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TWO FIXED POINT RESULTS FOR A CLASS OF MAPPINGS OF CONTRACTIVE TYPE

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Abstract. In this paper, we prove two fixed point results for a class of mappings of contractive type acting on a closed subset of a complete metric space.

Keywords. Contractive mapping; Fixed point; Metric space.

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1. INTRODUCTION

Since the publication of Banach's classical fixed point theorem [1], metric fixed point theory has been and continues to be an important part of nonlinear operator theory [2, 4, 5, 6, 7, 9, 11, 12, 13, 14, 15, 16]. For example, several results regarding the existence of fixed points for general nonexpansive mappings in special Banach spaces were presented in [4, 5], while for self-mappings of general complete metric spaces existence results were established for classes of contractive mappings in [3, 8, 10]. An extension of the existence result of [10] and several other existence results for certain mappings of contractive type have also been presented in [17].

In the present paper, we prove two fixed point results for a class of mappings of contractive type acting on a closed subset of a complete metric space.

Let (X,d) be a complete metric space and let K be a nonempty closed subset of X. For each $x \in X$ and each $r > 0$ set

$$B(x,r) = \{y \in X : \ d(x,y) \le r\}.$$

Let $T : K \to X$, $\alpha \in [0, 2^{-1})$ and let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function which satisfies the following property

(i) for each $M > s > 0$ there exists $\Delta(M,s) > 0$ such that

$$\psi(t) < t^2 - \Delta(M,s) \text{ for all } t \in [s,M]$$

and such that for each $x, y \in K$ [9],

$$d(T(x),T(y))^2 \le \alpha d(x,T(y))^2 + \alpha d(y,T(x))^2 + (1-2\alpha)\psi(d(x,y)). \tag{1.1}$$

Fix $\theta \in K$. The following theorem is our first main result. It is proved in Section 2.

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Theorem 1.1. *Assume that $M > 0$ and that a sequence*

$$\{x_n\}_{n=1}^{\infty} \subset K \cap B(\theta, M) \tag{1.2}$$

satisfies

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0. \tag{1.3}$$

Then there exists $\lim_{n\to\infty} x_n$ and $T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} x_n$.

Corollary 1.1. *Assume that $M > 0$ and that for each $\varepsilon > 0$ there exists*

$$x_\varepsilon \in K \cap B(\theta, M)$$

such that $d(x_\varepsilon, T(x_\varepsilon)) \le \varepsilon$. Then there exists

$$x \in K \cap B(\theta, M)$$

such that $T(x) = x$.

The next theorem is our second main result. It is proved in Section 3.

Theorem 1.2. *Let $M > 0$. Assume that for each natural number n and each $\varepsilon > 0$ there exists a sequence*

$$\{x_i\}_{i=0}^{n} \subset K \cap B(\theta, M)$$

such that for all integers $i = 0, \dots, n-1$,

$$d(x_{i+1}, T(x_i)) \le \varepsilon.$$

Then there exists $x \in B(\theta, M) \cap K$ such that $x = T(x)$.

It should be mentioned that when $K = X$ and X is a CAT(0) space, our mapping T is a special case of α-nonexpansive mappings studied in [9].

2. Proof of Theorem 1.1

We may assume without loss of generality that for all integers $n \ge 1$,

$$d(x_n, T(x_n)) \le 1 \tag{2.1}$$

and that $M > 1$.

Let $\varepsilon \in (0, 1)$. By property (i), there exists $\Delta_0 > 0$ such that

$$\psi(t) < t^2 - \Delta_0 \text{ for all } t \in [\varepsilon, 2M]. \tag{2.2}$$

Choose a number $\delta \in (0, \varepsilon)$ such that

$$\delta < 2^{-1}(1 - 2\alpha)\Delta_0(8M + 8)^{-1}. \tag{2.3}$$

By (1.3), there exists a natural number n_0 such that for all integers $n \ge n_0$,

$$d(x_n, T(x_n)) \le \delta. \tag{2.4}$$

Assume that $k, q \ge n_0$ are integers. We show that

$$d(x_k, x_q) \le \varepsilon. \tag{2.5}$$

In view of (2.4), we have

$$d(x_k, T(x_k)) \le \delta,\ d(x_q, T(x_q)) \le \delta. \tag{2.6}$$

It follows from (1.1) that

$$d(T(x_k),T(x_q))^2 \le \alpha d(x_k,T(x_q))^2 + \alpha d(x_q,T(x_k))^2 + (1-2\alpha)\psi(d(x_k,x_q)). \tag{2.7}$$

