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A KRASNOSELSKII-TYPE ALGORITHM FOR APPROXIMATING SOLUTIONS OF VARIATIONAL INEQUALITY PROBLEMS AND CONVEX FEASIBILITY PROBLEMS

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Abstract. In this paper, a Krasnoselskii-type algorithm for approximating a common element of the set of solutions of a variational inequality problem for a monotone, *k-Lipschitz* map and solutions of a convex feasibility problem involving a countable family of relatively nonexpansive maps is studied in a uniformly smooth and 2-uniformly convex real Banach space. A strong convergence theorem is proved. Finally, a numerical example is presented.

Keywords. Generalized projection; Monotone map; Relatively nonepxansive map; Subgradient method; Variational inequality problem.

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1. Introduction

Let E be a real Banach space and let E^* be the dual space of E. Let C be a nonempty, closed and convex subset of E and let $A : C \to E^*$ be a map. Recall that A is said to be

- *k-Lipschitz continuous* if there exists a constant $k \geq 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \ \forall x, y \in C. \tag{1.1}$$

- *monotone* if the following inequality holds:

$$\langle x - y, Ax - Ay \rangle \geq 0, \ \forall x, y \in C. \tag{1.2}$$

- δ*-inverse strongly monotone* if there exists a $\delta > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \delta \|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.3}$$

- *maximal monotone* if A is monotone and the graph of A is not properly contained in the graph of any other monotone map.

It is immediate that if A is δ*-inverse strongly monotone*, then A is monotone and Lipschitz continuous.

The problem of finding a point $u \in C := \bigcap_{i=1}^{\infty} C_i$, where C_i is a convex set for each i, is called a *convex feasibility problem*.

The problem of finding a point $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C, \tag{1.4}$$

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is called a *variational inequality problem*. We denote the set of solutions of variational inequality problem (1.4) by $VI(C,A)$.

Remark 1.1. It is easy to see that if u is a solution of the variational inequality problem (1.4) then,

$$\langle x-u, Ax\rangle \geq 0,\ \forall x \in C.$$

Variational inequality problems were formulated in the late 1960's by Lions and Stampacchia [19]. Since then, these have been studied extensively. In numerous models for solving real-life problems, such as, in signal processing, networking, resource allocation, image recovery, and so on, the constraints can be expressed as variational inequality problems and (or) as fixed point problems. Consequently, the problem of finding common elements of the set of solutions of variational inequality problems and the set of fixed points of nonlinear operators has become a flourishing area of contemporary research for numerous mathematicians working in nonlinear operator theory (see, for example, [8, 20, 21] and the references contained in them).

Numerous researchers have proposed and analyzed various iterative schemes for approximating solutions of variational inequality problems, approximating fixed points of nonexpansive maps and their generalizations (see, e.g., the following monographs: Alber [1], Berinde [4], Browder [5], Chidume [9], Gobel and Reich [13] and the references therein). In most of the early results on iterative methods for approximating solutions of variational inequality problem, the map A was often assumed to be *inverse-strongly monotone* (see, e.g., Buong [3], Censor *et al.* [6], Chidume *et al.* [10], and the references therein). To relax the inverse-strong monotonicity condition on A, Korpelevič [18] introduced, in a finite dimensional Euclidean space $\mathbb{R}^n$, the following *extragradient method*

$$\begin{cases} x_1 = x \in C; \\ x_{n+1} = P_C(x_n - \lambda A[P_C(x_n - \lambda Ax_n)]), \forall\, n \in \mathbb{N}. \end{cases} \tag{1.5}$$

where A was assumed to be monotone and Lipschitz. The extragradient method has since then been studied and improved on by many authors in various ways.

However, we observe that in the extragradient method, *two* projections onto a closed and convex subset C of H need to be computed at each step of the iteration process. As mentioned in [7], this may affect the efficiency of the method if the set C is not simple enough. Therefore, to improve on the extragradient method, Censor *et al.* [7] modified the extragradient method and proposed the following iterative algorithm:

$$\begin{cases} x_0 \in H; \\ y_n = P_C(x_n - \tau Ax_n); \\ T_n = \{w \in H : \langle x_n - \tau Ax_n - y_n, w - y_n\rangle \leq 0\}; \\ x_{n+1} = P_{T_n}(x_n - \tau Ay_n), \forall n \in \mathbb{N}. \end{cases} \tag{1.6}$$

The method (1.6) replaces the second projection onto the closed and convex subset C in (1.5) with a projection on to the *half-space* T_n. Algorithm (1.6) is the so called *subgradient extragradient method.* We note that, the set T_n is a half-space, and hence algorithm (1.6) is easier to implement than algorithm (1.5). Under some mild assumptions, Censor *et al.* in [7] proved that the sequence generated by algorithm (1.6) *converges weakly* to a solution of the variational inequality problem (1.4) *in a real Hilbert space.*

In order to obtain strong convergence, Kraikaew and Saejung [17] combined the subgradient extragradient method (1.5) with the Halpern method introduced in [12] and proposed the following iterative algorithm:

$$\begin{cases} x_0 \in H; \\ y_n = P_C(x_n - \tau A x_n); \\ T_n = \{w \in H : \langle x_n - \tau A x_n - y_n, w - y_n \rangle \leq 0\}; \\ z_n = \alpha_n x_0 + (1-\alpha_n) P_{T_n}(x_n - \tau A y_n); \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S z_n, \forall n \geq 0, \end{cases} \tag{1.7}$$

where $\beta_n \subset [a,b] \subset (0,1)$, for some $a, b \in (0,1)$ and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the sequence generated by algorithm (1.7) *converges strongly* to a point $u \in VI(C,A) \cap F(S)$ *in a real Hilbert space.*

In 2015, Nakajo [23] proposed and studied the following CQ method in a 2-uniformly convex and uniformly smooth real Banach space.

$$\begin{cases} x_1 = x \in E; \\ y_n = \Pi_C J^{-1}[Jx_n - \lambda_n A(x_n)]; \\ z_n = T y_n; \\ C_n = \{z \in C : \phi(z, z_n) \leq \phi(z, x_n) - \phi(y_n, x_n) - 2\lambda_n \langle y_n - z, Ax_n - Ay_n \rangle\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \ n \geq 0. \end{cases} \tag{1.8}$$

He proved, with the assumption that A is a monotone and L-Lipschitz map and T is relatively nonexpansive, that the sequence generated by (1.8) converges strongly to a point $q \in F(T) \cap VI(C,A)$.

