ON REICH’S STRICT FIXED POINT THEOREM FOR MULTI-VALUED OPERATORS IN COMPLETE METRIC SPACES

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Abstract. The aim of this paper to present an extended version of Reich’ strict fixed point theorem for multi-valued operators. Under the classical assumptions considered by Simeon Reich in 1972 (i.e., the completeness of the metric space \((X,d)\) and the \(\delta\)-contraction type condition on a self multi-valued operator on \(X\) having nonempty and bounded values) several other conclusions with respect to the strict fixed point problem are presented.

Keywords. Multi-valued operator; Complete metric space; Fixed point; Ulam-Hyers stability; Well-posedness.

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1. INTRODUCTION

The most important metric fixed point theorem for multi-valued contractions on a complete metric space was given by Nadler [7] and Covitz-Nadler [3]. In the same time, of the major interest is to obtain sufficient conditions for the existence (and uniqueness) of a strict fixed point of a multi-valued operator. In this direction, a very important result was proved by Reich in 1972, see [16]. Several related results and generalizations are known, see, for example, [1] and [5].

The aim of this paper to present an extended (by the conclusions point of view) version of Reich’ strict fixed point theorem for multi-valued operators in complete metric spaces. Some related mathematical phenomena will be considered: data dependence, Ulam-Hyers stability, well-posedness, and Ostrovski property. A local fixed point theorem is also proved.

2. PRELIMINARIES

Let us recall first some important preliminary concepts and results.

Let \((X,d)\) be a metric space and let \(P(X)\) be the family of all nonempty subsets of \(X\). We denote by \(P_c(X)\) the family of all nonempty closed subsets of \(X\) and by \(P_b(X)\) the family of all nonempty bounded subsets of \(X\). Also \(P_{b,c}(X) := P_b(X) \cap P_c(X)\). For \(x_0 \in X\) and \(r > 0\), we will also denote by \(B(x_0; r) := \{x \in X | d(x_0, x) < r\}\) the open ball, respectively by \(\bar{B}(x_0; r) := \{x \in X | d(x_0, x) \leq r\}\) the closed ball, centered in \(x_0\) with radius \(r\).

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We also recall, in the context of a metric space, the definitions of some important functionals in multi-valued analysis theory:

(a) the gap functional generated by $d$:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+, \quad D_d(A, B) := \inf\{d(a, b) \mid a \in A, \ b \in B\}.$$  

(b) the excess functional of $A$ over $B$ generated by $d$:

$$e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad e_d(A, B) := \sup\{D_d(a, B) \mid a \in A\}.$$  

(c) the Hausdorff-Pompeiu functional generated by $d$:

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}.$$  

(d) the diameter functional generated by $d$:

$$\delta_d : P(X) \times P(X) \to \mathbb{R}_+, \quad \delta_d(A, B) := \sup\{d(a, b) \mid a \in A, \ b \in B\}. $$

The diameter of a set $A \in P(X)$ will be denoted by $diam(A) := \delta_d(A, A)$. We will avoid the subscript $d$ if the context is clear.

Some useful properties of these functionals are re-called (see, for example, [2, 6, 9]) in the next lemmas.

**Lemma 2.1.** If $(X, d)$ is a metric space, then we have:

(a) $H$ is a metric in $P_{cl}(X)$;

(b) if $A \in P_{cl}(X)$ and $x \in X$ are such that $D(x, A) = 0$, then $x \in A$.

(c) if $A, B \in P(X)$ and $q > 1$, then, for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.

(d) if $A, B \in P_b(X)$ and $q < 1$, then, for every $a \in A$ there exists $b \in B$ such that $d(a, b) \geq q\delta(A, B)$.

(e) the functional $\delta$ has the following properties:

1) $\delta(A, B) = 0$ implies that $A = B = \{x^*\}$;

2) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$, for all $A, B, C \in P_b(X)$;

3) $\delta(A, B) = \delta(B, A)$, for all $A, B \in P_b(X)$;

4) if $A \in P_b(X)$ then $\delta(A, A) = 0$ if and only if $A$ is a singleton.

Finally, let us recall that if $X$ is a nonempty set and $F : X \to P(X)$ is a multi-valued operator, then we denote by $Fix(F) := \{x \in X : x \in F(x)\}$ the fixed point set for $F$, and by $SFix(F) := \{x \in X : \{x\} = F(x)\}$ the strict fixed point set for $F$. In some papers, instead of strict fixed point the terms stationary point or end-point are used. We also denote by $Graph(F) := \{(x, y) \in X \times X \mid y \in F(x)\}$ the graph of $F$.

