T. Chen, Linear System Theory and Design, Harcourt Brace Jovanovich, Orlando, 1984.

[6] R. F. Curtain, H. J. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer, New York, 1995.

[7] M. C. Delfour, S. K. Mitter, Controllability, observability and optimal feedback control of affine hereditary differential systems, SIAM J. Control 10 (1972), 298-328.

[8] M. G. Dmitriev, G. A. Kurina, Singular perturbations in control problems, Autom. Remote Control 67 (2006), 1-43.

[9] A. B. Filimonov, N. B. Filimonov, Robust correction of dynamic plants in automatic control systems, Optoelectronics, Instrumentation and Data Processing 51 (2015), 478–484.

[10] L. Fridman, Chattering on high gain control systems with fast actuator, Proceedings of the 36th IEEE Conference on Decision and Control 4 (1997), 3232–3233.

[11] R. Gabasov, F. M. Kirillova, V. V. Krakhotko, Controllability of multiloop systems with lumped parameters, Autom. Remote Control 32 (1971), 1710-1717.

[12] Z. Gajic, M.-T. Lim, Optimal Control of Singularly Perturbed Linear Systems and Applications. High Accuracy Techniques, Marsel Dekker, New York, 2001.

[13] F. R. Gantmacher, The Theory of Matrices, Nauka, Moscow, 1967. [in Russian]

[14] V. Y. Glizer, Euclidean space controllability of singularly perturbed linear systems with state delay, Systems and Control Letters 43 (2001), 181-191.

[15] V. Y. Glizer, Controllability of singularly perturbed linear time-dependent systems with small state delay, Dynamics and Control 11 (2001), 261-281.

[16] V. Y. Glizer, Controllability of nonstandard singularly perturbed systems with small state delay, IEEE Trans. Autom. Control 48 (2003), 1280-1285.

[17] V. Y. Glizer, Suboptimal solution of a cheap control problem for linear systems with multiple state delays, J. Dyn. Control Syst. 11 (2005), 527-574.

[18] V. Y. Glizer, L. M. Fridman, V. Turetsky, Cheap suboptimal control of an integral sliding mode for uncertain systems with state delays, IEEE Trans. Autom. Control 52 (2007), 1892-1898.

[19] V. Y. Glizer, Novel controllability conditions for a class of singularly perturbed systems with small state delays, J. Optim. Theory Appl. 137 (2008), 135-156.

[20] V. Y. Glizer, Infinite horizon cheap control problem for a class of systems with state delays, J. Nonlinear Convex Anal. 10 (2009), 199-233.

[21] V. Y. Glizer, H_∞ cheap control for a class of linear systems with state delays, J. Nonlinear Convex Anal. 10 (2009), 235-259.

[22] V. Y. Glizer, Stochastic singular optimal control problem with state delays: regularization, singular perturbation, and minimizing sequence, SIAM J. Control Optim. 50 (2012), 2862-2888.

[23] V. Y. Glizer, Cheap quadratic control of linear systems with state and control delays, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms 19 (2012), 277-301.

[24] V. Y. Glizer, Singular solution of an infinite horizon linear-quadratic optimal control problem with state delays, in "Variational and Optimal Control Problems on Unbounded Domains" (Gershon Wolansky and Alexander J. Zaslavski, Eds.), Contemporary Mathematics Series, Vol. 619, American Mathematical Society, Providence, RI, pp. 59-98, 2014.

[25] V. Y. Glizer, O. Kelis, Solution of a zero-sum linear quadratic differential game with singular control cost of minimizer, J. Control Decision 2 (2015), 155-184.

[26] V. Y. Glizer, Controllability conditions of linear singularly perturbed systems with small state and input delays, Math. Control Signals Systems 28 (2016), 1-29.

[27] V. Y. Glizer, O. Kelis, Singular infinite horizon zero-sum linear-quadratic differential game: saddle-point equilibrium sequence, Numer. Algebra, Control Optim. 7 (2017), 1–20.

[28] S. Hajji, M. M'Saad, M. Farza, A. Chaari, M. Kamoun, State feedback control design for a class of nonlinear systems, Int. J. Sci. Tech. Autom. Control Comput. Eng. 2 (2008), 748-767.

[29] A. Halanay, Differential Equations: Stability, Oscillations, Time Lags, Academic Press, New York, 1966.

[30] A. Halanay, Differential games with delay, SIAM J. Control 6 (1968), 579-593.

[31] J. Hammer, High-gain control of nonlinear minimum-phase systems, Proceedings of 2005 American Control Conference (2005), 147–152.