Motivated by the results of Kraikaew and Saejung [17], and Nakajo [23], we introduce in this paper a *Krasnoselskii-type algorithm* in a *uniformly smooth and 2-uniformly convex real Banach space* and prove *strong convergence* of the sequence generated by our algorithm to a point $q \in F(S) \cap VI(C,A)$. As an immediate consequence of this, we obtain a strong convergence theorem for approximating a *common element of solutions of a variational inequality problem* and a common fixed point of a *countable family of relatively nonexpansive maps.* Our theorems are improvements on some recent important results (see Remark 3.1 below).

2. PRELIMINARIES

Let J be the *normalized duality map* from E to 2^{E^*}. The following properties of J will be needed subsequently (see, e.g., Ibaraki and Takahashi [14]).

- If E is a reflexive, strictly convex and smooth real Banach space, then J is single-valued and bijective.
- In a Hilbert space H, the duality map J and its inverse J^{-1} are the identity maps on H.
- If E is uniformly smooth and uniformly convex, then the dual space E^* is also uniformly smooth and uniformly convex and the normalized duality map J and its inverse, J^{-1}, are both uniformly continuous on bounded sets.

Let E be a smooth real Banach space and $\phi : E \times E \to \mathbb{R}$ be defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall\, x,y \in E. \tag{2.1}$$

It was first introduced by Alber and has been extensively studied by many authors (see, for example, Alber [2]; Chidume [9], Chidume *et al.* [10]; Kamimura and Takahashi [16]; Nilsrakoo and Saejung [24]; Ofoedu and Shehu [26]; Reich [27]; Zegeye [31]; and the references cited therein). It is easy to see from the definition of ϕ that, in a real Hilbert space H, equation (2.1) reduces to $\phi(x,y) = \|x-y\|^2$, $\forall x,y \in H$.

Furthermore, given $x,y,z \in E$, and $\tau \in (0,1)$, we have the following properties and definitions (see, e.g., Nilsrakoo and Saejung [24]):

P1: $(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2$,
P2: $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle z-x, Jy - Jz\rangle$,
P3: $\phi(\tau x + (1-\tau)y, z) \le \tau\phi(x,z) + (1-\tau)\phi(y,z)$.

Definition 2.1. Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. The map $\Pi_C : E \to C$ defined by $\tilde{x} = \Pi_C(x) \in C$ such that $\phi(\tilde{x},x) = \inf_{y\in C}\phi(y,x)$ is called the *generalized projection* of x onto C. Clearly, in a real Hilbert space, the generalized projection Π_C coincides with the metric projection P_C from E onto C.

Definition 2.2. Let $S : C \to E$ be a map. Then, S is said to be *relatively nonexpansive* if the following conditions hold:

(i) $F(S) := \{x \in C : Sx = x\} \neq \emptyset$;
(ii) $\phi(u, Sv) \le \phi(u,v), \ \forall\, u \in F(S)\ v \in C$;
(iii) $(I-S)$ is demi-closed at zero, i.e., whenever a sequence $\{v_n\}$ in C converges weakly to u and $\{v_n - Sv_n\}$ converges strongly 0, then $u \in F(S)$.

Lemma 2.1 (Alber [1])**.** *Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space E. Then,*

(1) if $x \in E$ and $y \in C$, then $\tilde{x} = \Pi_C x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x}\rangle \ge 0$, for all $y \in C$,
(2) $\phi(y,\tilde{x}) + \phi(\tilde{x},x) \le \phi(y,x), \quad \forall x \in E, \ y \in C$.

Lemma 2.2 (Xu [30])**.** *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive constant α such that*

$$\alpha\|x-y\|^2 \le \phi(x,y), \ \forall\, x,y \in E. \tag{2.2}$$

Remark 2.1. Without loss of generality, we may assume $\alpha \in (0,1)$.

Lemma 2.3 (Xu [30])**.** *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a constant $c_2 > 0$ such that for every $x,y \in E$, the following inequality holds:*

$$\langle x-y, Jx - Jy\rangle \ge c_2\|x-y\|^2.$$

Lemma 2.4 (Kamimura and Takahashi [16])**.** *Let E be a uniformly convex and smooth real Banach space, and let $\{u_n\}$ and $\{v_n\}$ be two sequences of E. If either $\{u_n\}$ or $\{v_n\}$ is bounded and $\phi(u_n,v_n) \to 0$, then $\|u_n - v_n\| \to 0$.*

Lemma 2.5 (Nilsrakoo and Saejung [24]). *Let E be a uniformly smooth Banach space and $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0,2r] \to [0,\infty)$ such that $g(0) = 0$ and*

$$\phi\left(u, J^{-1}[\beta Jx + (1-\beta)Jy]\right) \le \beta\phi(u,x) + (1-\beta)\phi(u,y) - \beta(1-\beta)g(\|Jx - Jy\|)$$

for all $\beta \in [0,1]$, $u \in E$ and $x, y \in B_r$.

Lemma 2.6 (Rockafellar [28]). *Let C be a nonempty closed and convex subset of a reflexive space E and A, a monotone, hemicontinuous map of C into E^*. Let $T : E \to 2^{E^*}$ be an operator defined by:*

$$Tu = \begin{cases} Au + N_C(u), & u \in C, \\ \emptyset, & u \notin C, \end{cases} \tag{2.3}$$

where $N_C(u)$ is defined as follows:

$$N_C(u) = \{w^* \in E^* : \langle u - z, w^* \rangle \ge 0, \ \forall\, z \in C\}.$$

Then, T is maximal monotone and $T^{-1}0 = VI(C,A)$.

