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ON REICH'S STRICT FIXED POINT THEOREM FOR MULTI-VALUED OPERATORS IN COMPLETE METRIC SPACES

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Abstract. The aim of this paper to present an extended version of Reich' strict fixed point theorem for multi-valued operators. Under the classical assumptions considered by Simeon Reich in 1972 (i.e., the completeness of the metric space (X,d) and the δ-contraction type condition on a self multi-valued operator on X having nonempty and bounded values) several other conclusions with respect to the strict fixed point problem are presented.
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1. Introduction

The most important metric fixed point theorem for multi-valued contractions on a complete metric space was given by Nadler [7] and Covitz-Nadler [3]. In the same time, of the major interest is to obtain sufficient conditions for the existence (and uniqueness) of a strict fixed point of a multi-valued operator. In this direction, a very important result was proved by Reich in 1972, see [16]. Several related results and generalizations are known, see, for example, [1] and [5].

The aim of this paper to present an extended (by the conclusions point of view) version of Reich' strict fixed point theorem for multi-valued operators in complete metric spaces. Some related mathematical phenomena will be considered: data dependence, Ulam-Hyers stability, well-posedness, and Ostrovski property. A local fixed point theorem is also proved.

2. Preliminaries

Let us recall first some important preliminary concepts and results.

Let (X,d) be a metric space and let $P(X)$ be the family of all nonempty subsets of X. We denote by $P_{cl}(X)$ the family of all nonempty closed subsets of X and by $P_b(X)$ the family of all nonempty bounded subsets of X. Also $P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$. For $x_0 \in X$ and $r > 0$, we will also denote by $B(x_0;r) := \{x \in X | d(x_0,x) < r\}$ the open ball, respectively by $\tilde{B}(x_0;r) := \{x \in X | d(x_0,x) \leq r\}$ the closed ball, centered in x_0 with radius r.

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We also recall, in the context of a metric space, the definitions of some important functionals in multi-valued analysis theory:
(a) the gap functional generated by d:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+,\ D_d(A,B) := \inf\{d(a,b) \mid a \in A,\ b \in B\}.$$

(b) the excess functional of A over B generated by d:

$$e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ e_d(A,B) := \sup\{D_d(a,B) \mid a \in A\}.$$

(c) the Hausdorff-Pompeiu functional generated by d:

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A,B) = \max\{e_d(A,B), e_d(B,A)\}.$$

(d) the diameter functional generated by d:

$$\delta_d : P(X) \times P(X) \to \mathbb{R}_+,\ \delta_d(A,B) := \sup\{d(a,b) \mid a \in A,\ b \in B\}.$$

The diameter of a set $A \in P(X)$ will be denoted by $diam(A) := \delta_d(A,A)$. We will avoid the subscript $_d$ if the context is clear.

Some useful properties of these functionals are re-called (see, for example, [2, 6, 9]) in the next lemmas.

Lemma 2.1. *If (X,d) is a metric space, then we have:*
(a) H is a metric in $P_{b,cl}(X)$;
(b) if $A \in P_{cl}(X)$ and $x \in X$ are such that $D(x,A) = 0$, then $x \in A$.
(c) if $A, B \in P(X)$ and $q > 1$, then, for every $a \in A$ there exists $b \in B$ such that $d(a,b) \le qH(A,B)$.
(d) if $A, B \in P_b(X)$ and $q < 1$, then, for every $a \in A$ there exists $b \in B$ such that $d(a,b) \ge q\delta(A,B)$.
(e) the functional δ has the following properties:
(1) $\delta(A,B) = 0$ implies that $A = B = \{x^*\}$;
(2) $\delta(A,B) \le \delta(A,C) + \delta(C,B)$, for all $A, B, C \in P_b(X)$;
(3) $\delta(A,B) = \delta(B,A)$, for all $A, B \in P_b(X)$;
(4) if $A \in P_b(X)$ then $\delta(A,A) = 0$ if and only if A is a singleton.

