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A KRASNOSELSKII-TYPE ALGORITHM FOR APPROXIMATING ZEROS OF MONOTONE MAPS IN BANACH SPACES WITH APPLICATIONS

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Abstract. In this paper, a Krasnoselskii-type algorithm that approximates the unique zero of a Hölder continuous and strongly monotone map is constructed. Strong convergence of the sequence generated by this algorithm is established in strictly convex, reflexive and smooth real Banach spaces. Results obtained in this paper are applied to approximate a minimizer of some convex functionals and a solution of a Hammerstein integral equation.

Keywords. Hölder continuous map; Iterative algorithm; Lipschitz map; Monotone map; Strong convergence.

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1. Introduction

In this paper, E stands for a real Banach space with dual space E^* and H stands for a real Hilbert space except stated otherwise. A map $J : E \to 2^{E^*}$ defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\},$$

is called the normalized duality map on E, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between elements of E and E^*. It is well known that if E is smooth, then J is single-valued. If, in addition, E is reflexive and strictly convex, then $J^{-1} := J_* : E^* \to E$ exists and it is a duality map on E^*. Furthermore, if $E = H$, then $J = I$, the identity map on H. Several other properties of the normalized duality map abound in the literature (see, e.g., [3] and [10], and the references therein).

Let X and Y be normed spaces. A map $T : X \to Y$ is said to be

- *Hölder continuous* if for all $u, v \in X$, there exist $\alpha \in (0,1]$ and $H > 0$ such that $\|Tu - Tv\| \leq H\|u - v\|^\alpha$, where H is called the Hölder constant and α is the Hölder exponent;
- *Lipschitz* if for all $u, v \in X$, there exists $L > 0$ such that $\|Tu - Tv\| \leq L\|u - v\|$, where L is called the Lipschitz constant.

Remark 1.1. From the above definitions, it is easy to see that the class of Lipschitz maps is a proper subclass of the class of Hölder continuous maps.

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A map A from E to E is called accretive if, for each $x,y \in E$, there exists $j(x-y) \in J(x-y)$ such that $\langle Ax-Ay, j(x-y)\rangle \geq 0$.

$A : E \to E^*$ is said to be

- *monotone* if for each $x,y \in E$, we have that $\langle Ax-Ay, x-y\rangle \geq 0$;
- η*-strongly monotone* if for each $x,y \in E$, there exists $\eta > 0$ such that $\langle Ax-Ay, x-y\rangle \geq \eta\|x-y\|^2$.

In Hilbert spaces, accretive and monotone maps coincide, since in such situation J is the identity map on H and $H = H^*$. A point $x^* \in E$ is called a zero of A if $Ax^* = 0$. We shall denote the set of zeros of a map A by $A^{-1}(0) := \{x \in D(A) : Ax = 0\}$, where $D(A)$ denotes the domain of A. Several existence and uniqueness results for zeros of accretive and monotone maps and their types have been established in various Banach spaces (see, e.g., [11] and [16], and the references therein).

The approximation of zeros of monotone and accretive maps and their types assuming existence, is of great interest as they have numerous applications in economics, physics, optimization, evolution theory and so on. In Hilbert spaces, zeros of monotone maps can represent minimizers of some convex functionals, equilibrium states of some dynamical systems, solutions of some partial differential equations, etc. (see, e.g., [4], [14], [15], [21], [22] and [23], and the references therein). In general Banach spaces, if A is accretive, zeros of A correspond to equilibrium points of some dynamical system whereas if A is monotone, they correspond to minimizers of some convex functionals. Iterative methods have been utilized to approximate zeros of monotone and accretive maps and their types assuming existence, by numerous authors in various Banach spaces (see, e.g., [5], [6], [7], [9], [12], [17] and [18] and the references therein).