Lemma 2.7 (Xu [29]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\beta_n, \ n \ge 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

$$(i)\ \{\alpha_n\} \subset [0,1] \ \textit{and} \ \sum_{n=1}^{\infty}\alpha_n = \infty; \ (ii)\ \limsup_{n\to\infty}\beta_n \le 0. \ Then, \ \lim_{n\to\infty} a_n = 0.$$

Lemma 2.8 (Mainge [22]). *Let Γ_n be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\ge 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \ge 0$. Also, consider the sequence of integers $\{\tau(n)\}_{n\ge n_0}$ defined by*

$$\tau(n) = max\{k \le n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\ge n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\tau(n) = \infty$ and, for all $n \ge n_0$, it holds that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ and we have

$$\Gamma_n \le \Gamma_{\tau(n)+1}.$$

Lemma 2.9 (Alber [2]). *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then,*

$$V(u,u^*) + 2\langle J^{-1}u^* - u, v^* \rangle \le V(u, u^* + v^*), \tag{2.4}$$

for all $u \in E$ and $u^*, v^* \in E^*$.

Lemma 2.10 (Kohsaka and Takahash [15]). *Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E and let $(S_i)_{i=1}^{\infty}$, $S_i : C \to E$, for each $i \ge 1$, be a family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \ne \emptyset$. Let $(\eta_i)_{i=1}^{\infty} \subset (0,1)$ and $(\mu_i)_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty}\eta_i = 1$. Consider the map $T : C \to E$ defined by*

$$Tx = J^{-1}\left(\sum_{i=1}^{\infty}\eta_i(\mu_i Jx + (1-\mu_i)JS_i x)\right) \textit{ for each } x \in C. \tag{2.5}$$

Then, T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(S_i)$.

For the existence and convergence of (2.5), the reader is referred to [25]. The following result has recently been proved. For the sake of completeness, we reproduce the proof here.

Lemma 2.11 (Chidume and Otubo [11])**.** *Let E be a 2-uniformly convex and smooth real Banach space and C be a nonempty, closed and convex subset of E. Let $x_1,\ x_2 \in E$ be arbitrary and $\Pi_C : E \to C$ be the generalized projection. Then, the following inequality holds:*

$$\|\Pi_C x_1 - \Pi_C x_2\| \leq \frac{1}{c_2}\|Jx_1 - Jx_2\|, \tag{2.6}$$

where c_2 is the constant appearing in Lemma 2.3 and J is the normalized duality map on E.

Proof. By Lemma 2.1 (1), for any $x_1, x_2 \in E$, $x_1 \neq x_2$, we have

$$\langle \Pi_C x_2 - \Pi_C x_1, Jx_1 - J\Pi_C x_1\rangle \leq 0 \quad \text{and} \quad \langle \Pi_C x_1 - \Pi_C x_2, Jx_2 - J\Pi_C x_2\rangle \leq 0.$$

Adding these two inequalities, we obtain

$$\begin{aligned} &\langle \Pi_C x_1 - \Pi_C x_2, (Jx_2 - Jx_1) - (J\Pi_C x_2 - J\Pi_C x_1)\rangle \leq 0 \\ \Rightarrow\ &\langle \Pi_C x_1 - \Pi_C x_2, Jx_2 - Jx_1\rangle - \langle \Pi_C x_1 - \Pi_C x_2, J\Pi_C x_2 - J\Pi_C x_1\rangle \leq 0 \\ \Rightarrow\ &\langle \Pi_C x_1 - \Pi_C x_2, J\Pi_C x_1 - J\Pi_C x_2\rangle \leq \langle \Pi_C x_1 - \Pi_C x_2, Jx_1 - Jx_2\rangle. \end{aligned}$$

By Lemma 2.3, we obtain

$$\Longrightarrow \quad \begin{aligned} c_2\|\Pi_C x_1 - \Pi_C x_2\|^2 &\leq \|\Pi_C x_1 - \Pi_C x_2\| \cdot \|Jx_1 - Jx_2\|, \\ \|\Pi_C x_1 - \Pi_C x_2\| &\leq \frac{1}{c_2}\|Jx_1 - Jx_2\|. \end{aligned} \tag{2.7}$$

But this inequality also holds if $x_1 = x_2$. The proof is complete. □

Remark 2.2. Lemma 2.11 implies that the generalized projection Π_C is uniformly continuous whenever J is.

2.1. **Analytical representations of duality maps in L_p, l_p, and W_m^p, spaces, $1 < p < \infty$.** The analytical representations of duality maps are known in L_p, l_p, and W_m^p, $1 < p < \infty$. Precisely, in the spaces l_p, $L_p(G)$ and $W_m^p(G)$, $p \in (1,\infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$\begin{aligned} Jz &= \|z\|_{l_p}^{2-p} y \in l_q,\ y = \{|z_1|^{p-2}z_1, |z_2|^{p-2}z_2, ...\},\ z = \{z_1, z_2, ...\}, \\ J^{-1}z &= \|z\|_{l_q}^{2-q} y \in l_p,\ y = \{|z_1|^{q-2}z_1, |z_2|^{q-2}z_2, ...\},\ z = \{z_1, z_2, ...\}, \\ Jz &= \|z\|_{L_p}^{2-p}|z(s)|^{p-2}z(s) \in L_q(G),\ s \in G, \\ J^{-1}z &= \|z\|_{L_q}^{2-q}|z(s)|^{q-2}z(s) \in L_p(G),\ s \in G, \text{ and} \\ Jz &= \|z\|_{W_m^p}^{2-p} \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha(|D^\alpha z(s)|^{p-2} D^\alpha z(s)) \in W_{-m}^q(G), m > 0, s \in G \end{aligned}$$

(see e.g., Alber and Ryazantseva, [2]; p. 36).

3. Main Results

3.1. **Approximating a common element of solutions of a variational inequality problem and a fixed point of a relatively nonexpansive map.** We present a modified subgradient extragradient algorithm in Banach spaces for finding a solution of variational inequality problem (1.4) which is also a fixed point of a given relatively nonexpansive map.

Theorem 3.1. *Let E be a uniformly smooth and 2-uniformly convex real Banach space and let C be a nonempty, closed and convex subset of E. Let $A : E \to E^*$ be a monotone map on C and k-Lipschitz on E and let $S : E \to E$ be a relatively nonexpansive map. We define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in E; \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n); \\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle \le 0\}; \\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda Ay_n); \\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jt_n); \\ x_{n+1} = J^{-1}(\lambda Jx_n + (1-\lambda)JSz_n), \end{cases} \tag{3.1}$$

where $\lambda \in (0,1)$ such that $\lambda < \frac{\alpha}{k}$, α is the constant in Lemma 2.2 and $\{\alpha_n\} \subset [0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose $F(S) \cap VI(C,A) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the point $q = \Pi_{F(S)VI(C,A)} x_0$.