Finally, let us recall that if X is a nonempty set and $F : X \to P(X)$ is a multi-valued operator, then we denote by $Fix(F) := \{x \in X : x \in F(x)\}$ the fixed point set for F, and by $SFix(F) := \{x \in X : \{x\} = F(x)\}$ the strict fixed point set for F. In some papers, instead of strict fixed point the terms stationary point or end-point are used. We also denote by $Graph(F) := \{(x,y) \in X \times X | y \in F(x)\}$ the graph of F.

Moreover, for arbitrary $(x_0, x_1) \in Graph(F)$, the sequence $(x_n)_{n\in\mathbb{N}}$ with $x_{n+1} \in F(x_n)$ (for $n \in \{1,2,3,\cdots\}$) is called the sequence of successive approximations for F staring from (x_0, x_1).

Some typical conditions in fixed point theory for a multi-valued operator are given now.

Definition 2.1. Let (X,d) be a metric space. Then:
1) $F : X \to P_{cl}(X)$ is called (α,β)-contraction of Reich type if $\alpha, \beta \ge 0$, $\alpha + 2\beta < 1$ and

$$H(F(x_1), F(x_2)) \le \alpha d(x_1,x_2) + \beta\left(D_d(x_1,F(x_1)) + D_d(x_2,F(x_2))\right), \quad \forall x_1, x_2 \in X.$$

We notice that in [4] the case when $\alpha = 0$ is treated. This kind of mappings are called Kannan type multi-valued operators.

2) $F : X \to P_b(X)$ is called $(\alpha, \beta) - \delta$-contraction of Reich type if $\alpha, \beta \geq 0$, $\alpha + 2\beta < 1$ and

$$\delta(F(x_1), F(x_2)) \leq \alpha d(x_1, x_2) + \beta\left(\delta_d(x_1, F(x_1)) + \delta_d(x_2, F(x_2))\right), \quad \forall x_1, x_2 \in X;$$

The concepts of multi-valued weakly Picard operator and multi-valued Picard operator are very important in fixed point theory for multi-valued operators.

Definition 2.2. ([11, 20, 21]) Let (X, d) be a metric space. Then $F : X \to P(X)$ is called a multivalued weakly Picard operator (briefly, MWP operator) if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;
(iii) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Let us recall the following important notion.

Definition 2.3. Let (X, d) be a metric space and let $F : X \to P(X)$ be an MWP operator. Then we define the multivalued operator $F^\infty : Graph(F) \to P(Fix(F))$ by the formula $F^\infty(x, y) = \{z \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from (x, y) that converges to $z\}$.

An important concept is given by the following definition.

Definition 2.4. Let (X, d) be a metric space and $F : X \to P(X)$ an MWP operator. Then F is a ψ-multi-valued weakly Picard operator (briefly ψ-MWP operator) if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 with $\psi(0) = 0$ and there exists a selection f^∞ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)), \text{ for all } (x, y) \in Graph(F).$$

In particular, if $\psi(t) = ct$, we say that F is a c-multi-valued weakly Picard operator (briefly c-MWP operator).

Example 2.1. An (α, β)-contraction of Reich type is a c-MWP operator with $c := \frac{1-\beta}{1-(\alpha+\beta)}$.

Definition 2.5. ([11, 12]) We say that $F : X \to P(X)$ is a multi-valued Picard operator if:
(i) $SFix(F) = Fix(F) = \{x^*\}$;
(ii) $F^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

Several examples of Picard and weakly Picard operators, as well as, different applications of this theory are given, for example, in [9, 10, 11, 12, 14].

3. Reich's strict fixed point theorem

In this section, the study of the fixed point problem for $(\alpha, \beta) - \delta$-contractions of Reich type is considered.

In 1972, Reich proved the following strict fixed point principle (see [19] for other similar results).

Theorem 3.1. *(Reich's Theorem) Let (X, d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta \in \mathbb{R}_+$ with $0 < \alpha + 2\beta < 1$ such that*

$$\delta(F(x), F(y)) \leq \alpha d(x, y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right), \quad \forall x, y \in X.$$

Then there exists a unique strict fixed point $x^* \in X$ of F and $Fix(F) = SFix(F) = \{x^*\}$.

