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STABILITY OF FIXED POINTS IN GENERALIZED METRIC SPACES

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Abstract. Let (X,G) be a generalized metric space. Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of G-metrics on X and let $T_n : X_n \to X_n$, where $\{X_n\}$ is a sequence of nonempty subsets of X, be a (ψ,φ)-weakly contractive mapping with a fixed point x_n. In this paper, we study the the convergence of $\{T_n\}$ and the convergence of $\{x_n\}$. Various stability results are established.

Keywords. Fixed point; Stability; Altering distance function; G-metric space.

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1. Introduction

Recently, Mustafa and Sims [13, 14] introduced generalized metric space (or G-metric space) as a generalization of metric spaces where every triplet of an arbitrary set X is mapped into $\mathbb{R}^+$ (non-negative reals). Subsequently, a number of useful results on fixed and periodic points in these spaces for different classes of mappings were obtained [1, 5, 7, 8, 15, 16, 17, 18, 20, 22]. Barbet and Nachi [2, 3] and Nachi [19] obtained some stability results in a metric space using certain new notions of convergence over a variable domain. These results generalize the corresponding results in Bonsall [4], Fraser and Nadler [6] and Nadler [21]. Recently, Mishra *et al.* generalized Barbet-Nachi's results in different settings for various classes of mappings; see [9, 10, 11, 12] and the references therein. To the best of our knowledge, stability results have not been studied in the setting of G-metric spaces.

Let (X,G) be a generalized metric space. Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of G-metrics on X and let $T_n : X_n \to X_n$, where $\{X_n\}$ is a sequence of nonempty subsets of X, be a (ψ,φ)-weakly contractive mapping with a fixed point x_n. In this paper, we study the the convergence of $\{T_n\}$ and the convergence of $\{x_n\}$. Various stability results are established.

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $\overline{\mathbb{N}} = \mathbb{N}\cup\{\infty\}$. First we recall the following definitions and results from Mustafa and Sims [13, 14] which play an important role in this paper.

Definition 1.1. Let X be a nonempty set. Let $G : X\times X\times X \to \mathbb{R}^+$ be a mapping satisfying the following conditions:

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($\mathbf{G}_1$) $0 < G(x,x,y)$ for all $x,y \in X$, with $x \neq y$;
($\mathbf{G}_2$) $G(x,y,z) = 0$ if $x = y = z$;
($\mathbf{G}_3$) $G(x,y,z) \geq G(x,y,y)$ for all $x,y,z \in X$, with $y \neq z$;
($\mathbf{G}_4$) $G(x,y,z) = G(z,y,x) = G(z,x,y) = \ldots$, $x,y,z \in X$;
($\mathbf{G}_5$) $G(x,y,z) \leq G(x,u,u) + G(u,y,z)$ for all $x,y,z,u \in X$.

Then G is called a G-metric on X and the pair (X,G) is called a $G-$metric space. For simplicity, we us X to stand for the G-metric space.

Definition 1.2. [14] A sequence $\{x_n\}$ in a (X,G) is said to be

(i): *G-convergent* to $x^* \in X$ if, for any $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$, such that, for all $n,m \geq n_0$, $G(x^*,x_n,x_m) < \varepsilon$. This is denoted by $\lim_{n,m\to\infty} G(x^*,x_n,x_m) = 0$;

(ii): *G-Cauchy* if, for any $\varepsilon > 0$, we have an $n_0 \in \mathbb{N}$, that is, for all $n,m,l \geq n_0$, $G(x_n,x_m,x_l) < \varepsilon$. That is, $\lim_{n,m,l\to\infty} G(x_n,x_m,x_l) = 0$.

Further, if every G-Cauchy sequence in (X,G) is G-convergent, then (X,G) is called a G-complete.

Proposition 1.1. [14] *Let X be a G-metric space. The following assertions are equivalent*

(1) $\{x_n\}$ *is G-convergent to x;*
(2) $G(x_l,x_k,x) \to 0$ *as* $k,l \to \infty$;
(3) $G(x_l,x_l,x) \to 0$ *as* $l \to \infty$.
(4) $G(x_l,x,x) \to 0$ *as* $l \to \infty$.

Definition 1.3. [1] Let X be a G-metric space. X is said to be symmetric if $G(x,x,y) = G(x,y,y)$, $\forall x,y \in X$.