Proof. We divide the proof into two steps.

Step 1. We show that $\{x_n\}$ is bounded.

To do this, we first show that

$$\phi(u,t_n) \le \phi(u,x_n) - \Big(1 - \frac{k\lambda}{\alpha}\Big)\Big(\phi(y_n,x_n) + \phi(t_n,y_n)\Big).$$

Let $u \in F(S) \cap VI(C,A)$. Then

$$\begin{aligned} \phi(u,t_n) &\le \phi(u, J^{-1}[Jx_n - \lambda Ay_n]) - \phi(t_n, J^{-1}[Jx_n - \lambda Ay_n]) \\ &= \|u\|^2 - 2\langle u, Jx_n - \lambda Ay_n\rangle - \|t_n\|^2 + 2\langle t_n, Jx_n - \lambda Ay_n\rangle \\ &= \phi(u,x_n) - \phi(t_n,x_n) + 2\langle u - t_n, \lambda Ay_n\rangle \\ &= \phi(u,x_n) - \phi(t_n,x_n) + 2\lambda\langle u - y_n, Ay_n\rangle + 2\lambda\langle y_n - t_n, Ay_n\rangle. \end{aligned} \tag{3.2}$$

Using Remark 1.1 and Property P2, we have

$$\begin{aligned} \phi(u,t_n) &\le \phi(u,x_n) - \phi(t_n,x_n) + 2\lambda\langle y_n - t_n, Ay_n\rangle \\ &= \phi(u,x_n) - \phi(y_n,x_n) - \phi(t_n,y_n) + 2\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle. \end{aligned} \tag{3.3}$$

Now, we estimate $\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle$. Using the fact that $t_n \in T_n$, the Lipschitz continuity of A and Lemma 2.2, we obtain

$$\begin{aligned}\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle &= \langle t_n - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle + \lambda\langle t_n - y_n, Ax_n - Ay_n\rangle \\ &\leq \lambda\langle t_n - y_n, Ax_n - Ay_n\rangle \\ &\leq \lambda\|t_n - y_n\|\|Ax_n - Ay_n\| \\ &\leq \frac{k\lambda}{2}\Big(\|t_n - y_n\|^2 + \|x_n - y_n\|^2\Big) \\ &\leq \frac{k\lambda}{2\alpha}\Big(\phi(t_n, y_n) + \phi(y_n, x_n)\Big). \qquad (3.4)\end{aligned}$$

It follows that

$$\begin{aligned}\phi(u, t_n) &\leq \phi(u, x_n) - \phi(y_n, x_n) - \phi(t_n, y_n) + \frac{k\lambda}{\alpha}\Big(\phi(t_n, y_n) + \phi(y_n, x_n)\Big) \\ &= \phi(u, x_n) - \Big(1 - \frac{k\lambda}{\alpha}\Big)\Big(\phi(y_n, x_n) + \phi(t_n, y_n)\Big) \qquad (3.5) \\ &\leq \phi(u, x_n). \qquad (3.6)\end{aligned}$$

Now,

$$\begin{aligned}\phi(u, x_{n+1}) &\leq \lambda\phi(u, x_n) + (1-\lambda)\phi(u, z_n) \\ &\leq \lambda\phi(u, x_n) + (1-\lambda)\big(\alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, t_n)\big) \\ &\leq \lambda\phi(u, x_n) + (1-\lambda)\alpha_n\phi(u, x_0) + (1-\lambda)(1-\alpha_n)\phi(u, x_n) \qquad (3.7) \\ &= \big(1 - (1-\lambda)\alpha_n\big)\phi(u, x_n) + (1-\lambda)\alpha_n\phi(u, x_0) \\ &\leq max\{\phi(u, x_n), \phi(u, x_0)\}\end{aligned}$$

By induction, we have

$$\phi(u, x_{n+1}) \leq \phi(u, x_0).$$

Hence, $\{\phi(u, x_n)\}$ is bounded. By propert P1, $\{x_n\}$ is bounded. Furthermore, by Lemma 2.11, we have that $\{y_n\}$ is also bounded.

Step 2. We show that $\{x_n\}$ converges strongly to some point $q = \Pi_{F(S)\cap VI(C,A)}x_0$. To show this, we first establish the following:

(i) $\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|t_n - y_n\| = \lim_{n\to\infty}\|z_n - Sz_n\| = 0$;
(ii) $\Omega_w(x_n) \subset F(S)\cap VI(C,A)$.

For (i), we shall consider two cases.

Case 1. Suppose there exists an $n_0 \in \mathbb{N}$ such that

$$\phi(u, x_{n+1}) \leq \phi(u, x_n),\ \forall\ n \geq n_0.$$

Then, $\{\phi(u, x_n)\}$ is convergent.

Now, we estimate $\phi(u,x_{n+1})$ using inequality (3.5).

$$\begin{aligned}
\phi(u,x_{n+1}) &\leq \lambda\phi(u,x_n)+(1-\lambda)\phi(u,Sz_n)\\
&\leq \lambda\phi(u,x_n)+(1-\lambda)\phi(u,z_n)\\
&\leq \lambda\phi(u,x_n)+(1-\lambda)\big(\alpha_n\phi(u,x_0)+(1-\alpha_n)\phi(u,t_n)\big)\\
&\leq \lambda\phi(u,x_n)+(1-\lambda)\alpha_n\phi(u,x_0)+(1-\lambda)(1-\alpha_n)\Big(\phi(u,x_n)-\big(1-\frac{\lambda k}{\alpha}\big)\phi(y_n,x_n)\Big) \qquad (3.8)\\
&= \phi(u,x_n)+(1-\lambda)\alpha_n\phi(u,x_0)+\alpha_n(\lambda-1)\phi(u,x_n)-(1-\lambda)\left(1-\frac{\lambda k}{\alpha}\right)\phi(y_n,x_n)\\
&\quad +\alpha_n(1-\lambda)\left(1-\frac{\lambda k}{\alpha}\right)\phi(y_n,x_n).
\end{aligned}$$