Proof. *A. Reich's original proof.* Let $p := \sqrt{\alpha+2\beta} \in (0,1)$. Then, by Lemma 2.1, we can define a selection $f : X \to X$ of F, by letting to each point $x \in X$ the point $f(x) \in F(x)$ which satisfies $d(x, f(x)) \geq p\delta(x, F(x))$. Then, we have

$$\begin{aligned} d(f(x), f(y)) &\leq \delta(F(x), F(y)) \\ &\leq \alpha d(x,y) + \beta\left(\delta(x,F(x)) + \delta(y,F(y))\right) \\ &\leq \alpha d(x,y) + \beta p^{-1}\left(d(x,f(x)) + d(y,f(y))\right). \end{aligned}$$

Since $\alpha+2\beta p^{-1} < p^{-1}(\alpha+2\beta) = p < 1$, we obtain that f satisfies all the conditions of Círić-Reich-Rus' Theorem (see [15], [17]) and hence it has a unique fixed point $x^* \in X$. Thus, $x^* \in Fix(F)$. Moreover, since $0 = d(x^*, f(x^*)) \geq p\delta(x^*, F(x^*))$, we get $\delta(x^*, F(x^*)) = 0$ and so $F(x^*) = \{x^*\}$. Hence, $x^* \in SFix(F)$. We show now that $Fix(F) \subset SFix(F)$. Indeed, let $y \in Fix(F)$. If we If we suppose that $\delta(y, F(y)) > 0$, then

$$\delta(F(y), F(y)) \leq 2\beta\delta(y, F(y)) \leq \delta(y, F(y)),$$

which is a contradiction. Thus $F(y) = \{y\}$, i.e., $y \in SFix(F)$. For the uniqueness of the fixed point (and the strict fixed point too), we notice that, if $z \in X$ is another strict fixed point of F such that $x^* \neq z$, then we have

$$\begin{aligned} d(x^*, z) &\leq \delta(F(x^*), F(z)) \\ &\leq \alpha d(x^*, z) + \beta\left(\delta(x^*, F(x^*)) + \delta(z, F(z))\right) \\ &= \alpha d(x^*, z). \end{aligned}$$

Thus $z = x^*$.

B. An alternative proof. Let $q > 1$ and let $x_0 \in X$ be arbitrary. Then there exists $x_1 \in F(x_0)$ such that $\delta(x_0, F(x_0)) \leq q \cdot d(x_0, x_1)$. Thus, we have

$$\begin{aligned} \delta(x_1, F(x_1)) &\leq \delta(F(x_0), F(x_1)) \\ &\leq \alpha d(x_0, x_1) + \beta\left(\delta(x_0, F(x_0)) + \delta(x_1, F(x_1))\right) \\ &\leq \alpha d(x_0, x_1) + \beta q d(x_0, x_1) + \beta\delta(x_1, F(x_1)). \end{aligned}$$

Hence, we get $\delta(x_1, F(x_1)) \leq \frac{\alpha+\beta q}{1-\beta} d(x_0, x_1)$. By this approach we can construct a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for F, such that

$$d(x_n, x_{n+1}) \leq \delta(x_n, F(x_n)) \leq \left(\frac{\alpha+\beta q}{1-\beta}\right)^n d(x_0, x_1), \quad \forall n \in \mathbb{N}.$$

Choosing $q < \frac{1-\alpha-\beta}{\beta}$ we obtain $\frac{\alpha+\beta q}{1-\beta} < 1$. Hence $(x_n)_{n\in\mathbb{N}}$ ia a Cauchy sequence in the complete metric space (X, d). Let us denote by $x^* \in X$ its limit. We show that x^* is a strict fixed point for F, i.e., $F(x^*) = \{x^*\}$. Indeed, since

$$\begin{aligned} \delta(x^*, F(x^*)) &\leq d(x^*, x_{n+1}) + D(x_{n+1}, F(x_n)) + \delta(F(x_n), F(x^*)) \\ &\leq d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta\delta(x_n, F(x_n)) + \beta\delta(x^*, F(x^*)) \\ &\leq d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta\left(\frac{\alpha+\beta q}{1-\beta}\right)^n \cdot d(x_0, x_1) + \beta\delta(x^*, F(x^*)), \end{aligned}$$

we obtain that

$$\delta(x^*, F(x^*)) \leq \frac{1}{1-\beta}\left(d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta\left(\frac{\alpha+\beta q}{1-\beta}\right)^n \cdot d(x_0, x_1)\right).$$

