DISINTEGRATION OF YOUNG MEASURES AND CONDITIONAL PROBABILITIES

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Abstract. Existence of a version of conditional probabilies is proved by means of a general disintegration theorem of Young measures due to M. Valadier.

1. INTRODUCTION

In this brief note, the relationship between the concept of disintegrations of Young measures and that of regular conditional probabilities is clarified. Although the similarity between these two has been noticed in the long history of the theory of Young measures, some vagueness seems to have been left. Our target is to prove the existence of regular conditional probabilities through the disintegrability theorem. For a general theory of Young measures, the readers are referred to Valadier [5].

2. PRELIMINARIES

Let $(\Omega, \mathcal{E}, \mu)$ be a finite measure space and X a Hausdorff topological space endowed with the Borel σ-field $\mathcal{B}(X)$. The mapping $\pi_\Omega : \Omega \times X \to \Omega$ (resp. $\pi_X : \Omega \times X \to X$) denotes the projection of $\Omega \times X$ into Ω (resp. X).

A (positive) finite measure γ on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$ is called a **Young measure** if it satisfies

$$\gamma \circ \pi_\Omega^{-1} = \mu. \tag{2.1}$$

The set of all Young measures on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$ is denoted by $\mathfrak{Y}(\Omega, \mu; X)$.

A family $\{\nu_\omega | \omega \in \Omega\}$ of measures on $(X, \mathcal{B}(X))$ is called a **measurable family** if the mapping

$$\omega \mapsto \nu_\omega(B)$$

is measurable for any $B \in \mathcal{B}(X)$. Given a measurable family $\{\nu_\omega | \omega \in \Omega\}$ of finite measures on $(X, \mathcal{B}(X))$, the function

$$\omega \mapsto \int_X \chi_A(\omega, x) d\nu_\omega$$

is measurable for any $A \in \mathcal{E} \otimes \mathcal{B}(X)$. χ_A is the characteristic function of A. A set function γ defined by

$$\gamma(A) = \int_\Omega \left\{ \int_X \chi_A(\omega, x) d\nu_\omega \right\} d\mu, \quad A \in \mathcal{E} \otimes \mathcal{B}(X) \tag{2.2}$$

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is a measure on $(\Omega \times X, \mathscr{E} \otimes \mathscr{B}(X))$. In particular if every $v_\omega (\omega \in \Omega)$ is a probability measure, then γ defined by (2.2) is a Young measure since (2.1) is satisfied.

Conversely, is it possible to represent any Young measure on $\Omega \times X$ in the form of (2.2) for some measurable family $\{v_\omega | \omega \in \Omega\}$ of probability measures? The most general result concerning this problem is given by Valadier [4].

If a Young measure γ is representable in the form (2.2), its right-hand side is called the **disintegration** of γ. In this case, we symbolically express γ as

$$\gamma = \int_\Omega \delta_\omega \otimes v_\omega d\mu. \tag{2.3}$$

3. Valadier's Disintegrability Theorem

Valadier's quite general result on the disintegrability of Young measures is stated below.

Let $\mathscr{E}$ be a σ-field on Ω. The completion of $\mathscr{E}$ by a finite (positive) measure μ is denoted by $\mathscr{E}_\mu$. A function $f : \Omega \to \mathbb{R}$ is said to be $\mathscr{E}_\mu$-measurable if it is measurable with respect to $\mathscr{E}_\mu$.

Theorem 3.1 (Valadier's Disintegration Theorem (Valadier [4])). *Let $(\Omega, \mathscr{E}, \mu)$ be a finite (positive) measure space and X a Hausdorff topological space. Suppose that γ is a Young measure on $(\Omega \times X, \mathscr{E} \otimes \mathscr{B}(X))$ such that $\gamma \circ \pi_X^{-1}$ is a Radon measure on X. Then there exists a $\mathscr{E}_\mu$-measurable family $\{v_\omega | \omega \in \Omega\}$ of Radon probability measures on X which satisfies*

$$\gamma(A) = \int_\Omega \left\{ \int_X \chi_A(\omega, x) dv_\omega \right\} d\mu \quad for \quad A \in \mathscr{E}_\mu \otimes \mathscr{B}(X).$$

Remark 3.1. Valadier's proof makes use of the theory of liftings due to A. and C. Ionescu-Tulcea. In case that X is a metrizable space enjoying some additional properties, the proof becomes a little bit easy. (cf. Valadier [5] pp.178-182.)

Remark 3.2. If X is a Souslin space, any finite Borel measure on it is automatically Radon.

4. Conditional Probabilities

Let $(\Omega, \mathscr{E}, P)$ be a probability space and $\mathfrak{F}$ a sub-σ-field of $\mathscr{E}$. As anyone knows, the existence of the conditional probability $P(A|\mathfrak{F})(\omega)(A \in \mathscr{E})$ is assured by means of Radon-Nikodým theorem. $P(A|\mathfrak{F})(\omega)$ is said to be **regular** if (i) the mapping $\omega \mapsto P(A|\mathfrak{F})(\omega)$ is $\mathfrak{F}$-measurable for any $A \in \mathscr{E}$ and (ii) $P(\cdot|\mathfrak{F})(\omega)$ is a probability measure on $(\Omega, \mathscr{E})$ for a.e. ω. A similarity between the concept of regular conditional probabilities and that of measurable families is quite clear.