Moreover, for arbitrary $(x_0, x_1) \in Graph(F)$, the sequence $(x_n)_{n \in \mathbb{N}}$ with $x_{n+1} \in F(x_n)$ (for $n \in \{1, 2, 3, \cdots\}$) is called the sequence of successive approximations for $F$ staring from $(x_0, x_1)$.

Some typical conditions in fixed point theory for a multi-valued operator are given now.

**Definition 2.1.** Let $(X, d)$ be a metric space. Then:

1) $F : X \to P_{cl}(X)$ is called $(\alpha, \beta)$-contraction of Reich type if $\alpha, \beta \geq 0, \alpha + 2\beta < 1$ and

$$H(F(x_1), F(x_2)) \leq \alpha d(x_1, x_2) + \beta (D_d(x_1, F(x_1)) + D_d(x_2, F(x_2))), \quad \forall x_1, x_2 \in X.$$  

We notice that in [4] the case when $\alpha = 0$ is treated. This kind of mappings are called Kannan type multi-valued operators.
2) $F : X \rightarrow P_b(X)$ is called $(\alpha, \beta) - \delta$-contraction of Reich type if $\alpha, \beta \geq 0$, $\alpha + 2\beta < 1$ and
\[
\delta(F(x_1), F(x_2)) \leq \alpha d(x_1, x_2) + \beta (\delta_d(x_1, F(x_1)) + \delta_d(x_2, F(x_2))), \quad \forall x_1, x_2 \in X;
\]

The concepts of multi-valued weakly Picard operator and multi-valued Picard operator are very important in fixed point theory for multi-valued operators.

**Definition 2.2.** ([11, 20, 21]) Let $(X, d)$ be a metric space. Then $F : X \rightarrow P(X)$ is called a multi-valued weakly Picard operator (briefly, MWP operator) if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ such that

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;
(iii) the sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of $F$.

Let us recall the following important notion.

**Definition 2.3.** Let $(X, d)$ be a metric space and let $F : X \rightarrow P(X)$ be an MWP operator. Then we define the multi-valued operator $F^\infty : Graph(F) \rightarrow P(\text{Fix}(F))$ by the formula $F^\infty(x, y) = \{ z \in \text{Fix}(F) \mid \text{there exists a sequence of successive approximations of } F \text{ starting from } (x, y) \text{ that converges to } z \}$.

An important concept is given by the following definition.

**Definition 2.4.** Let $(X, d)$ be a metric space and $F : X \rightarrow P(X)$ an MWP operator. Then $F$ is a $\psi$-multi-valued weakly Picard operator (briefly $\psi$-MWP operator) if $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is increasing, continuous in 0 with $\psi(0) = 0$ and there exists a selection $f^\infty$ of $F^\infty$ such that
\[
d(x, f^\infty(x, y)) \leq \psi(d(x, y)), \quad \text{for all } (x, y) \in Graph(F).
\]
In particular, if $\psi(t) = ct$, we say that $F$ is a $c$-multi-valued weakly Picard operator (briefly $c$-MWP operator).

**Example 2.1.** An $(\alpha, \beta)$-contraction of Reich type is a $c$-MWP operator with $c := \frac{1-\beta}{1-(\alpha+\beta)}$.

**Definition 2.5.** ([11, 12]) We say that $F : X \rightarrow P(X)$ is a multi-valued Picard operator if:

(i) $\text{SFix}(F) = \text{Fix}(F) = \{ x^* \}$;
(ii) $F^n(x) \overset{H_\text{d}}{\to} \{ x^* \}$ as $n \to \infty$, for each $x \in X$.

Several examples of Picard and weakly Picard operators, as well as, different applications of this theory are given, for example, in [9, 10, 11, 12, 14].

3. Reich’s Strict Fixed Point Theorem

In this section, the study of the fixed point problem for $(\alpha, \beta) - \delta$-contractions of Reich type is considered.

In 1972, Reich proved the following strict fixed point principle (see [19] for other similar results).