Proposition 1.2. [14] *Let X be a G-metric space. Define* $d_G : X \times X \to \mathbb{R}^+$ *by*

$$d_G(x,y) = G(x,x,y) + G(x,y,y)$$

for all $x,y \in X$. *Then* d_G *is metric on* X. *Note that if* X *is symmetric G-metric, then*

$$d_G(x,y) = 2G(x,x,y)$$

for all $x,y \in X$. *If* X *is non symmetric, then*

$$\frac{3}{2}G(x,x,y) \leq d_G(x,y) \leq 3G(x,x,y)$$

for all $x,y \in X$.

Example 1.1. [14] Let $X = \{1,2\}$ and $G : X \times X \times X \to \mathbb{R}^+$ be defined as

(x,y,z)	$G(x,y,z)$
$(1,1,1),(2,2,2)$	0
$(1,2,1),(1,1,2),(2,1,1)$	3
$(1,2,2),(2,1,2),(2,2,1)$	6

Here G is a G-metric on X. Note that $G(1,1,2) \neq G(1,2,2)$. Therefore G is a non symmetric G-metric.

A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be an altering distance function, if φ is increasing and continuous with $\varphi(t) = 0$ if and only if $t = 0$. Let X be a G-metric space. A mapping $T : X \to X$ is said to be (ψ,φ)-*weakly contractive* if it satisfies

$$\psi(G(Tx,Ty,Tz)) \leq \psi(G(x,y,z)) - \varphi(G(x,y,z)), \quad \forall x,y,z \in X, \tag{1.1}$$

where ψ and φ are altering distance functions.

If $\psi(t) = t$ in (1.1), then T is said to be *weakly contractive,* that is,

$$G(Tx,Ty,Tz) \leq G(x,y,z) - \varphi(G(x,y,z)), \quad \forall x,y,z \in X. \tag{1.2}$$

Furthermore, if $\varphi(t) = (1-\lambda)t$ for $\lambda \in [0,1)$ and $\psi(t) = t$ in (1.1), then T is said to be λ-contractive, that is,

$$G(Tx,Ty,Tz) \leq \lambda G(x,y,z), \quad \forall x,y,z \in X. \tag{1.3}$$

Barbet and Nachi [19] (see also Mishra, Singh and Stofile [12]) studied the stability, the G-convergence and the H-convergence for family of mappings $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$. In case that X is a G-metric space, we get the corresponding definitions and result. Let X and Y be two nonempty sets and let $T : X \to Y$ be a mapping. Then the graph of T, denoted by $Gr(T)$, is defined by$Gr(T) = \{(x,Tx) : x \in X\}$.

Definition 1.4. Let X be a $G-$metric space, $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings, where $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets X. A mapping $T_\infty : X_\infty \to X$ is called:

(1) a (G*)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G*), if $Gr(T_\infty) \subset \liminf Gr(T_n)$.

(2) a (H*)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H*), if for all sequences $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$, we have a sequence $\{y_n\}$ in X_∞ with

$$\lim_n G(x_n,x_n,y_n) = 0 \text{ and } \lim_n G(T_nx_n,T_nx_n,T_ny_n) = 0.$$

Now we have the following result as a consequence of the above definition of the (G*) limit.

Lemma 1.1. *Let X be a G-metric space. Let $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings, where $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets X. If a mapping $T_\infty : X_\infty \to X$ is a (G*)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$, that is, $Gr(T_\infty) \subset \liminf Gr(T_n)$, then, for every $x \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that*

$$\lim_n G(x_n,x_n,x) = 0 \textit{ and } \lim_n G(T_nx_n,T_nx_n,T_\infty x) = 0.$$

Proof. Note that

$$Gr(T_n) = \{(x_n,T_nx) : x_n \in X_n\}, \quad n \in \mathbb{N}$$

and

$$Gr(T_\infty) = \{(x,T_\infty x) : x \in X_\infty\}. \tag{1.4}$$

Therefore

$$\liminf Gr(T_n) = \liminf\{(x_n,T_nx_n) : x_n \in X_n\},$$

which implies that

$$\liminf Gr(T_n) = \lim_{n\to\infty}[\inf\{(x_m,T_mx_m) : x_m \in X_m\},\ m \geq n].$$

Moreover, we have

$$\liminf G(T_n) = \sup[\inf\{(x_m,T_mx_m) : x_m \in X_m],\ m \geq n].$$

Hence,

$$\liminf G(T_n) = \cup_{n=1}^{\infty}[\cap_{n=m}^{\infty}\{(x_m, T_m x_m) : x_m \in X_m\}]. \tag{1.5}$$