As $n \to \infty$, we obtain that $\delta(x^*, F(x^*)) = 0$ and thus $F(x^*) = \{x^*\}$. The fact that $Fix(F) = SFix(F)$ and the uniqueness of the strict fixed point follow as before. □

Remark 3.1. 1) In fact, in [16], the following assumption is made: there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + \beta + \gamma < 1$ such that

$$\delta(F(x), F(y)) \leq \alpha d(x,y) + \beta \delta(x, F(x)) + \gamma \delta(y, F(y)), \quad \forall x, y \in X.$$

2) By the alternative proof, it also follows that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from arbitrary $x_0 \in X$, such that

$$d(x_n, x^*) \leq \frac{L^n}{1-L} d(x_0, x_1), \quad \forall n \in \mathbb{N},$$

where $L = \frac{\alpha+\beta q}{1-\beta}$, with any $q \in (1, \frac{1-\alpha-\beta}{\beta})$.

On the other hand, it is worth to notice that by Reich's original proof we also obtain, taking into account the proof of Círić-Reich-Rus' Theorem (see [15], [17]), that the sequence $u_n = f^n(x_0)$, for $n \in \mathbb{N}^*$ (where x_0 is arbitrary in X) converges to $x^* \in Fix(f)$ and the following apriori estimation holds

$$d(u_n, x^*) \leq \frac{s^n}{1-s} d(x_0, f(x_0)), \quad \forall n \in \mathbb{N},$$

where $s = \frac{\alpha+\beta}{1-\beta} < 1$. Hence, for the strict fixed point $x^* \in X$ the following estimation holds

$$d(u_n, x^*) \leq \frac{s^n}{1-s} d(x_0, x_1), \quad \forall n \in \mathbb{N},$$

where $u_n = f^n(x_0) \to x^*$ as $n \in \mathbb{N}^*$ (where x_0 is arbitrary in X and $f : X \to X$ is the selection constructed in Reich's original proof).

3) It is worth to notice that the following Kannan type assumption

$$H(F(x), F(y)) \leq \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right), \quad \forall x, y \in X,$$

(where $\beta \in (0, \frac{1}{2})$) does not imply, in general, the existence of a fixed point (see [16] Example 1). Moreover, if we impose the assumption

$$H(F(x), F(y)) \leq \alpha d(x,y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right), \quad \forall x, y \in X,$$

(where $\alpha + 2\beta < 1$) the existence of a fixed point for F was established just for the case when $X := \mathbb{R}$, see [16] Proposition 1.

In fact, the above remarks lead to some interesting open questions.

Moreover, we can prove the following interesting result, see also [10].

Theorem 3.2. *Let (X,d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta \in \mathbb{R}_+$ with $0 < \alpha + 2\beta < 1$, such that*

$$\delta(F(x), F(y)) \leq \alpha d(x,y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right), \quad \forall x, y \in X.$$

Then F is a MP operator.