Recently, Chidume, Bello and Usman [5] studied the following Krasnoselskii-type algorithm:

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad \forall n \geq 0, \tag{1.1}$$

where $A : E \to E^*$ is an η-strongly monotone and Lipschitz map. They proved, in L_p spaces, $1 < p \leq 2$, that the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique zero of A under suitable conditions on the control parameter λ. They also obtained, in L_p spaces, $2 \leq p < \infty$, a strong convergence theorem of their sequence $\{x_n\}$ to the unique zero of A when $A : L_p \to L_p^*$ is a Lipschitz map satisfying the following condition: for all $x,y \in E$ and for some constant $k \in (0,1)$,

$$\langle Ax-Ay, x-y\rangle \geq k\|x-y\|^{\frac{p}{p-1}}. \tag{1.2}$$

In this paper, we constructed a Krasnoselskii-type algorithm that approximates the unique zero of a Hölder continuous map satisfying certain monotonicity condition. Strong convergence of the sequence generated by this algorithm is established in **reflexive, strictly convex and smooth Banach spaces with normalized duality maps whose inverses are either Hölder continuous or Lipschitz**. Finally, we apply our results to solve a convex minimization problem and to approximate a solution of a Hammerstein integral equation. The results obtained in this paper mainly generalized, extended and improved those in Chidume, Bello and Usman [5].

2. PRELIMINARIES

Let E be a smooth real Banach space and let E^* be the dual space of E. The map $\phi : E\times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall\, x,y \in E, \tag{2.1}$$

will play a central role in what follows. It was introduced by Alber [1] and has been studied by [1], [2], [13], [18], [20] and a host of other authors.

If $E = H$, equation (2.1) reduces to $\phi(x,y) = \|x-y\|^2$, $\forall x,y \in H$. It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2, \quad \forall x,y \in E. \tag{2.2}$$

Let $V : E\times E^* \to \mathbb{R}$ be a map defined by $V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$. Then, it is easy to see that

$$V(x,x^*) = \phi(x, J^{-1}(x^*)),\ \forall\, x \in E,\ x^* \in E^*. \tag{2.3}$$

Lemma 2.1 (Alber, [1])**.** *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then,*

$$V(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^*+y^*), \quad \forall x \in E, x^*, y^* \in E^*. \tag{2.4}$$

3. MAIN RESULTS

Now, we are ready to give our main results.

3.1. Strong convergence theorems in Banach spaces with duality maps whose inverses are Lipchitzian.

In Theorem 3.1 below, λ is a constant in $(0,1)$ such that

$$\lambda \in \left(0,\ \frac{\eta}{2(LH^2+\eta)}\right),$$

where L and H are Lipschitz and Hölder constants of J^{-1} and A respectively, and $\eta > 0$, is some constant.

Theorem 3.1. *Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^* and let $J : E \to E^*$ be the normalized duality map on E such that J^{-1} is Lipschitz. Let $A : E \to E^*$ be an Hölder continuous map with $A^{-1}(0) \neq \emptyset$ such that*

$$\langle x-y, Ax-Ay\rangle \ge \eta\|x-y\|^{2\alpha},\ \forall\, x,y \in E, \tag{3.1}$$

where $\alpha \in (0,1]$ is the Hölder continuity exponent of A and $\eta > 0$ is some constant. Let $\{x_n\}_{n\ge 1}$ be a sequence in E defined by

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad \forall n \ge 1.$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to the unique $x^* \in A^{-1}(0)$.

Proof. Since $A^{-1}(0) \neq \emptyset$, we fix $x^* \in A^{-1}(0)$. Using Lemma 2.1 with $y^* = \lambda Ax_n$, we have that

$$\begin{aligned}
& \phi(x^*, x_{n+1}) \\
= \;& V(x^*, Jx_n - \lambda Ax_n) \\
\leq \;& \phi(x^*, x_n) - 2\lambda \langle J^{-1}(Jx_n - \lambda Ax_n) - x^*, Ax_n \rangle \\
= \;& \phi(x^*, x_n) - 2\lambda \langle J^{-1}(Jx_n - \lambda Ax_n) - x^*, Ax_n - Ax^* \rangle \\
= \;& \phi(x^*, x_n) - 2\lambda \langle x_n - x^*, Ax_n - Ax^* \rangle - 2\lambda \langle J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n), Ax_n - Ax^* \rangle \\
\leq \;& \phi(x^*, x_n) - 2\lambda \langle x_n - x^*, Ax_n - Ax^* \rangle + 2\lambda \| J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n) \| \| Ax_n - Ax^* \|.
\end{aligned}$$

Using condition (3.1), the Hölder continuity of A and the Lipschitz condition of J^{-1}, respectively, for some $\eta > 0, \alpha \in (0,1]$ and $L > 0$, we find that