Thus,

$$\sigma\phi(y_n,x_n)\leq \phi(u,x_n)-\phi(u,x_{n+1})+(1-\lambda)\alpha_n\phi(u,x_0)+\alpha_n(\lambda-1)\phi(u,x_n)+\alpha_n\sigma\phi(y_n,x_n),$$

where $\sigma=(1-\lambda)\big(1-\frac{\lambda k}{\alpha}\big)$. Using the fact that $\alpha_n\to 0$, the boundedness of $\{x_n\}$ and $\{y_n\}$, we deduce that $\phi(y_n,x_n)\to 0$, as $n\to\infty$. Hence, by Lemma 2.4, we have $\lim_{n\to\infty}\|x_n-y_n\|=0$. Using (3.5), we have

$$\begin{aligned}
\phi(u,x_{n+1}) &\leq \lambda\phi(u,x_n)+(1-\lambda)\alpha_n\phi(u,x_0)+(1-\lambda)(1-\alpha_n)\phi(u,t_n)\\
&= \lambda\phi(u,x_n)+(1-\lambda)\alpha_n\phi(u,x_0)+(1-\lambda)\phi(u,t_n)-(1-\lambda)\alpha_n\phi(u,t_n) \qquad (3.9)\\
&\leq \lambda\phi(u,x_n)+(1-\lambda)\phi(u,x_n)+(1-\lambda)\alpha_n\phi(u,x_0)-(1-\lambda)\alpha_n\phi(u,t_n).
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\big(\phi(u,x_n)-\phi(u,t_n)\big)=0.$$

It also follows from (3.5) that

$$0\leq\left(1-\frac{\lambda k}{\alpha}\right)\phi(t_n,y_n)\leq\phi(u,x_n)-\phi(u,t_n)-\left(1-\frac{\lambda k}{\alpha}\right)\phi(y_n,x_n).$$

Thus, $\phi(t_n,y_n)\to 0$, as $n\to\infty$. By Lemma 2.4, $\|t_n-y_n\|\to 0$, as $n\to\infty$. It follows that $\lim_{n\to\infty}\|x_n-t_n\|=0$.

Next, observe that

$$\phi(x_n,z_n)\leq\alpha_n\phi(x_n,x_0)+(1-\alpha_n)\phi(x_n,t_n).$$

Using the fact that $\alpha_n\to 0$, as $n\to\infty$, boundedness of $\{x_n\}$, $\{t_n\}$ and $\|x_n-t_n\|\to 0$, we have $\phi(x_n,z_n)\to 0$, as $n\to\infty$. Thus, $\|x_n-z_n\|\to 0$ as $n\to\infty$. Using Lemma 2.5, we have

$$\begin{aligned}
\phi(u,x_{n+1}) &\leq \lambda\phi(u,x_n)+(1-\lambda)\phi(u,Sz_n)-\lambda(1-\lambda)g(\|Jx_n-JSz_n\|)\\
&\leq \lambda\phi(u,x_n)+(1-\lambda)\big(\alpha_n\phi(u,x_0)+(1-\alpha_n)\phi(u,t_n)\big)-\lambda(1-\lambda)g(\|Jx_n-JSz_n\|)\\
&\leq \lambda\phi(u,x_n)+(1-\lambda)\alpha_n\phi(u,x_0)+(1-\lambda)(1-\alpha_n)\phi(u,x_n)-\lambda(1-\lambda)g(\|Jx_n-JSz_n\|)\\
&= \phi(u,x_n)+(1-\lambda)\alpha_n\big(\phi(u,x_0)-\phi(u,x_n)\big)-\lambda(1-\lambda)g(\|Jx_n-JSz_n\|).
\end{aligned}$$

Thus

$$0\leq\lambda(1-\lambda)g(\|Jx_n-JSz_n\|)\leq\phi(u,x_n)-\phi(u,x_{n+1})+\alpha_n(1-\lambda)\big(\phi(u,x_0)-\phi(u,x_n)\big). \qquad (3.10)$$

Since $\alpha_n \to 0$, $\{x_n\}$ is bounded and $\lim_{n\to\infty}\phi(u,x_n)$ exists, we have that $g(\|Jx_n - JSz_n\|) \to 0$, as $n \to \infty$. This implies that $\|Jx_n - JSz_n\| \to 0$. By the uniform continuity of J^{-1} on bounded sets, we have $\|x_n - Sz_n\| \to 0$. Hence, $\|z_n - Sz_n\| \to 0$ since

$$\|z_n - Sz_n\| \le \|z_n - x_n\| + \|x_n - Sz_n\|.$$

Now, we show that $\Omega_w(x_n) \subset F(S) \cap VI(C,A)$ where $\Omega_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}$. Since $\{x_n\}$ is bounded, $\Omega_w(x_n) \neq \emptyset$. Let $u \in \Omega_w(x_n)$. Then, there exists a subssequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u$. This implies that $z_{n_k} \rightharpoonup u$, as $k \to \infty$. Since $\lim_{k\to\infty}\|z_{n_k} - Sz_{n_k}\| = 0$, it follows that $u \in F(S)$. Next we show that $u \in VI(C,A)$. Let

$$Tv = \begin{cases} Av + N_C(v), & if\ v \in C, \\ \emptyset, & if\ v \notin C, \end{cases}$$

be as defined in Lemma 2.6. Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$. It is known that if T is maximal monotone, then given $(x,x^*) \in E \times E^*$ such that if $\langle x-y, x^*-y^*\rangle \ge 0$, $\forall\ (y,y^*) \in G(T)$, one has $x^* \in Tx$.

Claim: $(u,0) \in G(T)$.

Let $(v,u^*) \in G(T)$. To establish the claim, it suffices to show that $\langle v-u, u^*\rangle \ge 0$. Now, $(v,u^*) \in G(T) \Rightarrow u^* \in Tv = Av + N_C(v) \Rightarrow u^* - Av \in N_C(v)$. Therefore,

$$\langle v-y, u^* - Av\rangle \ge 0,\ \forall\ y \in C.$$

Since $y_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n)$ and $v \in C$, we have by Lemma 2.1 (1) that

$$\langle y_n - v, Jx_n - \lambda Ax_n - Jy_n\rangle \ge 0.$$

Thus,

$$\left\langle v - y_n, \frac{Jy_n - Jx_n}{\lambda} + Ax_n \right\rangle \ge 0.$$

Using the fact that $y_n \in C$ and $u^* - Av \in N_C(v)$, we have

$$\begin{aligned} \langle v - y_{n_k}, u^*\rangle &\ge \langle v - y_{n_k}, Av\rangle \\ &\ge \langle v - y_{n_k}, Av\rangle - \left\langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda} + Ax_{n_k} \right\rangle \\ &= \langle v - y_{n_k}, Av - Ay_{n_k}\rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle - \left\langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda} \right\rangle \\ &\ge \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle - \left\langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda} \right\rangle. \end{aligned}$$

Using the Lipschitz continuity of A, and uniform continuity of J on bounded sets, we have

$$\langle v-u, u^*\rangle \ge 0.$$

Therefore, $\Omega_w(x_n) \subset VI(C,A)$. Thus, $\Omega_w(x_n) \subset F(S) \cap VI(C,A)$.