Proof. By Theorem 3.1 we know that $Fix(F) = SFix(F) = \{x^*\}$. We have to prove that $F^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$. We have, for every $x \in X$, that

$$\begin{aligned}\delta(F(x),x^*) &= \delta(F(x),F(x^*)) \\ &\leq \alpha d(x,x^*) + \beta\,(\delta(x,F(x)) + \delta(x^*,F(x^*))) \\ &= \alpha d(x,x^*) + \beta\delta(x,F(x)) \\ &\leq \alpha d(x,x^*) + \beta(d(x,x^*) + \delta(x^*,F(x))).\end{aligned}$$

Thus

$$\delta(F(x),x^*) \leq \frac{\alpha+\beta}{1-\beta} d(x,x^*), \quad \forall x \in X.$$

It follows that

$$\delta(F^2(x),x^*) = \sup_{y\in F(x)} \delta(F(y),x^*) \leq \sup_{y\in F(x)} \left(\frac{\alpha+\beta}{1-\beta}\right) d(y,x^*) \leq \left(\frac{\alpha+\beta}{1-\beta}\right)^2 d(x,x^*).$$

By mathematical induction, we get that

$$\delta(F^n(x),x^*) \leq \left(\frac{\alpha+\beta}{1-\beta}\right)^n d(x,x^*) \to 0 \text{ as } n \to +\infty, \text{ for each } x \in X.$$

The proof is now complete. □

In some situations, it is important to get a localization of the (strict) fixed point for a multi-valued operator. In the case of $(\alpha,\beta)-\delta$-contractions of Reich type we have the following local fixed point theorem.

Theorem 3.3. *Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. Suppose that $F : \tilde{B}(x_0;r) \to P_b(X)$ is a multi-valued operator for which:*

(a) there exist $\alpha, \beta \in \mathbb{R}_+$ with $0 < \alpha + 2\beta < 1$ such that

$$\delta(F(x),F(y)) \leq \alpha d(x,y) + \beta\,(\delta(x,F(x)) + \delta(y,F(y))), \quad \forall \tilde{B}(x_0;r) \in X;$$

(b) $\delta(x_0,F(x_0)) \leq \frac{1-\alpha-2\beta}{1+\beta} r$.

Then there exists a unique $x^* \in \tilde{B}(x_0;r)$ such that $SFix(F) = Fix(F) = \{x^*\}$. In particular, if $\beta > 0$, then $x^* \in B(x_0;r)$.

Proof. We will show that the closed ball $\tilde{B}(x_0;r)$ is invariant with respect to F, i.e., $F(\tilde{B}(x_0;r)) \subseteq \tilde{B}(x_0;r)$. For this purpose, let $x \in \tilde{B}(x_0;r)$ and $y \in F(x)$ be arbitrary chosen. Then we have

$$d(y,x_0) \leq \delta(F(x),F(x_0)) + \delta(x_0,F(x_0)).$$

On the other hand,

$$\begin{aligned}\delta(F(x),F(x_0)) &\leq \alpha r + \beta\,(\delta(x,F(x)) + \delta(x_0,F(x_0))) \\ &\leq \alpha r + \beta\,(d(x,x_0) + \delta(x_0,F(x_0)) + \delta(F(x_0),F(x)) + \delta(x_0,F(x_0))) \\ &= (\alpha+\beta)r + 2\beta\delta(x_0,F(x_0)) + \beta\delta(F(x_0),F(x)).\end{aligned}$$

Hence

$$\delta(F(x),F(x_0)) \leq \frac{(\alpha+\beta)r + 2\beta\delta(x_0,F(x_0))}{1-\beta}.$$

Hence, going back to our first relation, we get

$$\begin{aligned} d(y,x_0) &\leq \frac{(\alpha+\beta)r+\beta\delta(x_0,F(x_0))}{1-\beta}+\delta(x_0,F(x_0)) \\ &= \frac{(\alpha+\beta)r+(\beta+1)\delta(x_0,F(x_0))}{1-\beta}. \end{aligned}$$

By (b) we obtain that $d(y,x_0) \leq r$, proving that the closed ball $\tilde{B}(x_0;r)$ is invariant with respect to F. The conclusion follows now by Theorem 3.1.