[32] A. Ilchmann, H. Logemann, Adaptive λ–tracking for a class of infinite-dimensional systems, Systems Control Lett. 34 (1998), 11-21.

[33] A. Ilchmann, E. P. Ryan, C. J. Sangwin, Systems of controlled functional differential equations and adaptive tracking, SIAM J. Control Optim. 40 (2002), 1746-1764.

[34] A. Ilchmann, E. P. Ryan, High-gain control without identification: a survey GAMM-Mitt, 31 (2008), 115-125.

[35] T. Kaczorek, Linear Control Systems, Research Studies Press and John Wiley, New York, 1993.

[36] T. Kaczorek, K. Rogowski, Fractional Linear Systems and Electrical Circuits, Studies in Systems, Decision and Control series, Vol. 13, Springer, Berlin, 2015.

[37] A. I. Kalinin, L. I. Lavrinovich, Singular perturbations in the linear-quadratic optimal control problem, Doklady of the National Academy of Sciences of Belarus 60 (2016), 31–34. [in Russian]

[38] R. E. Kalman, Contributions to the theory of optimal control, Bol. Soc. Mat. Mexicana, 5 (1960), 102-119.

[39] J. Klamka, Controllability of Dynamical Systems, Kluwer Academic Publishers, Dordrecht, 1991.

[40] J. Klamka, Controllability of dynamical systems, Mat. Stosow. 9 (2008), 57-75.

[41] J. Klamka, Stochastic controllability and minimum energy control of systems with multiple delays in control, Appl. Math. Comput. 206 (2008), 704–715.

[42] J. Klamka, Controllability of dynamical systems. A survey, Bull. Pol. Acad. Sci. Tech. Sci. 61 (2013), 335-342.

[43] P. V. Kokotovic, A. H. Haddad, Controllability and time-optimal control of systems with slow and fast modes, IEEE Trans. Autom. Control 20 (1975), 111-113.

[44] P. V. Kokotovic, H. K. Khalil, J. O'Reilly, Singular Perturbation Methods in Control: Analysis and Design, Academic Press, London, 1986.

[45] P. V. Kokotovic, A. Bensoussan, G. L. Blankenship, editors, Singular Perturbations and Asymptotic Analysis in Control Systems, Lecture Notes in Control and Information Sciences 90, Springer-Verlag, Berlin, 1987.

[46] T. B. Kopeikina, Controllability of singularly perturbed linear systems with delay, Differ. Equ. 25 (1989), 1055-1064.

[47] T. B. Kopeikina, Unified method of investigating controllability and observability problems of time-variable differential systems, Funct. Differ. Equ. 13 (2006), 463-481.

[48] P. Krishnamurthy, F. Khorram, High-gain output-feedback control for nonlinear systems based on multiple time scaling, Systems Control Lett. 56 (2007), 7-15.

[49] Kuehn C. Multiple Time Scale Dynamics, Springer, New York, 2015.

[50] G. A. Kurina, Complete controllability of singularly perturbed systems with slow and fast modes, Math. Notes 52 (1992), 1029-1033.

[51] W. H. Kwon, A. E. Pearson, Feedback stabilization of linear systems with delayed control, IEEE Trans. Autom. Control 25 (1980), 266-269.

[52] P. Loepelmann, B. Baker, High-gain feedback stability of a nonlinear drivetrain system, Proceedings of 2013 European Control Conference (2013), 908–913.

[53] R. Marino, High-gain feedback in non-linear control systems, Int. J. Control 42 (1985), 1369-1385.

[54] R. Michino, I. Mizumoto, Z. Iwai, M. Kumon, Robust high gain adaptive output feedback control for nonlinear systems with uncertain nonlinearities in control input term, Int. J. Control Autom. Sys. 1 (2003), 19-27.

[55] J. W. Polderman, I. M. Y. Mareels, Robustification of high gain adaptive control, Proceedings of 2001 European Control Conference (2001), 2428–2432.

[56] P. Sannuti, On the controllability of singularly perturbed systems, IEEE Trans. Autom. Control 22 (1977), 622-624.

[57] P. Sannuti, On the controllability of some singularly perturbed nonlinear systems, J. Math. Anal. Appl. 64 (1978), 579-591.

[58] Ya. Z. Tsypkin, B. T. Polyak, High-gain robust control, Eur. J. Control 5 (1999), 3-9.

[59] S.-T. Wu, On digital high-gain and sliding-mode control, Int. J. Control 66 (1997), 65-84.

[60] Y. Zhang, D. S. Naidu, C. Cai, Y. Zou, Singular perturbations and time scales in control theory and applications: an overview, Int. J. Inf. Syst. Sci. 1 (2014), 1-36.