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) - 2\lambda\eta ||x_n - x^*||^{2\alpha} + 2\lambda^2 L ||Ax_n|| ||Ax_n - Ax^*|| \\
&= \phi(x^*, x_n) - 2\lambda\eta ||x_n - x^*||^{2\alpha} + 2\lambda^2 L ||Ax_n - Ax^*|| ||Ax_n - Ax^*|| \\
&\leq \phi(x^*, x_n) - 2\lambda\eta ||x_n - x^*||^{2\alpha} + 2\lambda^2 LH^2 ||x_n - x^*||^{2\alpha} \\
&\leq \phi(x^*, x_n) - 2\lambda\eta ||x_n - x^*||^{2\alpha} + 2\lambda^2 (LH^2 + \eta) ||x_n - x^*||^{2\alpha}.
\end{aligned}$$

Using the condition that $\lambda \in \left(0, \dfrac{\eta}{2(LH^2+\eta)}\right)$, we obtain that

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - \lambda\eta ||x_n - x^*||^{2\alpha} \leq \phi(x^*, x_n). \tag{3.2}$$

Hence, $\{\phi(x^*, x_n)\}$ is a monotone nonincreasing sequence of real numbers bounded below by 0, thus it converges. Consequently, by inequality (2.2), we obtain that $\{x_n\}$ is bounded. Furthermore, using inequality (3.2), we have that

$$\lambda\eta \|x_n - x^*\|^{2\alpha} \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}). \tag{3.3}$$

By taking limit across inequality (3.3), we obtained that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Corollary 3.1. *Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^* such that J^{-1} is Lipschitz. Let $A : E \to E^*$ be an η-strongly monotone and M-Lipschitz map. Let $\{x_n\}_{n \geq 1}$ be a sequence in E defined by*

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad \forall n \geq 1. \tag{3.4}$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to the unique $x^* \in A^{-1}(0)$.

Corollary (3.2) below is an immediate application of Theorem 3.1.

Corollary 3.2. *Let $E = L_p$ space, $1 < p \leq 2$, with dual space E^* and let $J : E \to E^*$ be the normalized duality map on E. Let $A : E \to E^*$ be an Hölder continuous map with $A^{-1}(0) \neq \emptyset$ such that*

$$\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^{2\alpha}, \ \forall\, x, y \in E, \tag{3.5}$$

where $\alpha \in (0,1]$ is the Hölder continuity exponent of A and $\eta > 0$ is some constant. Let $\{x_n\}_{n \geq 1}$ be a sequence in E defined by

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad \forall n \geq 1.$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to the unique $x^* \in A^{-1}(0)$.

If $\alpha = 1$ in Corollary 3.2, then we have the following corollary.

Corollary 3.3. *Let $E = L_p$ space, $1 < p \leq 2$, with dual space E^* and let $J : E \to E^*$ be the normalized duality map on E. Let $A : E \to E^*$ be an η-strongly and L-Lipschitzian map with $A^{-1}(0) \neq \emptyset$. Let $\{x_n\}_{n\geq 1}$ be a sequence in E defined by*

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad \forall n \geq 1.$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to the unique $x^* \in A^{-1}(0)$, where $\lambda \in (0,1)$ is a constant such that $\lambda \in \left(0, \dfrac{\eta}{2(LL_1^2+\eta)}\right)$, $L > 0$ and $L_1 > 0$ are the Lipschitz constants of A and J^{-1} respectively.

3.2. **Strong convergence theorems in Banach spaces with duality maps whose inverses are HÖLDER continuous.** In this section, we prove strong convergence theorems in Banach spaces with duality maps that are Hölder continuous. Consequently, we shall obtain as a corollary strong convergence theorem in L_p spaces, $2 \leq p < \infty$.

In Theorem 3.2 below, λ is a constant in $(0,1)$ such that

$$\lambda \in \left(0, \left(\frac{\eta}{2(H_1 H_2^{1+\alpha}+\eta)}\right)^{\frac{1}{\alpha}}\right),$$

where H_1 and H_2 are Hölder constants of J^{-1} and A respectively, and $\eta > 0$, $\alpha \in (0,1]$, are some constants.

Theorem 3.2. *Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^* and let $J : E \to E^*$ be the normalized duality map on E such that J^{-1} is Hölder continuous. Let $A : E \to E^*$ be an Hölder continuous map with $A^{-1}(0) \neq \emptyset$ such that for some $\eta > 0$*

$$\langle x-y, Ax-Ay\rangle \geq \eta \|x-y\|^{\gamma(1+\alpha)},\ \forall\, x,y \in E, \tag{3.6}$$

where $\alpha, \gamma \in (0,1]$ are the Hölder exponent of J^{-1} and A respectively. Let $\{x_n\}_{n\geq 1}$ be a sequence in E defined by

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad \forall n \geq 1.$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to the unique $x^* \in A^{-1}(0)$.