**Theorem 3.1.** (Reich’s Theorem) Let $(X, d)$ be a complete metric space and $F : X \rightarrow P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta \in \mathbb{R}_+$ with $0 < \alpha + 2\beta < 1$ such that
\[
\delta(F(x), F(y)) \leq \alpha d(x, y) + \beta (\delta(F(x), F(x)) + \delta(F(y), F(y))), \quad \forall x, y \in X.
\]
Then there exists a unique strict fixed point $x^* \in X$ of $F$ and $\text{Fix}(F) = \text{SFix}(F) = \{ x^* \}$. 

\textbf{Proof.} \textit{A. Reich's original proof.} Let \( p := \sqrt{\alpha + 2\beta} \in (0, 1) \). Then, by Lemma 2.1, we can define a selection \( f : X \to X \) of \( F \), by letting to each point \( x \in X \) the point \( f(x) \in F(x) \) which satisfies \( d(x, f(x)) \geq p \delta(x, F(x)) \). Then, we have
\[
d(f(x), f(y)) \leq \delta(F(x), F(y)) \\
\leq \alpha d(x, y) + \beta (\delta(x, F(x)) + \delta(y, F(y))) \\
\leq \alpha d(x, y) + \beta p^{-1} (d(x, f(x)) + d(y, f(y))).
\]
Since \( \alpha + 2\beta p^{-1} < p^{-1}(\alpha + 2\beta) = p < 1 \), we obtain that \( f \) satisfies all the conditions of Cirić-Reich-Rus’ Theorem (see [15], [17]) and hence it has a unique fixed point \( \delta \). Thus, we have
\[
\delta(F(x), F(y)) \leq 2\beta \delta(y, F(y)) \leq \delta(y, F(y)),
\]
which is a contradiction. Thus \( F(y) = \{ y \} \), i.e., \( y \in SFix(F) \). For the uniqueness of the fixed point (and the strict fixed point too), we notice that, if \( z \in X \) is another strict fixed point of \( F \) such that \( x^* \neq z \), then we have
\[
d(x^*, z) \leq \delta(F(x^*), F(z)) \\
\leq \alpha d(x^*, z) + \beta (\delta(x^*, F(x^*)) + \delta(z, F(z))) \\
= \alpha d(x^*, z).
\]
Thus \( z = x^* \).

\textit{B. An alternative proof.} Let \( q > 1 \) and let \( x_0 \in X \) be arbitrary. Then there exists \( x_1 \in F(x_0) \) such that \( \delta(x_0, F(x_0)) \leq q \cdot d(x_0, x_1) \). Thus, we have
\[
\delta(x_1, F(x_1)) \leq \delta(F(x_0), F(x_1)) \\
\leq \alpha d(x_0, x_1) + \beta (\delta(x_0, F(x_0)) + \delta(x_1, F(x_1))) \\
\leq \alpha d(x_0, x_1) + \beta q d(x_0, x_1) + \beta \delta(x_1, F(x_1)).
\]
Hence, we get \( \delta(x_1, F(x_1)) \leq \frac{\alpha + \beta q}{1 - \beta} d(x_0, x_1) \). By this approach we can construct a sequence \( (x_n)_{n \in \mathbb{N}} \subset X \) of successive approximations for \( F \), such that
\[
d(x_n, x_{n+1}) \leq \delta(x_n, F(x_n)) \leq \left( \frac{\alpha + \beta q}{1 - \beta} \right)^n d(x_0, x_1), \quad \forall n \in \mathbb{N}.
\]
Choosing \( q < \frac{1 - \alpha - \beta}{\beta} \), we obtain \( \frac{\alpha + \beta q}{1 - \beta} < 1 \). Hence \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in the complete metric space \((X, d)\). Let us denote by \( x^* \in X \) its limit. We show that \( x^* \) is a strict fixed point for \( F \), i.e., \( F(x^*) = \{ x^* \} \). Indeed, since
\[
\delta(x^*, F(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, F(x_n)) + \delta(F(x_n), F(x^*)) \\
\leq d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta \delta(x_n, F(x_n)) + \beta \delta(x^*, F(x^*)) \\
\leq d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta \left( \frac{\alpha + \beta q}{1 - \beta} \right)^n d(x_0, x_1) + \beta \delta(x^*, F(x^*)),
\]
we obtain that
\[
\delta(x^*, F(x^*)) \leq \frac{1}{1 - \beta} d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta \left( \frac{\alpha + \beta q}{1 - \beta} \right)^n d(x_0, x_1)).
\]
As \( n \to \infty \), we obtain that \( \delta(x^*, F(x^*)) = 0 \) and thus \( F(x^*) = \{x^*\} \). The fact that \( \text{Fix}(F) = S\text{Fix}(F) \) and the uniqueness of the strict fixed point follow as before.