We now proceed to the existence problem of regular conditional probabilities. The following theorem is basically obtained by Hoffmann-Jørgensen [2] and Chatterji [1]. However we now give an alternative proof based upon Valadier's theorem. (See also Schwartz [3].)

Theorem 4.1. *Let $(\Omega, \mathscr{E})$ be a measurable space and X a Hausdorff topological space with a Radon probability measure θ. A function $p : X \to \Omega$ is assumed to be $(\mathscr{B}(X), \mathscr{E})$-measurable. A measure μ on $(\Omega, \mathscr{E})$ is defined by $\mu = \theta \circ p^{-1}$. Then there exists a $\mathscr{E}_\mu$-measurable family $\{v_\omega | \omega \in \Omega\}$ of Radon probabilities on X which satisfies*

$$\int_\Omega v_\omega(B \cap p^{-1}(A)) d\mu = \theta(B \cap p^{-1}(A)) \quad for \quad A \in \mathscr{E}, B \in \mathscr{B}(X).$$

(The extension of μ is also denoted by the same notation.)

Proof. To start with, we define a measure γ on $(\Omega \times X, \mathscr{E} \otimes \mathscr{B}(X))$ by

$$\gamma(E) = \int_\Omega \left\{ \int_X \chi_E(\omega,x)d\theta \right\}d\mu \quad \text{for} \quad E \in \mathscr{E}_\mu \otimes \mathscr{B}(X).$$

We can easily confirm that γ is a Young measure, taking account of the relation

$$\gamma \circ \pi_\Omega^{-1}(A) = \int_\Omega \left\{ \int_X \chi_{A \times X}(\omega,x)d\theta \right\}d\mu$$
$$= \int_\Omega \chi_A(\omega)d\mu$$
$$= \mu(A) \quad \text{for} \quad A \in \mathscr{E}.$$

We also note that

$$\gamma \circ \pi_X^{-1}(B) = \int_\Omega \left\{ \int_X \chi_{\Omega \times B}(\omega,x)d\theta \right\}d\mu$$
$$= \int_\Omega \theta(B)d\mu \qquad\qquad (4.1)$$
$$= \theta(B) \quad \text{for} \quad B \in \mathscr{B}(X).$$

Hence $\gamma \circ \pi_X^{-1}$ is a Radon measure on X.

By Valadier's theorem, there exists a $\mathscr{E}_\mu$-measurable family $\{v_\omega | \omega \in \Omega\}$ of Radon probabilities on X which satisfies

$$\gamma(E) = \int_\Omega \left\{ \int_X \chi_E(\omega,x)dv_\omega \right\}d\mu \quad \text{for} \quad E \in \mathscr{E}_\mu \otimes \mathscr{B}(X).$$

If $E = A \times B \in \mathscr{E} \otimes \mathscr{B}(X)$ in particular, we obtain

$$\gamma(A \times B) = \int_\Omega \left\{ \int_X \chi_{A \times B}(\omega,x)dv_\omega \right\}d\mu$$
$$= \int_\Omega \chi_A(\omega)v_\omega(B)d\mu \qquad\qquad (4.2)$$
$$= \int_A v_\omega(B)d\mu.$$

Since $\theta = \gamma \circ \pi_X^{-1}$ by (4.1), we find that

$$\theta(B \cap p^{-1}(A)) = \gamma \circ \pi_X^{-1}(B \cap p^{-1}(A))$$
$$= \gamma(\Omega \times (B \cap p^{-1}(A)))$$
$$= \int_\Omega v_\omega(B \cap p^{-1}(A))d\mu$$

by (4.2). This completes the proof.   □

The following result is actually a corollary of Theorem 4.1.

Theorem 4.2. *Let X be a Hausdorff topological space endowed with the Borel σ-field $\mathscr{B}(X)$. μ is a Radon probability measure on X. For any sub-σ-field $\mathscr{E}$ of $\mathscr{B}(X)$, there exists a $\mathscr{E}_\mu$-measurable family $\{v_\omega | \omega \in X\}$ which satisfies*

$$\mu(A \cap B) = \int_A v_\omega(B)d\mu \quad \text{for} \quad A \in \mathscr{E}, B \in \mathscr{B}(X).$$

(The extension of μ is also denoted by the same notation.)

Proof. Let Ω be a copy of X (i.e. $\Omega = X$) endowed with $\mathscr{E}$. We specify a mapping $p : X \to \Omega$ as the identity $p(\omega) = \omega$, which is $(\mathscr{B}(X), \mathscr{E})$-measurable.

Since θ in Theorem 4.1 corresponds to the Radon measure μ on $(X, \mathscr{B}(X))$ here, $\mu = \theta \circ p^{-1}$ in Theorem 4.1 should be interpreted as $\mu|_{\mathscr{E}}$, the restriction of μ to $\mathscr{E}$.

Applying Theorem 4.1 to this simple setting, there exists a $\mathscr{E}_\mu$-measurable family $\{v_\omega | \omega \in \Omega\}$ of Radon probability measures on X which satisfies

$$\int_\Omega \chi_A(\omega) v_\omega(B) d\mu = \mu(B \cap p^{-1}(A)) = \mu(A \cap B)$$

$$\text{for } A \in \mathscr{E}, B \in \mathscr{B}(X).$$

This proves Theorem 4.2. □

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