Proof. Since $A^{-1}(0) \neq \emptyset$, we fix $x^* \in A^{-1}(0)$. Then, by the same method of proof as in the proof of Theorem 3.1, we have that

$$\begin{aligned} &\phi(x^*, x_{n+1}) \\ \leq\ & \phi(x^*, x_n) - 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle + 2\lambda \|J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n)\| \|Ax_n - Ax^*\|. \end{aligned}$$

Using condition (3.6), the Hölder continuity of J^{-1} and A respectively, in the above inequality, we find that, for some $\alpha, \gamma \in (0,1]$, that

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) - 2\lambda\eta ||x_n - x^*||^{\gamma(1+\alpha)} + 2\lambda^{1+\alpha} H_1 ||Ax_n - Ax^*||^{\alpha} ||Ax_n - Ax^*|| \\ &\leq \phi(x^*, x_n) - 2\lambda\eta ||x_n - x^*||^{\gamma(1+\alpha)} + 2\lambda^{1+\alpha} H_1 H_2^{1+\alpha} ||x_n - x^*||^{\gamma(1+\alpha)} \\ &\leq \phi(x^*, x_n) - 2\lambda\eta ||x_n - x^*||^{\gamma(1+\alpha)} + 2\lambda^{\alpha}\lambda(H_1 H_2^{1+\alpha} + \eta) ||x_n - x^*||^{\gamma(1+\alpha)}. \end{aligned}$$

In view of

$$\lambda \in \left(0, \left(\frac{\eta}{2(H_1H_2^{1+\alpha}+\eta)}\right)^{\frac{1}{\alpha}}\right),$$

we obtain that

$$\phi(x^*,x_{n+1}) \leq \phi(x^*,x_n) - \lambda\eta||x_n - x^*||^{\gamma(1+\alpha)} \leq \phi(x^*,x_n). \quad (3.7)$$

Hence, $\{\phi(x^*,x_n)\}$ is a monotone nonincreasing sequence of real numbers bounded below by 0, thus it converges. Therefore, by inequality (2.2), we obtain that $\{x_n\}$ is bounded. Also, from inequality (3.7), we have that

$$\lambda\eta||x_n - x^*||^{\gamma(1+\alpha)} \leq \phi(x^*,x_n) - \phi(x^*,x_{n+1}). \quad (3.8)$$

By taking limit across inequality (3.8), we obtained that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Corollary 3.4. *Let $E = L_p$ space, $2 \leq p < \infty$ and let E^* be its dual space. Let $A : E \to E^*$ be an Hölder continuous map with $A^{-1}(0) \neq \emptyset$ such that for some $\eta > 0$*

$$\langle x-y, Ax-Ay\rangle \geq \eta\|x-y\|^{\gamma(1+\alpha)},\ \forall\, x,y \in E, \quad (3.9)$$

where $\gamma \in (0,1]$ is the Hölder exponent of A and $\alpha \in (0,1]$ is some constant. Let $\{x_n\}_{n\geq 1}$ be a sequence in E defined by

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad \forall n \geq 1.$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to the unique $x^* \in A^{-1}(0)$.

Proof. It is well known that, in L_p spaces, $2 \leq p < \infty$, the inverse of the normalized duality map, J^{-1} is Hölder continuous with Hölder constant H and Hölder exponent $\alpha \in (0,1]$, say. Hence, the rest of the proof follows as in the proof of Theorem 3.2. □

If $\gamma = 1$ in Corollary 3.4, then Corollary 3.5 below is immediate.

Corollary 3.5. *Let $E = L_p$ space, $2 \leq p < \infty$ and let E^* be its dual space. Let $A : E \to E^*$ be a Lipschitz map with $A^{-1}(0) \neq \emptyset$ such that for some $\eta > 0$*

$$\langle x-y, Ax-Ay\rangle \geq \eta\|x-y\|^{1+\alpha},\ \forall\, x,y \in E. \quad (3.10)$$

Let $\{x_n\}_{n\geq 1}$ be a sequence in E defined by

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), \quad \forall n \geq 1.$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to the unique $x^* \in A^{-1}(0)$.