\( \square \)

**Remark 3.1.**  
1) In fact, in [16], the following assumption is made: there exist \( \alpha, \beta, \gamma \in \mathbb{R}_+ \) with \( \alpha + \beta + \gamma < 1 \) such that

\[ \delta(F(x), F(y)) \leq \alpha d(x, y) + \beta \delta(x, F(x)) + \gamma \delta(y, F(y)), \quad \forall x, y \in X. \]

2) By the alternative proof, it also follows that there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) of successive approximations for \( F \) starting from arbitrary \( x_0 \in X \), such that

\[ d(x_n, x^*) \leq \frac{L^n}{1 - L} d(x_0, x_1), \quad \forall n \in \mathbb{N}, \]

where \( L = \frac{\alpha + \beta \alpha}{1 - \beta} \), with any \( q \in (1, \frac{1 - \alpha - \beta}{\beta}) \).

On the other hand, it is worth to notice that by Reich’s original proof we also obtain, taking into account the proof of Cirić-Reich-Rus’ Theorem (see [15], [17]), that the sequence \( u_n = f^n(x_0) \), for \( n \in \mathbb{N}^* \) (where \( x_0 \) is arbitrary in \( X \)) converges to \( x^* \in \text{Fix}(f) \) and the following apriori estimation holds

\[ d(u_n, x^*) \leq \frac{s^n}{1 - s} d(x_0, f(x_0)), \quad \forall n \in \mathbb{N}, \]

where \( s = \frac{\alpha + \beta}{1 - \beta} < 1 \). Hence, for the strict fixed point \( x^* \in X \) the following estimation holds

\[ d(u_n, x^*) \leq \frac{s^n}{1 - s} d(x_0, x_1), \quad \forall n \in \mathbb{N}, \]

where \( u_n = f^n(x_0) \to x^* \) as \( n \in \mathbb{N}^* \) (where \( x_0 \) is arbitrary in \( X \) and \( f : X \to X \) is the selection constructed in Reich’s original proof).

3) It is worth to notice that the following Kannan type assumption

\[ H(F(x), F(y)) \leq \beta \left( \delta(x, F(x)) + \delta(y, F(y)) \right), \quad \forall x, y \in X, \]

(where \( \beta \in (0, \frac{1}{2}) \)) does not imply, in general, the existence of a fixed point (see [16] Example 1). Moreover, if we impose the assumption

\[ H(F(x), F(y)) \leq \alpha d(x, y) + \beta \left( \delta(x, F(x)) + \delta(y, F(y)) \right), \quad \forall x, y \in X, \]

(where \( \alpha + 2\beta < 1 \)) the existence of a fixed point for \( F \) was established just for the case when \( X := \mathbb{R} \), see [16] Proposition 1.

In fact, the above remarks lead to some interesting open questions.

Moreover, we can prove the following interesting result, see also [10].

**Theorem 3.2.** Let \( (X, d) \) be a complete metric space and \( F : X \to P_b(X) \) be a multi-valued operator for which there exist \( \alpha, \beta \in \mathbb{R}_+ \) with \( 0 < \alpha + \beta < 1 \), such that

\[ \delta(F(x), F(y)) \leq \alpha d(x, y) + \beta \left( \delta(x, F(x)) + \delta(y, F(y)) \right), \quad \forall x, y \in X. \]

Then \( F \) is a MP operator.
Proof. By Theorem 3.1 we know that Fix(F) = SFix(F) = \{x^*\}. We have to prove that \(F^n(x) \overset{H}{\to} \{x^*\}\) as \(n \to \infty\), for each \(x \in X\). We have, for every \(x \in X\), that
\[
\delta(F(x),x^*) = \delta(F(x),F(x^*)) \\
\leq \alpha d(x,x^*) + \beta \left( \delta(x,F(x)) + \delta(x^*,F(x^*)) \right) \\
= \alpha d(x,x^*) + \beta \delta(x,F(x)) \\
\leq \alpha d(x,x^*) + \beta (d(x,x^*) + \delta(x,F(x))).
\]
Thus
\[
\delta(F(x),x^*) \leq \frac{\alpha + \beta}{1 - \beta} d(x,x^*), \quad \forall x \in X.
\]
It follows that
\[
\delta(F^2(x),x^*) = \sup_{y \in F(x)} \delta(F(y),x^*) \leq \sup_{y \in F(x)} \left( \frac{\alpha + \beta}{1 - \beta} \right) d(y, x^*) \leq \left( \frac{\alpha + \beta}{1 - \beta} \right)^2 d(x, x^*).
\]
By mathematical induction, we get that
\[
\delta(F^n(x),x^*) \leq \left( \frac{\alpha + \beta}{1 - \beta} \right)^n d(x,x^*) \to 0 \text{ as } n \to +\infty, \text{ for each } x \in X.
\]
The proof is now complete. \(\square\)