If $\beta > 0$, then we can show that $x^* \in B(x_0;r)$. Indeed, suppose, by contradiction, that $d(x^*,x_0) = r$. Then we have

$$\begin{aligned} r &= d(x^*,x_0) \\ &\leq \delta(F(x^*),F(x_0))+\delta(x_0,F(x_0)) \\ &\leq \alpha d(x^*,x_0)+(\beta+1)\delta(x_0,F(x_0)) \\ &\leq \alpha r+(\beta+1)\frac{1-\alpha-2\beta}{1+\beta}r = (1-2\beta)r, \end{aligned}$$

which gives the necessary contradiction. The proof is now complete. □

We will discuss now the well-posedness of the strict fixed point problem. For the well-posedness concept in the single-valued case see the paper Reich-Zaslavski [18], while the multi-valued case is considered in [13].

Definition 3.1. Let (X,d) be a metric space and $F: X \to P_b(X)$ be a multivalued operator. The strict fixed point problem

$$\{x\} = F(x),\ x \in X \tag{3.1}$$

is well-posed for F if:

(a_2) $SFix(F) = \{x^*\}$

(b_2) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\delta_d(x_n,F(x_n)) \to 0$ as $n \to +\infty$, then $x_n \to x^*$ as $n \to +\infty$.

In this respect, we have the following result.

Theorem 3.4. *Let (X,d) be a complete metric space and $F: X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha,\beta \in \mathbb{R}_+$ with $0 < \alpha+2\beta < 1$ such that*

$$\delta(F(x),F(y)) \leq \alpha d(x,y)+\beta\left(\delta(x,F(x))+\delta(y,F(y))\right), \quad \forall x,y \in X.$$

Then the strict fixed point problem is well-posed for F.

Proof. By Theorem 3.1 we know that $Fix(F) = SFix(F) = \{x^*\}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\delta_d(x_n,F(x_n)) \to 0$ as $n \to +\infty$. We will prove that $x_n \to x^*$ as $n \to +\infty$. For this purpose, we have

$$\begin{aligned} d(x_n,x^*) &\leq \delta(x_n,F(x_n))+\delta(F(x_n),F(x^*)) \\ &\leq \delta(x_n,F(x_n))+\alpha d(x_n,x^*)+\beta\left(\delta(x_n,F(x_n))+\delta(x^*,F(x^*))\right) \\ &= (1+\beta)\delta(x_n,F(x_n))+\alpha d(x_n,x^*). \end{aligned}$$

Letting $n \to \infty$, we obtain the desired conclusion. □

We will continue our study by presenting the concept of Ulam-Hyers stability for the strict fixed point problem. For related definitions and results, see [8].

Definition 3.2. Let (X,d) be a metric space and $F : X \to P_b(X)$ be a multi-valued operator. The strict fixed point problem (3.1) is called Ulam-Hyers stable if there exists $c > 0$ such that for each $\varepsilon > 0$ and for each ε-solution $y \in X$ of the strict fixed point problem, i.e.,

$$\delta(y,F(y)) \leq \varepsilon, \tag{3.2}$$

there exists a solution $x^* \in X$ of the strict fixed point inclusion (3.1) such that

$$d(y,x^*) \leq c\varepsilon.$$

We have the following result concerning the Ulam-Hyers stability of the strict fixed point problem.

Theorem 3.5. *Let (X,d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha,\beta \in \mathbb{R}_+$ with $0 < \alpha + 2\beta < 1$ such that*

$$\delta(F(x),F(y)) \leq \alpha d(x,y) + \beta\left(\delta(x,F(x)) + \delta(y,F(y))\right), \quad \forall x,y \in X.$$

Then the strict fixed point problem is Ulam-Hyers stable.

Proof. By Theorem 3.1, we know that $Fix(F) = SFix(F) = \{x^*\}$. Let $\varepsilon > 0$ and $y \in X$ such that $\delta(y,F(y)) \leq \varepsilon$. Then, we have

$$\begin{aligned} d(y,x^*) &\leq \delta(y,F(y)) + \delta(F(y),F(x^*)) \\ &\leq \delta(y,F(y)) + \alpha d(y,x^*) + \beta\left(\delta(y,F(y)) + \delta(x^*,F(x^*))\right) \\ &= (1+\beta)\delta(y,F(y)) + \alpha d(y,x^*). \end{aligned}$$

Thus

$$d(y,x^*) \leq \frac{1+\beta}{1-\alpha}\delta(y,F(y)) \leq \frac{1+\beta}{1-\alpha}\varepsilon.$$

The proof is complete. □

Another stability concept is given in the next definition.