4. Applications

4.1. **An application to convex optimization problem.** In this section, we apply our theorem in solving the problem of finding the minimizers of convex functionals defined on Banach spaces. We shall first start with the following well known definitions and results.

Definition 4.1. A function $f : E \to \mathbb{R}$ is said to be

- *convex* if, for every $x,y \in E$ $f(\gamma x + (1-\gamma)y) \leq \gamma f(x) + (1-\gamma)f(y)$, for some $\gamma \in (0,1)$;
- *strictly convex* if, for every $x,y \in E$, $x \neq y$, $f(\gamma x + (1-\gamma)y) < \gamma f(x) + (1-\gamma)f(y)$, for some $\gamma \in (0,1)$;

- *strongly convex* if there exists $\eta > 0$ such that, for every $x, y \in E,\ x \neq y,\ f(\gamma x + (1-\gamma)y) \leq \gamma f(x) + (1-\gamma)f(y) - \eta\|x-y\|^2$ for some $\gamma \in (0,1)$.

Definition 4.2. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex functional. The subdifferential of f, denoted by ∂f, is a map $\partial f : E \to 2^{E^*}$ defined by $\partial f(x) = \{u^* \in E^* : f(y) - f(x) \geq \langle y - x, u^* \rangle\}$.

Lemma 4.1. *Let E be a real Banach space with E^* as its dual space and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex functional. Then, $p \in E$ is a minimizer of f if and only if $0 \in \partial f(p)$.*

Lemma 4.2. *Let E be a real normed space with dual space E^* and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper strongly convex function. Then, ∂f is strongly monotone.*

Proof. Let $x, y \in E$ and let $x^* \in \partial f(x), y^* \in \partial f(y)$. Then,

$$f(x) - f(z) \leq \langle x - z, x^* \rangle\ \forall\, z \in E \textit{ and } f(y) - f(w) \leq \langle y - w, y^* \rangle\ \forall\, w \in E.$$

For $\gamma \in (0,1)$, take $z = \gamma y + (1-\gamma)x$ and $w = \gamma x + (1-\gamma)y$. Then,

$$f(x) - f(\gamma y + (1-\gamma)x) \leq \gamma \langle x - y, x^* \rangle; \tag{4.1}$$

$$f(y) - f(\gamma x + (1-\gamma)y) \leq \gamma \langle y - x, y^* \rangle. \tag{4.2}$$

Adding inequalities (4.1) and (4.2) and using the strong convexity of f, we have that, for some $\eta > 0$, $\langle x - y, x^* - y^* \rangle \geq \dfrac{2\eta}{\gamma}\|x-y\|^2$. Therefore, ∂f is k-strongly monotone, where $k = \dfrac{2\eta}{\gamma}$. □

Lemma 4.3. *Let E be a real normed space with dual space E^* and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper function such that, for all $x, y \in E, x \neq y$, and some $\alpha \in (0,1],\ \gamma \in (0,1),\ \eta > 0$, the following inequality holds:*

$$f(\gamma x + (1-\gamma)y) \leq \gamma f(x) + (1-\gamma)f(y) - \eta\|x-y\|^{1+\alpha}. \tag{4.3}$$

Then, ∂f satisfies the condition: for all $x, y \in E, x^* \in \partial f(x), y^* \in \partial f(y)$ and some $\alpha \in (0,1), \eta > 0$

$$\langle x - y, x^* - y^* \rangle \geq \eta\|x-y\|^{1+\alpha} \tag{4.4}$$

Proof. The proof follows the same pattern as in the proof of Lemma 4.2. □

Lemma 4.4. *Let E be a real normed space with dual space E^* and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper function such that, for all $x, y \in E, x \neq y$ and some $\alpha \in (0,1],\ \gamma \in (0,1),\ \eta > 0$, the following inequality holds:*

$$f(\gamma x + (1-\gamma)y) \leq \gamma f(x) + (1-\gamma)f(y) - \eta\|x-y\|^{2\alpha}. \tag{4.5}$$

Then, ∂f satisfies the condition: for all $x, y \in E$, $x^* \in \partial f(x), y^* \in \partial f(y)$ and some $\alpha \in (0,1], \eta > 0$

$$\langle x - y, x^* - y^* \rangle \geq \eta\|x-y\|^{2\alpha}. \tag{4.6}$$

Remark 4.1. If $f : E \to \mathbb{R} \cup \{\infty\}$ is a proper, convex and differentiable function, then the differential of f, $df : E \to E^*$ and ∂f coincide.