In some situations, it is important to get a localization of the (strict) fixed point for a multi-valued operator. In the case of \((\alpha, \beta) - \delta\)-contractions of Reich type we have the following local fixed point theorem.

**Theorem 3.3.** Let \((X,d)\) be a complete metric space, \(x_0 \in X\) and \(r > 0\). Suppose that \(F : \bar{B}(x_0; r) \to P_b(X)\) is a multi-valued operator for which:

(a) there exist \(\alpha, \beta \in \mathbb{R}_+\) with \(0 < \alpha + 2\beta < 1\) such that
\[
\delta(F(x), F(y)) \leq \alpha d(x,y) + \beta \left( \delta(x,F(x)) + \delta(y,F(y)) \right), \quad \forall \bar{B}(x_0; r) \in X;
\]

(b) \(\delta(x_0, F(x_0)) \leq \frac{1-\alpha - 2\beta}{1+\beta} r.\)

Then there exists a unique \(x^* \in \bar{B}(x_0; r)\) such that \(S\text{Fix}(F) = \text{Fix}(F) = \{x^*\}\). In particular, if \(\beta > 0\), then \(x^* \in \bar{B}(x_0; r)\).

**Proof.** We will show that the closed ball \(\bar{B}(x_0; r)\) is invariant with respect to \(F\), i.e., \(F(\bar{B}(x_0; r)) \subseteq \bar{B}(x_0; r)\).

For this purpose, let \(x \in \bar{B}(x_0; r)\) and \(y \in F(x)\) be arbitrary chosen. Then we have
\[
d(y,x_0) \leq \delta(F(x), F(x_0)) + \delta(x_0, F(x_0)).
\]

On the other hand,
\[
\delta(F(x), F(x_0)) \leq \alpha r + \beta \left( \delta(x,F(x)) + \delta(x_0,F(x_0)) \right) \\
\leq \alpha r + \beta \left( d(x,x_0) + \delta(x_0,F(x_0)) + \delta(x_0,x) + \delta(F(x_0),F(x)) + \delta(x_0,F(x_0)) \right) \\
= (\alpha + \beta)r + 2\beta \delta(x_0,F(x_0)) + \beta \delta(F(x_0),F(x)).
\]

Hence
\[
\delta(F(x), F(x_0)) \leq \frac{(\alpha + \beta)r + 2\beta \delta(x_0,F(x_0))}{1 - \beta}.
\]
Hence, going back to our first relation, we get
\[
d(y, x_0) \leq \frac{(\alpha + \beta r + \beta \delta(x_0, F(x_0)))}{1 - \beta} + \delta(x_0, F(x_0))
\]
\[
= \frac{(\alpha + \beta r + (\beta + 1) \delta(x_0, F(x_0)))}{1 - \beta}.
\]
By (b) we obtain that \(d(y, x_0) \leq r\), proving that the closed ball \(\tilde{B}(x_0; r)\) is invariant with respect to \(F\). The conclusion follows now by Theorem 3.1.

If \(\beta > 0\), then we can show that \(x^* \in B(x_0; r)\). Indeed, suppose, by contradiction, that \(d(x^*, x_0) = r\). Then we have
\[
r = d(x^*, x_0)
\]
\[
\leq \delta(F(x^*), F(x_0)) + \delta(x_0, F(x_0))
\]
\[
\leq \alpha d(x^*, x_0) + (\beta + 1) \delta(x_0, F(x_0))
\]
\[
\leq \alpha r + (\beta + 1) \frac{1 - \alpha - 2\beta}{1 + \beta} r = (1 - 2\beta)r,
\]
which gives the necessary contradiction. The proof is now complete. □

We will discuss now the well-posedness of the strict fixed point problem. For the well-posedness concept in the single-valued case see the paper Reich-Zaslavski [18], while the multi-valued case is considered in [13].