Next, we show that $\{x_n\}$ converges strongly to the point $q = \Pi_{F(S)\cap VI(C,A)}x_0$. Since $\{x_n\}$ is bounded, then, there exixts a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \rightharpoonup z$ and

$$\lim_{k\to\infty}\langle x_{n_k} - q, Jx_0 - Jq\rangle = \limsup_{n\to\infty}\langle x_n - q, Jx_0 - Jq\rangle = \limsup_{n\to\infty}\langle z_n - q, Jx_0 - Jq\rangle.$$

Since $\Omega_w(x_n) \subset F(S) \cap VI(C,A)$, we have

$$\lim\langle x_{n_k} - q, Jx_0 - Jq\rangle = \langle z - q, Jx_0 - Jq\rangle \leq 0.$$

Hence, we deduce that

$$\limsup_{n\to\infty}\langle z_n - q, Jx_0 - Jq\rangle \leq 0. \tag{3.11}$$

But, from Lemma 2.9, we have

$$\begin{aligned}
\phi(q,x_{n+1}) &= \phi(q, J^{-1}\big(\lambda Jx_n + (1-\lambda)JSz_n\big) \\
&\leq \lambda\phi(q,x_n) + (1-\lambda)\phi(q,Sz_n) \\
&\leq \lambda\phi(q,x_n) + (1-\lambda)\phi\Big(q, J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)Jt_n\big)\Big) \\
&= \lambda\phi(q,x_n) + (1-\lambda)V\big(q, \alpha_n Jx_0 + (1-\alpha_n)Jt_n\big) \\
&\leq \lambda\phi(q,x_n) + (1-\lambda)\Big(V\big(q, \alpha_n Jx_0 + (1-\alpha_n)Jt_n - \alpha_n(Jx_0 - Jq)\big) + 2\alpha_n\langle z_n - q, Jx_0 - Jq\rangle\Big) \\
&= \lambda\phi(q,x_n) + (1-\lambda)\Big(V\big(q, \alpha_n Jq + (1-\alpha_n)Jt_n\big) + 2\alpha_n\langle z_n - q, Jx_0 - Jq\rangle\Big) \\
&\leq \lambda\phi(q,x_n) + (1-\lambda)(1-\alpha_n)V(q,Jt_n) + 2(1-\lambda)\alpha_n\langle z_n - q, Jx_0 - Jq\rangle \\
&\leq \lambda\phi(q,x_n) + (1-\lambda)(1-\alpha_n)\phi(q,x_n) + 2(1-\lambda)\alpha_n\langle z_n - q, Jx_0 - Jq\rangle \\
&= \big(1-(1-\lambda)\alpha_n\big)\phi(q,x_n) + 2(1-\lambda)\alpha_n\langle z_n - q, Jx_0 - Jq\rangle.
\end{aligned} \tag{3.12}$$

Using (3.11) and Lemma 2.7, we have $\phi(q,x_n) \to 0$. Hence, by Lemma 2.4, we have $x_n \to q$.

Case 2. If Case 1 does not hold, then there exists a subsequence $\{x_{m_j}\} \subset \{x_n\}$ such that $\phi(u,x_{m_j+1}) > \phi(u,x_{m_j})$, for all $j \in \mathbb{N}$. From Lemma 2.8, there exists a nondecreasing sequence $\{n_k\} \subset \mathbb{N}$, such that $\lim_{k\to\infty} n_k = \infty$ and the following inequalities hold

$$\phi(u,x_{n_k}) \leq \phi(u,x_{n_k+1}) \text{ and } \phi(u,x_k) \leq \phi(u,x_{n_k+1}), \text{ for each } k \in \mathbb{N}.$$

Now,

$$\begin{aligned}
\phi(u,x_{n_k}) \leq \phi(u,x_{n_k+1}) &\Rightarrow\ 0 \leq \phi(u,x_{n_k+1}) - \phi(u,x_{n_k}) \\
&\Rightarrow\ 0 \leq \liminf_{k\to\infty}\big(\phi(u,x_{n_k+1}) - \phi(u,x_{n_k})\big) \\
&\leq \liminf_{k\to\infty}\Big(\lambda\phi(u,x_{n_k}) + (1-\lambda)\big(\alpha_{n_k}\phi(u,x_0) \\
&\qquad + (1-\alpha_{n_k})\phi(u,t_{n_k})\big) - \phi(u,x_{n_k})\Big) \\
&= \liminf_{k\to\infty}(1-\lambda)\big(\phi(u,t_{n_k}) - \phi(u,x_{n_k})\big).
\end{aligned}$$

Since $\phi(u,t_{n_k}) \leq \phi(u,x_{n_k})$, $\forall k \geq 0$, one has

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}\big(\phi(u,t_{n_k}) - \phi(u,x_{n_k})\big) \\
&\leq \limsup_{k\to\infty}\big(\phi(u,t_{n_k}) - \phi(u,x_{n_k})\big) \\
&\leq 0.
\end{aligned}$$

Hence, $\lim_{k\to\infty}\big(\phi(u,t_{n_k}) - \phi(u,x_{n_k})\big) = 0$. Using a similar arguement as in Case 1 above, we obtain that

- $\|t_{n_k} - y_{n_k}\| \to 0$, $\|y_{n_k} - x_{n_k}\| \to 0$, as $k \to \infty$;

- $\|x_{n_k} - z_{n_k}\| \to 0, \ \|Sz_{n_k} - z_{n_k}\| \to 0$, as $k \to \infty$;
- $\Omega_w(x_{n_k}) \subset F(S) \cap VI(C,A)$.