Definition 3.3. Let (X,d) be a metric space and $F : X \to P(X)$ be a multi-valued operator with $SFix(F) = \{x^*\}$. If $(y_n)_{n\in\mathbb{N}}$ is a sequence in X such that the following implication holds

$$D(y_{n+1},F(y_n)) \to 0 \text{ as } n \to \infty \;\Rightarrow\; y_n \to x^* \text{ as } n \to \infty,$$

then we say that the strict fixed point problem (3.1) has the Ostrovski property.

Theorem 3.6. *Let (X,d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha,\beta \in \mathbb{R}_+$ with $0 < \alpha + 2\beta < 1$ such that*

$$\delta(F(x),F(y)) \leq \alpha d(x,y) + \beta\left(\delta(x,F(x)) + \delta(y,F(y))\right), \quad \forall x,y \in X.$$

Then the strict fixed point problem as the Ostrovski property.

Proof. By Theorem 3.1, we know that $Fix(F) = SFix(F) = \{x^*\}$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(y_{n+1},F(y_n)) \to 0$ as $n \to \infty$. Next, we have

$$d(y_{n+1},x^*) \leq D(y_{n+1},F(y_n)) + \delta(F(y_n),x^*).$$

On the other hand, we observe that

$$\begin{aligned}\delta(F(y_n),x^*) &= \delta(F(y_n),F(x^*)) \leq \alpha d(y_n,x^*) + \beta\delta(y_n,F(y_n))\\ &\leq \alpha d(y_n,x^*) + \beta\left(d(y_n,x^*) + \delta(x^*,F(y_n))\right)\\ &= (\alpha+\beta)d(y_n,x^*) + \beta\delta(x^*,F(y_n)).\end{aligned}$$

Hence

$$\delta(F(y_n),x^*) \leq \frac{\alpha+\beta}{1-\beta}d(y_n,x^*), \quad \forall n \in \mathbb{N}.$$

Denote $p := \frac{\alpha+\beta}{1-\beta} \in (0,1)$. As a consequence, we obtain

$$\begin{aligned}d(y_{n+1},x^*) &\leq D(y_{n+1},F(y_n)) + pd(y_n,x^*)\\ &\leq \cdots\\ &\leq \sum_{k=0}^{n} p^k D(y_{n-k+1},F(y_{n-k})) + p^{n+1}d(y_0,x^*).\end{aligned}$$

By Cauchy's Lemma (see [14]), we obtain the desired conclusion. □

Finally, we will present a data dependence theorem for the strict fixed point problem.

Theorem 3.7. *Let (X,d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha,\beta \in \mathbb{R}_+$ with $0 < \alpha+2\beta < 1$ such that*

$$\delta(F(x),F(y)) \leq \alpha d(x,y) + \beta\left(\delta(x,F(x)) + \delta(y,F(y))\right), \quad \forall x,y \in X.$$

Suppose that $G : X \to P_b(X)$ is a multi-valued operator such that $SFix(G) \neq \emptyset$ and there exists $\eta > 0$ such that $\delta(F(x),G(x)) \leq \eta$, for every $x \in X$. Then

$$\delta(SFix(F),SFix(G)) \leq \frac{\eta}{1-\alpha}.$$

Proof. By Theorem 3.1, we know that $Fix(F) = SFix(F) = \{x^*\}$. Let $y \in SFix(G)$ be arbitrary chosen. Then, we also have

$$\begin{aligned}d(y,x^*) &= \delta(G(y),F(x^*))\\ &\leq \delta(G(y),F(y)) + \delta(F(y),F(x^*))\\ &\leq \eta + \alpha d(y,x^*).\end{aligned}$$

Thus $d(y,x^*) \leq \frac{\eta}{1-\alpha}$, which gives immediately the desired conclusion. □

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