We now prove the main theorems of this section.

Theorem 4.1. *Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^* and let $J : E \to E^*$ be the normalized duality map on E such that J^{-1} is Lipschitz. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, differentiable and coercieve function satisfying condition (4.5). Then, f has a unique minimizer $x^* \in E$, say. Suppose that, in addition, $df : E \to E^*$ is Hölder continuous. Let $\{x_n\}_{n\geq 1}$ be a sequence in E defined*

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda df(x_n)), \quad \forall n \geq 1. \tag{4.7}$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to x^*, where $\lambda \in (0,1)$ is a constant such that $\lambda \in \left(0, \dfrac{\eta}{2(LH^2+\eta)}\right)$, $L > 0$ and $H > 0$ are the Lipschitz constant and Hölder constant of J^{-1} and A respectively, and $\eta > 0$ is some positive constant.

Proof. By condition (4.5), f is strictly convex. From the hypotheses of the theorem, f is proper, coercieve and continuous. Since E is reflexive, then f has a unique minimizer x^*, say, characterized by $df(x^*) = 0$ (Lemma 4.1). Furthermore, since df satisfies condition (4.6), the proof follows from Theorem 3.1. □

Theorem 4.2. *Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^* and let $J : E \to E^*$ be the normalized duality map on E such that J^{-1} is Hölder continuous. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, differentiable and coercieve function satisfying condition (4.3). Then, f has a unique minimizer $x^* \in E$, say. Suppose that, in addition, $df : E \to E^*$ is Hölder continuous. Let $\{x_n\}_{n\geq 1}$ be a sequence in E defined*

$$x_1 \in E, \quad x_{n+1} = J^{-1}(Jx_n - \lambda df(x_n)), \quad \forall n \geq 1. \tag{4.8}$$

Then the sequence $\{x_n\}$ generated in the above algorithm converges strongly to x^*, where $\lambda \in (0,1)$ is a constant such that $\lambda \in \left(0, \left(\dfrac{\eta}{2(H_1 H_2^{1+\alpha}+\eta)}\right)^{\frac{1}{\alpha}}\right)$, H_1 and H_2 are Hölder constants of J^{-1} and A respectively, and $\eta > 0$, $\alpha \in (0,1]$, some constants

Proof. By condition (4.3), f is strictly convex. From the hypotheses of the theorem, f is proper, coercieve and continuous. Since E is reflexive, then f has a unique minimizer x^*, say, characterized by $df(x^*) = 0$ (Lemma 4.1). Furthermore, since df satisfies condition (4.4), the proof follows from Theorem 3.2. □

4.2. **An application to the Hammerstein integral equations.**

Definition 4.3. Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation of Hammerstein-type has the form

$$u(x) + \int_\Omega k(x,y) f(y,u(y))dy = w(x), \tag{4.9}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions.

By simple transformation, (4.9) can put in the form

$$u + KFu = w. \tag{4.10}$$

which, without loss of generality, can be written as

$$u + KFu = 0. \tag{4.11}$$

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part posses Green's function can, as a rule, be put in the form (4.9) (see e.g., Pascali and Sburian [16], chapter p. 164).

Next, we apply Corollary 3.1 to approximate a solution of (4.11). The following lemmas are needed in what follows.

Lemma 4.5 (Chidume and Idu [8])**.** *Let X, Y be real uniformly convex and uniformly smooth spaces. Let $E = X \times Y$ with the norm $\|z\|_E = (\|u\|_X^q + \|v\|_Y^q)^{\frac{1}{q}}$, for arbitrary $z = [u,v] \in E$. Let $E^* = X^* \times Y^*$ denote the dual space of E. For arbitrary $x = [x_1, x_2] \in E$, define the map $j_q^E : E \to E^*$ by*

$$j_q^E(x) = j_q^E[x_1, x_2] := [j_q^X(x_1), j_q^Y(x_2)],$$

so that for arbitrary $z_1 = [u_1, v_1], z_2 = [u_2, v_2]$ in E, the duality pairing $\langle \cdot, \cdot \rangle$ is given by

$$\langle z_1, j_q^E z_2 \rangle := \langle u_1, j_q^X(u_2) \rangle + \langle v_1, j_q^Y(v_2) \rangle.$$