Next, we show that $\{x_k\}$ converges strongly to $q = \Pi_{F(S)\cap VI(C,A)}x_0$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$, as $j \to \infty$ and

$$\lim_{j\to\infty}\langle x_{n_{k_j}} - q, Jx_0 - Jq\rangle = \limsup_{k\to\infty}\langle x_{n_k} - q, Jx_0 - Jq\rangle = \limsup_{k\to\infty}\langle z_{n_k} - q, Jx_0 - Jq\rangle.$$

Since $\Omega_w(x_{n_k}) \subset F(S) \cap VI(C,A))$, we have $\limsup_{k\to\infty}\langle z_{n_k} - q, Jx_0 - Jq\rangle \le 0$. From inequality (3.12), we have

$$\begin{aligned}\phi(q, x_{n_k+1}) &\le \big(1-(1-\lambda)\alpha_{n_k}\big)\phi(q,x_{n_k}) + 2(1-\lambda)\alpha_{n_k}\langle z_{n_k} - q, Jx_0 - Jq\rangle\\ &\le \big(1-(1-\lambda)\alpha_{n_k}\big)\phi(q,x_{n_k+1}) + 2(1-\lambda)\alpha_{n_k}\langle z_{n_k} - q, Jx_0 - Jq\rangle.\end{aligned}$$

Since $(1-\lambda)\alpha_{n_k} > 0$ for all $k \ge 0$, we have

$$\begin{aligned}\phi(q,x_k) \le \phi&(q,x_{n_k+1})\\ &\le 2\langle z_{n_k} - q, Jx_0 - Jq\rangle\\ &\Rightarrow\\ &\limsup_{k\to\infty}\phi(q,x_k) \le \limsup_{k\to\infty} 2\langle z_{n_k} - q, Jx_0 - Jq\rangle.\end{aligned}$$

Thus, $\limsup_{k\to\infty}\phi(q,x_k) \le 0$. Therefore, $x_k \to q$, as $k \to \infty$. □

3.2. **Approximating a common element of variational inequality and convex feasibility problem.** We prove the following theorem.

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Let $A : E \to E^*$ be a monotone map on C and k-Lipschitz on E and let $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : E \to E$, $\forall\ i$. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and let $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty}\eta_i = 1$. Define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in E;\\ y_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n);\\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle \le 0\};\\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda Ay_n);\\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jt_n);\\ x_{n+1} = J^{-1}(\lambda Jx_n + (1-\lambda)JSz_n), \end{cases} \tag{3.13}$$

where $Sx = J^{-1}\Big(\sum_{i=1}^{\infty}\eta_i(\mu_i Jx + (1-\mu_i)JS_i x\Big)$ for each $x \in E$, $\lambda \in (0, \frac{\alpha}{k})$ and $\{\alpha_n\} \subset [0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Suppose $\Big(\bigcap_{i=1}^{\infty}F(S_i)\Big) \cap V(C,A) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by (3.13) converges strongly to the point $q = \Pi_{F(S)\cap VI(C,A)}x_0$.

Proof. By Lemma 2.10, S is relatively nonexpansive and $F(S) = \bigcap_{i=1}^{\infty}F(S_i)$. The conclusion follows from Theorem 3.1. □

Corollary 3.1. *Let C be a nonempty, closed and convex subset of $E = L_p\ \Big(or\ l_p\ or\ W_p^m(\Omega)\Big)$, $1 < p \leq 2$. Let $A : E \to E^*$ be a monotone map on C, k-Lipschitz on E and let $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : E \to E$, $\forall$ i. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in E; \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n); \\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda A x_n - J y_n\rangle \leq 0\}; \\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda A y_n); \\ z_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J t_n); \\ x_{n+1} = J^{-1}(\lambda J x_n + (1-\lambda) J S z_n), \end{cases} \tag{3.14}$$

where $Sx = J^{-1}\Big(\sum_{i=1}^{\infty} \eta_i\big(\mu_i Jx + (1-\mu_i) JS_i x\big)\Big)$ for each $x \in E$, $\lambda \in (0, \frac{\alpha}{k})$ and $\{\alpha_n\} \subset [0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose $\Big(\bigcap_{i=1}^{\infty} F(S_i)\Big) \cap V(C,A) \neq \emptyset$. Then, the sequences $\{x_n\}$ generated by (3.14) converges strongly to the point $q = \Pi_{F(S)\cap VI(C,A)} x_0$.

Proof. $L_p\ \Big(or\ l_p\ or\ W_p^m(\Omega)\Big)$, $1 < p \leq 2$, are uniformly smooth and 2-uniformly convex. Hence, the conclusion follows from Theorem 3.2. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : H \to H$ be a monotone map on C and k-Lipschitz on E and let $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : H \to H$, $\forall$ i. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and let $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in E; \\ y_n = P_C J(x_n - \lambda A x_n); \\ T_n = \{z \in E : \langle z - y_n, x_n - \lambda A x_n - y_n\rangle \leq 0\}; \\ t_n = P_{T_n}(x_n - \lambda A y_n); \\ z_n = \alpha_n x_0 + (1-\alpha_n) t_n); \\ x_{n+1} = \lambda x_n + (1-\lambda) S z_n, \end{cases} \tag{3.15}$$

where $Sx = \Big(\sum_{i=1}^{\infty} \eta_i\big(\mu_i x + (1-\mu_i) S_i x\big)\Big)$ for each $x \in E$, $\lambda \in (0, \frac{\alpha}{k})$ and $\{\alpha_n\} \subset [0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose $\Big(\bigcap_{i=1}^{\infty} F(S_i)\Big) \cap V(C,A) \neq \emptyset$. Then, the sequences $\{x_n\}$ generated by (3.15) converges strongly to the point $q = \Pi_{F(S)\cap VI(C,A)} x_0$.