**Definition 3.1.** Let \((X, d)\) be a metric space and \(F : X \rightarrow P_b(X)\) be a multivalued operator. The strict fixed point problem
\[
\{ x \} = F(x), \ x \in X
\]
(3.1)

is well-posed for \(F\) if:

1. \((a_2)\) \(SFix(F) = \{x^*\}\)
2. \((b_2)\) If \((x_n)_{n \in \mathbb{N}}\) is a sequence in \(X\) such that \(\delta_d(x_n, F(x_n)) \rightarrow 0\) as \(n \rightarrow +\infty\), then \(x_n \rightarrow x^*\) as \(n \rightarrow +\infty\).

In this respect, we have the following result.

**Theorem 3.4.** Let \((X, d)\) be a complete metric space and \(F : X \rightarrow P_b(X)\) be a multi-valued operator for which there exist \(\alpha, \beta \in \mathbb{R}_+\) with \(0 < \alpha + 2\beta < 1\) such that
\[
\delta(F(x), F(y)) \leq \alpha d(x, y) + \beta (\delta(x, F(x)) + \delta(y, F(y))), \ \forall x, y \in X.
\]

Then the strict fixed point problem is well-posed for \(F\).

**Proof.** By Theorem 3.1 we know that \(Fix(F) = SFix(F) = \{x^*\}\). Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(X\) such that \(\delta_d(x_n, F(x_n)) \rightarrow 0\) as \(n \rightarrow +\infty\). We will prove that \(x_n \rightarrow x^*\) as \(n \rightarrow +\infty\). For this purpose, we have
\[
d(x_n, x^*) \leq \delta(x_n, F(x_n)) + \delta(F(x_n), F(x^*))
\]
\[
\leq \delta(x_n, F(x_n)) + \alpha d(x_n, x^*) + \beta (\delta(x_n, F(x_n)) + \delta(x^*, F(x^*)))
\]
\[
= (1 + \beta) \delta(x_n, F(x_n)) + \alpha d(x_n, x^*).
\]
Letting \(n \rightarrow \infty\), we obtain the desired conclusion. □

We will continue our study by presenting the concept of Ulam-Hyers stability for the strict fixed point problem. For related definitions and results, see [8].
**Definition 3.2.** Let \((X, d)\) be a metric space and \(F : X \to P_b(X)\) be a multi-valued operator. The strict fixed point problem \((3.1)\) is called Ulam-Hyers stable if there exists \(c > 0\) such that for each \(\varepsilon > 0\) and for each \(\varepsilon\)-solution \(y \in X\) of the strict fixed point problem, i.e.,
\[
\delta(y, F(y)) \leq \varepsilon, \quad (3.2)
\]
there exists a solution \(x^* \in X\) of the strict fixed point inclusion \((3.1)\) such that
\[
d(y, x^*) \leq c\varepsilon.
\]
We have the following result concerning the Ulam-Hyers stability of the strict fixed point problem.

**Theorem 3.5.** Let \((X, d)\) be a complete metric space and \(F : X \to P_b(X)\) be a multi-valued operator for which there exist \(\alpha, \beta \in \mathbb{R}_+\) with \(0 < \alpha + 2\beta < 1\) such that
\[
\delta(F(x), F(y)) \leq \alpha d(x,y) + \beta (\delta(x, F(x)) + \delta(y, F(y))), \quad \forall x, y \in X.
\]
Then the strict fixed point problem is Ulam-Hyers stable.

**Proof.** By Theorem 3.1, we know that \(\text{Fix}(F) = S\text{Fix}(F) = \{x^*\}\). Let \(\varepsilon > 0\) and \(y \in X\) such that \(\delta(y, F(y)) \leq \varepsilon\). Then, we have
\[
d(y, x^*) \leq \delta(y, F(y)) + \delta(F(y), F(x^*))
\]
\[
\leq \delta(y, F(y)) + \alpha d(y, x^*) + \beta (\delta(y, F(y)) + \delta(x^*, F(x^*))
\]
\[
= (1 + \beta) \delta(y, F(y)) + \alpha d(y, x^*).
\]
Thus
\[
d(y, x^*) \leq \frac{1 + \beta}{1 - \alpha} \delta(y, F(y)) \leq \frac{1 + \beta}{1 - \alpha} \varepsilon.
\]
The proof is complete. \(\square\)

Another stability concept is given in the next definition.