Assume that (2.5) does not hold. Then

$$d(x_k,x_q) > \varepsilon. \tag{2.8}$$

By (1.2), one has

$$d(x_k,x_q) \le 2M. \tag{2.9}$$

It follows from (2.6) and (2.9) that

$$\begin{aligned} d(x_k,T(x_q))^2 &\le (d(x_k,x_q)+d(x_q,T(x_q)))^2 \\ &\le (d(x_k,x_q)+\delta)^2 \le d(x_k,x_q)^2+\delta(\delta+2d(x_k,x_q)) \\ &\le d(x_k,x_q)^2+\delta(4M+1) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} d(x_q,T(x_k))^2 &\le (d(x_q,x_k)+d(x_k,T(x_k)))^2 \\ &\le (d(x_q,x_k)+\delta)^2 \\ &\le d(x_q,x_k)^2+\delta(\delta+2d(x_q,x_k)) \\ &\le d(x_q,x_k)^2+\delta(4M+1). \end{aligned} \tag{2.11}$$

In view of (2.2), (2.8) and (2.9), we have

$$\psi(d(x_k,x_q)) < d(x_k,x_q)^2 - \Delta_0. \tag{2.12}$$

By (2.2), (2.10), (2.11) and (2.12), we have

$$\begin{aligned} d(T(x_k),T(x_q))^2 &\le \alpha(d(x_k,x_q)^2+\delta(4M+1)) + \alpha(d(x_q,x_k)^2 \\ &\quad + \delta(4M+1)) + (1-2\alpha)(d(x_k,x_q)^2-\Delta_0) \\ &= d(x_k,x_q)^2 + 2\delta(4M+1) - (1-2\alpha)\Delta_0. \end{aligned} \tag{2.13}$$

It follows from (2.1), (2.6) and (2.9) that

$$\begin{aligned} d(x_k,x_q)^2 &\le (d(x_k,T(x_k))+d(T(x_k),T(x_q))+d(x_q,T(x_q)))^2 \\ &\le (2\delta+d(T(x_k),T(x_q)))^2 \\ &\le d(T(x_k),T(x_q))^2+2\delta(2\delta+2d(T(x_k),T(x_q))) \\ &\le d(T(x_k),T(x_q))^2+2\delta(6+2d(x_k,x_q)) \\ &\le d(T(x_k),T(x_q))^2+2\delta(6+4M). \end{aligned} \tag{2.14}$$

By (2.13) and (2.14), one has

$$\begin{aligned} d(x_k,x_q)^2 &\le d(T(x_k),T(x_q))^2+2\delta(6+4M) \\ &\le d(x_q,x_k)^2+2\delta(4M+1)-(1-2\alpha)\Delta_0+2\delta(4M+6) \end{aligned}$$

and

$$(1-2\alpha)\Delta_0 \le 2\delta(8M+8).$$

This contradicts (2.3). The contradiction we have reached proves that (2.5) holds for all integers $q,k \geq n_0$. Since ε is any element of the interval $(0,1)$ we conclude that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and that there exists

$$x_* = \lim_{n\to\infty} x_n. \tag{2.15}$$

By (2.1) and property (i), for each natural number n, we have

$$d(T(x_n),T(x_*))^2 \leq \alpha d(x_n,T(x_*))^2 + \alpha d(x_*,T(x_n))^2 + (1-2\alpha)d(x_n,x_*)^2. \tag{2.16}$$

In view of (1.2) and (2.15), we have

$$x_* \in K \cap B(\theta,M). \tag{2.17}$$

Let $\varepsilon \in (0,1)$. It follows from (1.3) and (2.15) that there exists a natural number n_ε such that for all integers $n \geq n_\varepsilon$,

$$d(x_n,x_*) \leq \varepsilon, \tag{2.18}$$

$$d(x_n,T(x_n)) \leq \varepsilon. \tag{2.19}$$

Let an integer $n \geq n_\varepsilon$. Relations (2.18) and (2.19) imply that

$$d(x_*,T(x_n))^2 \leq (d(x_*,x_n)+d(x_n,T(x_n)))^2 \leq 4\varepsilon^2. \tag{2.20}$$