From (1.4) and (1.5) we have

$$\text{for all } x \in X_\infty,\ (x, T_\infty x) \in \cup_{n=1}^{\infty}[\cap_{n=m}^{\infty}\{(x_m, T_m x_m) : x_m \in X_m\}],$$

which implies that, for all $x \in X_\infty$, there exists $n \in \mathbb{N}$, $m \geq n$, $(x, T_\infty x) = (x_m, T_m x_m)$, $x_m \in X_m$. Thus, for all $x \in X_\infty$, $x_n \in \prod_{n\in\mathbb{N}} X_n$ such that, for all $m \geq n$, $x = x_m$ and $T_\infty x = T_m x_m$. For every $x \in X_\infty$, we have a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ with

$$\lim_{n\to\infty} G(x_n, x_n, x) = 0 \text{ and } \lim_{n\to\infty} G(T_n x_n, T_n x_n, T_\infty x) = 0.$$

□

Definition 1.5. Let X be a G-metric space. Let $\{G_n\}_{n\in\mathbb{N}}$ a sequence of G-metrics on X and Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X.

- **(A):** For $\{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ and $x \in X_\infty$ such that $\lim_n G_n(x_n, x_n, x) = 0$ if and only if $\lim_n G(x_n, x_n, x) = 0$.
- **(A$_0$):** For $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $x \in X_\infty$ such that $\lim_n G_n(x_n, x_n, x) = 0$ if and only if $\lim_n G(x_n, x_n, x) = 0$.
- **(B):** For $\{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, there is $\{y_n\}$ in X_∞ such that $\lim_n G_n(x_n,\ x_n, y_n) = 0$ if and only if $\lim_n G(x_n, x_n, y_n) = 0$.
- **(B$_0$):** For $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $x \in X_\infty$ such that $\lim_n G_n(x_n, x_n, x) = 0$ if and only if $\lim_n G(x_n, x_n, x) = 0$.

2. Stability of Fixed Points

In this section, we present various stability results for $\{T_n\}_{n\in\ \mathbb{N}}$, a sequence of (ψ, φ)-weakly contractive mappings in G-metric spaces. First, we establish the following result which ensures the existence of the (G*)-limit.

Proposition 2.1. *Let X be a G-metric space and Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subset of X. Suppose that $\{T_n : X_n \to X_n\}_{n\in\overline{\mathbb{N}}}$ is sequence of (ψ, φ)-weakly contractive mappings. If $T_\infty : X_\infty \to X$ is a (G^*)-limit of a sequence $\{T_n\}$, then T_∞ is the unique (G^*)-limit of $\{T_n\}$.*

Proof. Assume that $T_\infty, T_\infty^* : X_\infty \to X$ are two (G^*)-limits of $\{T_n\}$. For $x \in X_\infty$, we have two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that $\{x_n\}$ and $\{y_n\}$ G-converge to x and the sequences $\{T_n x_n\}$ and $\{T_n y_n\}$ G-converge to $T_\infty x$ and $T_\infty^* x$, respectively. Therefore,

$$G(x_n, x_n, y_n) \leq G(x_n, x_n, x) + G(y_n, x, x) \to 0 \text{ as } n \to \infty. \tag{2.1}$$

Further, one has

$$G(T_\infty x, T_\infty x, T_\infty^* x) \leq G(T_\infty x, T_\infty x, T_n x_n) + G(T_n x_n, T_n x_n, T_n y_n) + G(T_n y_n, T_n y_n, T_\infty^* x). \tag{2.2}$$

Since T_n is a (ψ, φ)-weakly contractive mapping for each $n \in \mathbb{N}$, one has

$$\psi(G(T_n x_n, T_n x_n, T_n y_n)) \leq \psi(G(x_n, x_n, y_n)) - \varphi(G(x_n, x_n, y_n)) < \psi(G(x_n, x_n, y_n)). \tag{2.3}$$

By the monotonicity of ψ, we obtain

$$G(T_n x_n, T_n x_n, T_n y_n) \leq G(x_n, x_n, y_n).$$

By (2.2), we have

$$G(T_\infty x, T_\infty x, T_\infty^* x) \leq G(T_\infty x, T_\infty x, T_n x_n) + G(x_n, x_n, y_n) + G(T_n y_n, T_n y_n, T_\infty^* x).$$

Taking limit as $n \to \infty$, we obtain $G(T_\infty x, T_\infty x, T_\infty^* x) = 0$. This completes the proof. □

The following stability result presents a generalization of Fraser and Nadler [6, Theorem 2], and Nachi [19, Theorem 8.4] in G-metric spaces.