**Definition 3.3.** Let \((X, d)\) be a metric space and \(F : X \to P(X)\) be a multi-valued operator with \(S\text{Fix}(F) = \{x^*\}\). If \((y_n)_{n \in \mathbb{N}}\) is a sequence in \(X\) such that the following implication holds
\[
D(y_{n+1}, F(y_n)) \to 0 \text{ as } n \to \infty \Rightarrow y_n \to x^* \text{ as } n \to \infty,
\]
then we say that the strict fixed point problem \((3.1)\) has the Ostrovski property.

**Theorem 3.6.** Let \((X, d)\) be a complete metric space and \(F : X \to P_b(X)\) be a multi-valued operator for which there exist \(\alpha, \beta \in \mathbb{R}_+\) with \(0 < \alpha + 2\beta < 1\) such that
\[
\delta(F(x), F(y)) \leq \alpha d(x,y) + \beta (\delta(x, F(x)) + \delta(y, F(y))), \quad \forall x, y \in X.
\]
Then the strict fixed point problem as the Ostrovski property.

**Proof.** By Theorem 3.1, we know that \(\text{Fix}(F) = S\text{Fix}(F) = \{x^*\}\). Let \((y_n)_{n \in \mathbb{N}}\) be a sequence in \(X\) such that \(D(y_{n+1}, F(y_n)) \to 0 \text{ as } n \to \infty\). Next, we have
\[
d(y_{n+1}, x^*) \leq D(y_{n+1}, F(y_n)) + \delta(F(y_n), x^*).
\]
On the other hand, we observe that  
\[ \delta(F(y_n), x^s) = \delta(F(y_n), F(x^s)) \leq \alpha d(y_n, x^s) + \beta \delta(y_n, F(y_n)) \]
\[ \leq \alpha d(y_n, x^s) + \beta (d(y_n, x^s) + \delta(x^s, F(y_n))) \]
\[ = (\alpha + \beta) d(y_n, x^s) + \beta \delta(x^s, F(y_n)). \]

Hence  
\[ \delta(F(y_n), x^s) \leq \frac{\alpha + \beta}{1 - \beta} d(y_n, x^s), \quad \forall n \in \mathbb{N}. \]

Denote  \( p := \frac{\alpha + \beta}{1 - \beta} \in (0, 1) \). As a consequence, we obtain  
\[ d(y_{n+1}, x^s) \leq D(y_{n+1}, F(y_n)) + pd(y_n, x^s) \]
\[ \leq \cdots \]
\[ \leq \sum_{k=0}^{n} p^k D(y_{n-k+1}, F(y_{n-k})) + p^{n+1} d(y_0, x^s). \]

By Cauchy’s Lemma (see [14]), we obtain the desired conclusion. \( \square \)

Finally, we will present a data dependence theorem for the strict fixed point problem.

**Theorem 3.7.** Let \((X, d)\) be a complete metric space and \(F : X \to P_b(X)\) be a multi-valued operator for which there exist \( \alpha, \beta \in \mathbb{R}_+ \) with \( 0 < \alpha + 2\beta < 1 \) such that  
\[ \delta(F(x), F(y)) \leq \alpha d(x, y) + \beta (\delta(x, F(x)) + \delta(y, F(y))), \quad \forall x, y \in X. \]

Suppose that \( G : X \to P_b(X) \) is a multi-valued operator such that \( SFix(G) \neq \emptyset \) and there exists \( \eta > 0 \) such that \( \delta(F(x), G(x)) \leq \eta \), for every \( x \in X \). Then  
\[ \delta(SFix(F), SFix(G)) \leq \frac{\eta}{1 - \alpha}. \]

**Proof.** By Theorem 3.1, we know that \( Fix(F) = SFix(F) = \{ x^s \} \). Let \( y \in SFix(G) \) be arbitrary chosen. Then, we also have  
\[ d(y, x^s) = \delta(G(y), F(x^s)) \]
\[ \leq \delta(G(y), F(y)) + \delta(F(y), F(x^s)) \]
\[ \leq \eta + \alpha d(y, x^s). \]

Thus \( d(y, x^s) \leq \frac{\eta}{1 - \alpha} \), which gives immediately the desired conclusion. \( \square \)

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**References**