Remark 3.1. Our theorems are improvements of the results of Kraikaew and Saejung [17], and Nakajo [23], in the following sense:

(1) Algorithm (1.8) studied in Nakajo [23] requires at *each step of the iteration process*, the computation of two subsets C_n and Q_n of C; their intersection $C_n \cap Q_n$, and the projection of the initial vector onto this intersection. In our algorithm (3.13), these subsets have been dispensed with and replaced with one half-space, T_n. Furthermore, Nakajo [23] proved a strong convergence theorem for a monotone and *k-Lipschitz* map and *one* relatively nonexpansive map, $S : E \to E$. In

our theorem, strong convergence is proved for a monotone and *k-Lipschitz* map and a *countable family* of relatively nonexpansive maps, $S_i : E \to E$.

(2) In the result of Kraikaew and Saejung [17], the iteration parameter β_n used in their algorithm (1.7), which is to be computed at each step of the iteration process has been replaced by a fixed constant λ in our algorithm (3.13). This λ is to be computed once and used at each step of the iteration process. Consequently, our algorithm reduces computational cost and possible computational complexity and errors. Furthermore, the theorem of Kraikaew and Saejung [17] is proved in a *real Hilbert space,* while our theorem is proved in the much more general *uniformly smooth and 2-uniformly convex real Banach spaces.*

(3) Finally, we remark that in some algorithms, the use of general sequences as iteration parameters instead of fixed constants may provide more general iteration algorithms. For example, the well-known Mann iteration process: $x_0 \in K,\ x_{n+1} = (1-c_n)x_n + c_nTx_n,\ n \geq 0,$ where (i) $\lim_{n\to\infty} c_n = 0$ and (ii) $\sum_{n=0}^{\infty} c_n = \infty$, provides a more general iteration scheme than the Krasnoselskii scheme: $x_0 \in K,\ x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\ n \geq 0,$ where $\lambda \in (0,1)$. While in this case, it is known that whenever the Krasnoselskii scheme converges, it is prefered to the Mann scheme because it involves less computation than the Mann scheme and converges as fast as a geometric progression, slightly faster than the convergence obtainable from any Mann sequence. However, there are problems where the Krasnoselskii scheme is not applicable but the Mann schheme is. It is a fact that whenever a general sequence β_n is introduced as an iteration parameter in any algorithm, it does not, in general, translate to a more general algorithm than an algorithm with a fixed constant β. If, for example, the general sequence β_n introduced is bounded away from 0 and 1, it is easy to show that whenever the algorithm with β_n converges, the same algorithm with such β_n replaced by $\beta \in (0,1)$ converges. Thus, the use of β_n in such algorithm only increases computational cost and possible computational complexity and errors, and is therefore totally undesirable. The use of a constant iteration parameter $\beta \in (0,1)$ is certainly prefered in such a case.

4. Numerical Illustration

In this section, we give a numerical example to compare the computational cost of our algorithm (3.1) with the algorithm (1.8) studied in Nakajo [23].

Example 4.1. Let $E = \mathbb{R}$, $C = [\alpha,\beta]$, $\alpha,\beta \in \mathbb{R}$. Clearly, for $x \in \mathbb{R}$,

$$P_C x = \begin{cases} \alpha, & if\ x < \alpha, \\ x, & if\ x \in C, \\ \beta, & if\ x > \beta. \end{cases}$$

Now, in algorithms (1.8) and (3.1), set $Ax = \frac{x}{3}$, $Sx = \sin x$, $C = [-1,1]$. Then, it is easy to see that A is monotone and $\frac{1}{3}$-Lipschitz and S is relatively nonexpansive. It is also easy to see that $F(S) \cap VI(C,A) = \{0\}$. Furthermore, we take $x_1 = 5$, $\lambda_n = \frac{n}{2n+1}$ in (1.8) and $x_0 = 5$, $\lambda = \frac{1}{2}$, $\alpha_n = \frac{1}{2^n}$ in (3.1), $n = 0,1,2,\cdots$, as our parameters. Using a tolerance of 10^{-8}, the numerical results are sketched in Figure 4.1 below, where the y-axis represents the value of $|x_n - 0|$ while the x-axis represents the number of iteration (n).

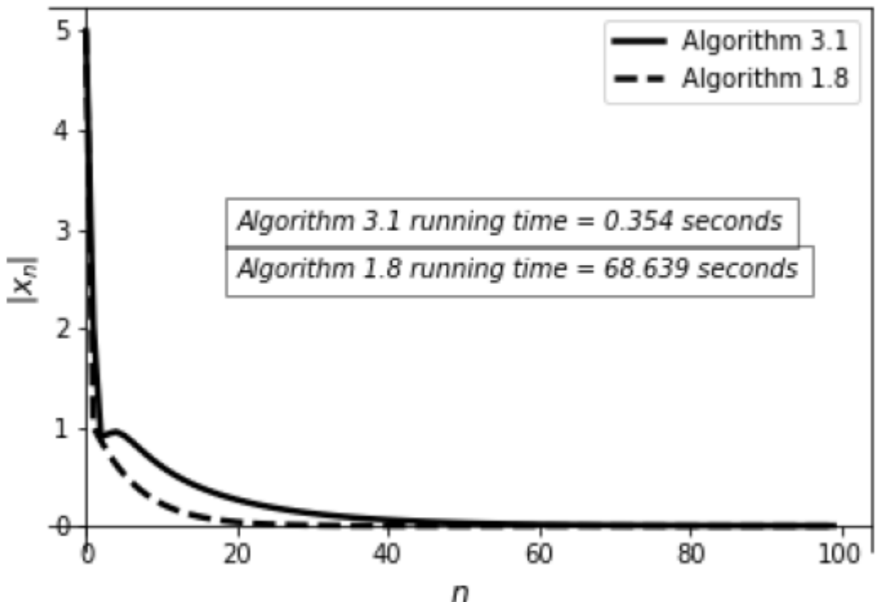

Figure 4.1

Remark 4.1. Figure 4.1 compares the computational cost of our algorithm (3.1) with the algorithm (1.8) studied in Nakajo [23]. All computations and graphs were implemented in python 3.6 using some abstractions developed at *AUST* and other open source python library such as numpy and matplotlib on Zinox with intel core i7 4Gb RAM.

5. Conclusion

It was observed that the number of iterations using algorithm (3.1) is greater than the number of iterations using algorithm (1.8). However, it took 0.354 seconds to obtain convergence for (3.1) while it took 68.639 seconds to obtain convergence for (1.8) using the same tolerance error. Consequently, looking at the time difference, we deduce that algorithm (1.8) requires much more computation time than algorithm (3.1).

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