By (2.18) and (2.19), we have

$$\begin{aligned}
d(x_n,T(x_*))^2 &\leq (d(x_n,T(x_n))+d(T(x_n),T(x_*)))^2 \\
&\leq (\varepsilon + d(T(x_n),T(x_*)))^2 \\
&\leq d(T(x_n),T(x_*))^2 + 2\varepsilon(\varepsilon + d(T(x_n),T(x_*))) \\
&\leq d(T(x_n),T(x_*))^2 + 2\varepsilon(1 + d(T(x_n),T(x_*))) \\
&\leq d(T(x_n),T(x_*))^2 + 2\varepsilon(1 + d(T(x_n),x_n) + d(x_n,x_*) + d(x_*,T(x_*))) \\
&\leq d(T(x_n),T(x_*))^2 + 2\varepsilon(1 + 2\varepsilon + d(x_*,T(x_*))) \\
&\leq d(T(x_n),T(x_*))^2 + 2\varepsilon(3 + d(x_*,T(x_*))).
\end{aligned} \tag{2.21}$$

It follows from (2.16), (2.18), (2.20) and (2.21) that

$$\begin{aligned}
d(x_n,T(x_*))^2 &\leq d(T(x_n),T(x_*))^2 + 2\varepsilon(3 + d(x_*,T(x_*))) \\
&\leq \alpha d(x_n,T(x_*))^2 + 4\alpha\varepsilon^2 + (1-2\alpha)\varepsilon^2 + 2\varepsilon(3 + d(x_*,T(x_*)))
\end{aligned}$$

and

$$(1-\alpha)d(x_n,T(x_*))^2 \leq 4\alpha\varepsilon^2 + (1-2\alpha)\varepsilon^2 + 2\varepsilon(3 + d(x_*,T(x_*))).$$

Since the relation above holds for every integer $n \geq n_\varepsilon$, we conclude that

$$\limsup_{n\to\infty} d(x_n,T(x_*))^2 \leq (1-\alpha)^{-1}(4\alpha\varepsilon^2 + (1-2\alpha)\varepsilon^2 + 2\varepsilon(3 + d(x_*,T(x_*)))).$$

Since ε is any element of the interval $(0,1)$, we conclude that

$$\limsup_{n\to\infty} d(x_n,T(x_*))^2 \leq 0,$$

$$\lim_{n\to\infty} d(x_n,T(x_*)) = 0.$$

Together with (2.15) this implies that $x_* = T(x_*)$. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In view of Corollary 1.1, in order to prove the theorem it is sufficient to show that for each $\varepsilon \in (0,1)$ there exists

$$x_\varepsilon \in K \cap B(\theta, M)$$

such that $d(x_\varepsilon, T(x_\varepsilon)) \le \varepsilon$.

We may assume without loss of generality that $M > 1$. Let $\varepsilon \in (0,1)$. By property (i), there exists

$$\delta \in (0, \varepsilon/4)$$

such that

$$\psi(t) < t^2 - \delta \text{ for each } t \in [\varepsilon/4, 2M]. \tag{3.1}$$

Choose a natural number $n > 4$ such that

$$8n^{-1}(4 + 4M) < \delta(1 - 2\alpha), \tag{3.2}$$

$$n > 8M^2\delta^{-1}(1 - 2\alpha)^{-1} + 4. \tag{3.3}$$

By the assumption of the theorem there exists a sequence

$$\{x_i\}_{i=0}^n \subset K \cap B(\theta, M) \tag{3.4}$$

such that for all integers $i = 0, \dots, n-1$,

$$d(x_{i+1}, T(x_i)) \le n^{-1}. \tag{3.5}$$

In order to complete the proof of the theorem it is sufficient to show that

$$d(x_{n-1}, T(x_{n-1})) \le \varepsilon. \tag{3.6}$$

Assume the contrary. Then

$$d(x_{n-1}, T(x_{n-1})) > \varepsilon. \tag{3.7}$$

By (3.5) and (3.7), we have

$$d(x_{n-1}, x_n) \ge d(x_{n-1}, T(x_{n-1})) - d(T(x_{n-1}), x_n) > \varepsilon - n^{-1} > \varepsilon/2. \tag{3.8}$$

For each integer $k \in \{0, \dots, n-1\}$, set

$$a_k = \sup\{d(x_i, x_j) : \ i, j \in \{k, \dots, n\}\}. \tag{3.9}$$

It follows from (3.4), (3.8) and (3.9) that

$$2M \ge a_0 \ge a_1 \ge \dots\dots \ge a_{n-1} = d(x_{n-1}, x_n) > \varepsilon/2. \tag{3.10}$$