Theorem 2.1. *Let X be a G-metric space let and $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subset of X equipped with a sequence of generalized metrics $\{G_n\}_{n\in\mathbb{N}}$ satisfying the property (A). Suppose that $\{T_n : X_n \to X_n\}_{n\in\overline{\mathbb{N}}}$ is sequence of (ψ,φ)-weakly contractive mappings on (X_n, G_n) $G-$converging in the sense of (G^*) to a (ψ,φ)-weakly contractive mapping $T_\infty : X_\infty \to X$. If $x_n \in X_n$ is a fixed point of T_n for each $n \in \overline{\mathbb{N}}$, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence G-converging to a point $x_\infty \in X_\infty$, then x_∞ is a fixed point of T_∞.*

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ $G-$converging to x_∞ in X_∞. By property (G*), we have a sequence $\{y_n\} \in \prod_{n\in\mathbb{N}} X_n$ implies

$$\lim_n G(y_n, y_n, x_\infty) = 0 \text{ and } \lim_n G(T_n y_n, T_n y_n, T_\infty x_\infty) = 0.$$

Using (A), we have

$$\lim_n G_n(y_n, y_n, x_\infty) = 0 \text{ and } \lim_n G_n(T_n y_n, T_n y_n, T_\infty x_\infty) = 0. \tag{2.4}$$

If we define a sequence $\{z_n\}$ such that

$$z_{n_k} = x_{n_k} \text{ for all } k \in \mathbb{N}, \text{ and } z_n = y_n \text{ if } n \neq n_k, \text{ for any } k \in \mathbb{N},$$

then $\lim_n G(z_n, z_n, x_\infty) = 0$. So $\lim_n G_n(z_n, z_n, x_\infty) = 0$. In view of

$$G(z_n, z_n, y_n) \leq G(z_n, z_n, x_\infty) + G(y_n, x_\infty, x_\infty) \to 0 \text{ as } n \to \infty,$$

we have

$$\lim_n G_n(z_n, z_n, y_n) = 0. \tag{2.5}$$

It follows that

$$G_{n_k}(T_{n_k} z_{n_k}, T_{n_k} z_{n_k}, T_\infty x_\infty) \leq G_{n_k}(T_{n_k} z_{n_k}, T_{n_k} z_{n_k}, T_{n_k} y_{n_k}) + G_{n_k}(T_{n_k} y_{n_k}, T_{n_k} y_{n_k}, T_\infty x_\infty). \tag{2.6}$$

Since T_{n_k} is a (ψ,φ)-weak contraction on (X_n, G_n) for each $k \in \mathbb{N}$, we have

$$\psi(G_{n_k}(T_{n_k} z_{n_k}, T_{n_k} z_{n_k}, T_{n_k} y_{n_k})) \leq \psi(G_{n_k}(z_{n_k}, z_{n_k}, y_{n_k})) - \varphi(G_{n_k}(z_{n_k}, z_{n_k}, y_{n_k})) < \psi(G_{n_k}(z_{n_k}, z_{n_k}, y_{n_k})). \tag{2.7}$$

Since ψ is increasing, we obtain that

$$G_{n_k}(T_{n_k} z_{n_k}, T_{n_k} z_{n_k}, T_{n_k} y_{n_k}) \leq G_{n_k}(z_{n_k}, z_{n_k}, y_{n_k}). \tag{2.8}$$

Using (2.6), we arrive at

$$G_{n_k}(T_{n_k} z_{n_k}, T_{n_k} z_{n_k}, T_\infty x_\infty) \leq G_{n_k}(z_{n_k}, z_{n_k}, y_{n_k}) + G_{n_k}(T_{n_k} y_{n_k}, T_{n_k} y_{n_k}, T_\infty x_\infty).$$

Letting $n_k \to \infty$, we have

$$\begin{aligned}\lim_{n_k} G_{n_k}(T_{n_k}z_{n_k}, T_{n_k}z_{n_k}, T_\infty x_\infty) &\leq \lim_{n_k} G_{n_k}(z_{n_k}, z_{n_k}, y_{n_k}) \\ &\quad + \lim_{n_k} G_{n_k}(T_{n_k}y_{n_k}, T_{n_k}y_{n_k}, T_\infty x_\infty) \\ &= 0.\end{aligned}$$

Since $T_{n_k}x_{n_k} = x_{n_k}(= z_{n_k})$ and $x_{n_k} \to x_\infty$ as $k \to \infty$, we obtain that $T_\infty x_\infty = x_\infty$. This completes the proof. □

Corollary 2.1. *Let X be a G-metric space and let $\{G_n\}_{n\in\mathbb{N}}$ a sequence of G-metrics on X satisfying the property (A_0). Suppose that $\{T_n : X \to X\}_{n\in\mathbb{N}}$ is the sequence of (ψ,φ)-weakly contractive mappings on (X, G_n) G-converging in the sense of (G^*) to a mapping $T_\infty : X_\infty \to X$. If $x_n \in X$ is a fixed point of T_n for each $n \in \overline{\mathbb{N}}$, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence G-converging to a point $x_\infty \in X_\infty$, then x_∞ is a fixed point of T_∞.*

Proof. By taking $X_n = X$ for $n \in \overline{\mathbb{N}}$ in Theorem 2.1, we find the desired result immediately. □

The following stability result improve and generalizes the results of Fraser and Nadler [6, Theorem 3], and Nachi [19, Theorem 8.5].

Theorem 2.2. *Let X be a G-metric space and let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subset of X equipped with a sequence of generalized metrics $\{G_n\}_{n\in\mathbb{N}}$ satisfying the property (A). Suppose that $\{T_n : X_n \to X_n\}_{n\in\overline{\mathbb{N}}}$ is sequence of (ψ,φ)-weakly contractive mappings on (X_n, G_n) G-converging in the sense of (G^*) to a (ψ,φ)-weakly contractive mapping $T_\infty : X_\infty \to X$. If $x_n \in X_n$ is a fixed point of T_n for each $n \in \overline{\mathbb{N}}$, then $\{x_n\}_{n\in\mathbb{N}}$ G-converges to x_∞.*

Proof. Let x_∞ be any element in X_∞. By (G^*), we have a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_n G(y_n, y_n, x_\infty) = 0 \text{ and } \lim_n G(T_n y_n, T_n y_n, T_\infty x_\infty) = 0.$$

By property (A), we deduce that

$$\lim_n G_n(y_n, y_n, x_\infty) = 0 \text{ and } \lim_n G_n(T_n y_n, T_n y_n, T_\infty x_\infty) = 0. \tag{2.9}$$

By the monotonicity of ψ and (2.9), we get

$$\begin{aligned}\lim_n \psi(G_n(x_n, x_n, x_\infty)) &\leq \lim_n \psi(G_n(T_n x_n, T_n x_n, T_\infty x_\infty)) \\ &\leq \lim_n \psi(G_n(T_n x_n, T_n x_n, T_n y_n) + G_n(T_n y_n, T_n y_n, T_\infty x_\infty)) \\ &= \lim_n \psi(G_n(T_n x_n, T_n x_n, T_n y_n)).\end{aligned} \tag{2.10}$$

Since T_n is a (ψ,φ)-weakly contractive on (X_n, G_n) and $\lim_n G(y_n, y_n, x_\infty) = 0$, we have

$$\begin{aligned}&\lim_n \psi(G_n(T_n x_n, T_n x_n, T_n y_n)) \\ &\leq \lim_n \psi(G_n(x_n, x_n, y_n)) - \varphi(G_n(x_n, x_n, y_n)) \\ &\leq \lim_n \psi(G_n(x_n, x_n, x_\infty) + G_n(y_n, y_n, x_\infty)) - \lim_n \varphi(G_n(x_n, x_n, y_n)) \\ &\leq \lim_n \psi(G_n(x_n, x_n, x_\infty)) - \lim_n \varphi(G_n(x_n, x_n, y_n)) \\ &\leq \lim_n \psi(G_n(x_n, x_n, x_\infty)).\end{aligned} \tag{2.11}$$

Let $\lim_n G(x_n, x_n, x_\infty) = t$, for $t \in \mathbb{R}_+$. If $t = 0$, then it is done. If $t > 0$, then we find from (2.10) and (2.11) that

$$\psi(t) \le \psi(t) - \lim_n \varphi(G_n(x_n, x_n, y_n)),$$

which implies $\lim_n \varphi(G_n(x_n, x_n, y_n)) = 0$, a contradiction. Hence, $\lim_n G(x_n, x_n, x_\infty) = 0$, that is, $\{x_n\}_{n\in\mathbb{N}}$ G-converges to x_∞. □