Let an integer k satisfy

$$1 \le k \le n-1. \tag{3.11}$$

In view of (3.9), there are integers

$$q, p \in \{k, \dots, n\} \tag{3.12}$$

such that

$$a_k = d(x_q, x_p). \tag{3.13}$$

By (3.9), (3.11) and (3.12), we have

$$q-1, \ p-1 \ge k-1 \ge 0,$$

$$d(x_{q-1}, x_{p-1}) \leq a_{k-1}. \tag{3.14}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} d(x_q, x_p)^2 &\leq (d(x_q, T(x_{q-1})) + d(T(x_{q-1}), T(x_{p-1})) + d(T(x_{p-1}), x_p))^2 \\ &\leq (2n^{-1} + d(T(x_{q-1}), T(x_{p-1})))^2 \\ &\leq d(T(x_{q-1}), T(x_{p-1}))^2 + 4n^{-1}(n^{-1} + d(T(x_{q-1}), T(x_{p-1}))) \\ &\leq d(T(x_{q-1}), T(x_{p-1}))^2 + 4n^{-1}(3 + d(x_p, x_q)) \\ &\leq d(T(x_{q-1}), T(x_{p-1}))^2 + 4n^{-1}(3 + 2M). \end{aligned} \tag{3.15}$$

By (1.1), we have

$$d(T(x_{q-1}), T(x_{p-1}))^2 \leq \alpha d(x_{q-1}, T(x_{p-1}))^2 + \alpha d(x_{p-1}, T(x_{q-1}))^2 + (1-2\alpha)\psi(d(x_{q-1}, x_{p-1})). \tag{3.16}$$

It follows form (3.4), (3.5), (3.9) and (3.12) that

$$\begin{aligned} d(x_{q-1}, T(x_{p-1}))^2 &\leq (d(x_{q-1}, x_p) + d(x_p, T(x_{p-1})))^2 \\ &\leq (d(x_{q-1}, x_p) + n^{-1})^2 \\ &\leq d(x_{q-1}, x_p)^2 + 2n^{-1}(n^{-1} + d(x_{q-1}, x_p)) \\ &\leq d(x_{q-1}, x_p)^2 + 2n^{-1}(1 + 2M) \\ &\leq a_{k-1}^2 + 2n^{-1}(1 + 2M) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} d(x_{p-1}, T(x_{q-1}))^2 &\leq (d(x_{p-1}, x_q) + d(x_q, T(x_{q-1})))^2 \\ &\leq (d(x_{p-1}, x_q) + n^{-1})^2 \\ &\leq d(x_{p-1}, x_q)^2 + 2n^{-1}(n^{-1} + d(x_{p-1}, x_q)) \\ &\leq d(x_{p-1}, x_q)^2 + 2n^{-1}(1 + 2M) \\ &\leq a_{k-1}^2 + 2n^{-1}(1 + 2M). \end{aligned} \tag{3.18}$$

By the monotonicity of ψ, (3.1), (3.9), (3.10), (3.12) and (3.14), we have

$$\psi(d(x_{q-1}, x_{p-1})) \leq \psi(a_{k-1}) < a_{k-1}^2 - \delta. \tag{3.19}$$

It follows from (3.16)-(3.19) that

$$\begin{aligned} d(T(x_{q-1}), T(x_{p-1}))^2 &\leq 2\alpha(a_{k-1}^2 + 2n^{-1}(1 + 2M)) + (1-2\alpha)(a_{k-1}^2 - \delta) \\ &= a_{k-1}^2 + 4\alpha n^{-1}(1 + 2M) - \delta(1 - 2\alpha). \end{aligned} \tag{3.20}$$

By (3.2), (3.13), (3.15) and (3.20), we have

$$\begin{aligned} a_k^2 &= d(x_q, x_p)^2 \\ &\leq d(T(x_{q-1}), T(x_{p-1}))^2 + 4n^{-1}(3 + 2M) \\ &\leq a_{k-1}^2 + 4\alpha n^{-1}(1 + 2M) - \delta(1 - 2\alpha) + 4n^{-1}(3 + 2M) \\ &\leq a_{k-1}^2 - \delta(1 - 2\alpha) + 4n^{-1}(4 + 4M) \\ &\leq a_{k-1}^2 - \delta(1 - 2\alpha)/2. \end{aligned} \tag{3.21}$$

It follows (3.10) and (3.21) which holds for every integer k satisfying (3.11) that

$$\begin{aligned} 4M^2 &\geq a_0^2 \\ &\geq a_0^2 - a_{n-1}^2 \\ &= \sum_{i=0}^{n-2} (a_i^2 - a_{i+1}^2) \\ &\geq (n-1)\delta(1-2\alpha)/2 \end{aligned}$$

and

$$n \leq 8M^2\delta^{-1}(1-2\alpha)^{-1} + 1.$$

This contradicts (3.3). The contradiction we reached proves (3.6) and Theorem 1.2 itself.

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