Corollary 2.2. *Let X be a G-metric space and let $\{G_n\}_{n\in\mathbb{N}}$ a sequence of G-metrics on X satisfying the property (A_0). Suppose that $\{T_n : X \to X\}_{n\in\mathbb{N}}$ is the sequence of (ψ, φ)-weakly contractive mappings on (X, G_n) G-converging to a mapping $T_\infty : X_\infty \to X$. If $x_n \in X$ is a fixed point of T_n for each $n \in \overline{\mathbb{N}}$, then $\{x_n\}_{n\in\mathbb{N}}$ G-converges to x_∞.*

Proof. By taking $X_n = X$ for $n \in \overline{\mathbb{N}}$ in Theorem 2.2, we find the desired conclusion easily. □

Theorem 2.3. *Let X be a G-metric space and let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subset of X equipped with a sequence of generalized metrics $\{G_n\}_{n\in\mathbb{N}}$ satisfying the property (B). Suppose that $\{T_n : X_n \to X_n\}_{n\in\overline{\mathbb{N}}}$ is sequence of mappings on (X_n, G_n) satisfies the property (H^*) and G-converging to a (ψ, φ)-weakly contractive mapping $T_\infty : X_\infty \to X$. If $x_n \in X_n$ is a fixed point of T_n for each $n \in \overline{\mathbb{N}}$, then $\{x_n\}_{n\in\mathbb{N}}$ G-converges to x_∞.*

Proof. By (H^*), we find a sequence $\{y_n\}$ in X_∞ having

$$\lim_n G(x_n, x_n, y_n) = 0 \text{ and } \lim_n G(T_n x_n, T_n x_n, T_\infty y_n) = 0.$$

Using property (B), we obtain

$$\lim_n G_n(x_n, x_n, y_n) = 0 \text{ and } \lim_n G_n(T_n x_n, T_n x_n, T_\infty y_{n_n}) = 0. \tag{2.12}$$

By the monotonicity of ψ and (2.12), we have

$$\psi(G_n(x_n, x_n, x_\infty)) \le \psi(G_n(T_n x_n, T_n x_n, T_\infty y_n) + G_n(T_\infty y_n, T_\infty y_n, T_\infty x_\infty)) \le \psi(G_n(T_\infty y_n, T_\infty y_n, T_\infty x_\infty)). \tag{2.13}$$

As T_∞ is (ψ, φ)-weakly contractive, we see that

$$\begin{aligned}\lim_n \psi(G_n(T_\infty y_n, T_\infty y_n, T_\infty x_\infty)) &\le \lim_n \psi(G_n(y_n, y_n, x_\infty)) - \lim_n \varphi(G_n(y_n, y_n, x_\infty))\\ &\le \lim_n \psi(G_n(y_n, y_n, x_n) + G_n(x_n, x_n, x_\infty))\\ &\quad - \lim_n \varphi(G_n(y_n, y_n, x_\infty))\\ &\le \lim_n \psi(G_n(x_n, x_n, x_\infty)) - \lim_n \varphi(G_n(y_n, y_n, x_\infty))\\ &\le \lim_n \psi(G_n(x_n, x_n, x_\infty)).\end{aligned} \tag{2.14}$$

Let $\lim_n G(x_n, x_n, x_\infty) = t$, for $t \in \mathbb{R}_+$. If $t = 0$, then it is done. If $t > 0$, then we find from (2.13) and (2.14) that

$$\psi(t) \le \psi(t) - \lim_n \varphi(G_n(y_n, y_n, x_\infty)),$$

which implies that $\lim_n \varphi(G_n(y_n, y_n, x_\infty)) = 0$. Hence, $\lim_n G(x_n, x_n, x_\infty) = 0$, that is, $\{x_n\}_{n\in\mathbb{N}}$ G-converges to x_∞. □

Corollary 2.3. *Let X be a G-metric space and let $\{G_n\}_{n\in\mathbb{N}}$ a sequence of G-metrics on X satisfying the property (B_0). Suppose that $\{T_n : X \to X\}$ is a sequence of mappings on (X, G_n) G-converging to a (ψ,φ)-weakly contractive mapping $T_\infty : X \to X$. If $x_n \in X$ is a fixed point of T_n for each $n \in \overline{\mathbb{N}}$, then $\{x_n\}_{n\in\mathbb{N}}$ G-converges to x_∞.*

Proof. By taking $X_n = X$ for all $n \in \overline{\mathbb{N}}$ in Theorem 2.3, we obtain the desired conclusion immediately. □

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