Then,

- *$(a.)$ E is uniformly smooth and uniformly convex,*
- *$(b.)$ j_q^E is single-valued duality mapping on E.*

Lemma 4.6. *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^* and let $X = E \times E^*$. Let $F : E \to E^*$ and $K : E^* \to E$ be Lipschitz strongly monotone maps with Lipschitz constants M_1 and M_2, respectively, and strongly monotonicity constants η_1 and η_2, respectively. Let $A : X \to X^*$ be defined by $A([u,v]) = [Fu - v, Kv + u]$. Then,*

- *(1) A is η-stongly monotone, where $\eta := \min\{\eta_1, \eta_2\}$;*
- *(2) A is M^*-Lipschitzian, where $M^* = \sqrt{M^2 + 2M + 1}$ and $M = \max\{M_1, M_2\}$.*

Proof. Letting $[u_1, v_1], [u_2, v_2] \in X$, one has

$$\begin{aligned}
& \Big\langle A([u_1,v_1]) - A([u_2,v_2]), [u_1,v_1] - [u_2,v_2] \Big\rangle \\
= \ & \Big\langle \big[F(u_1) - F(u_2) + v_2 - v_1, K(v_1) - K(v_2) + u_1 - u_2\big], \big[u_1 - u_2, v_1 - v_2\big] \Big\rangle \\
= \ & \langle F(u_1) - F(u_2), u_1 - u_2 \rangle + \langle K(v_1) - K(v_2), v_1 - v_2 \rangle \\
\geq \ & \eta(\|u_1 - u_2\|_E^2 + \|v_1 - v_2\|_{E^*}^2) = \eta \Big\| [u_1,v_1] - [u_2,v_2] \Big\|_X^2,
\end{aligned}$$

Also,

$$\begin{aligned}
& \|A([u_1,v_1]) - A([u_2,v_2])\|_X^2 \\
= & \big\| \big[F(u_1) - F(u_2) + v_2 - v_1, K(v_1) - K(v_2) + u_1 - u_2\big] \big\|_X^2 \\
= & \|F(u_1) - F(u_2) + v_2 - v_1\|_{E^*}^2 + \|K(v_1) - K(v_2) + u_1 - u_2\|_E^2 \\
\leq & \big(M_1\|u_1 - u_2\|_E + \|v_1 - v_2\|_{E^*}\big)^2 + \big(M_2\|v_1 - v_2\|_{E^*} + \|u_1 - u_2\|_E\big)^2 \\
\leq & (M_1^2 + M_1 + M_2 + 1)\|u_1 - u_2\|_E^2 + (M_2^2 + M_2 + M_1 + 1)\|v_1 - v_2\|_{E^*}^2 \\
\leq & (M^2 + 2M + 1)\Big(\|u_1 - u_2\|_E^2 + \|v_1 - v_2\|_{E^*}^2\Big) \\
= & (M^2 + 2M + 1)\big\|[u_1,v_1] - [u_2,v_2]\big\|_X^2.
\end{aligned}$$

□

Remark 4.2. For A defined in Lemma 4.6, $[u^*,v^*]$ is a zero of A if and only if u^* solves (4.11), where $v^* = Fu^*$.

We now prove the following theorem.

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^* such that J^{-1} is Lipschitz. Let $F : E \to E^*$ and $K : E^* \to E$ be as in Lemma 4.6 such that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^* respectively, defined from arbitrary points $u_1 \in E$ and $v_1 \in E^*$ by*

$$u_{n+1} = J^{-1}\Big(Ju_n - \lambda(Fu_n - v_n)\Big),\ n \geq 1; \qquad v_{n+1} = J_*^{-1}\Big(J_*u_n - \lambda(Kv_n + u_n)\Big),\ n \geq 1, \tag{4.12}$$

converge strongly to u^* and v^*, respectively, where u^* is the unique solution of the equation $u + KFu = 0$ and $v^* = Fu^*$.

Proof. Set $X = E \times E^*$ and define $A : X \to X^*$ by $A([u,v]) = [Fu - v, Kv + u]$. Then, by lemma 4.6, A is η-strongly monotone and M-Lipschitzian, where M and η are defined in Lemma 4.6. Also, equation (4.12) follows easily from (3.4). From Theorem 3.1 and Remark 4.2, we get the desired conclusion immediately. □

Remark 4.3. (see e.g., Alber and Ryazantseva, [3]; page 36) The analytical representations of duality maps are known in l^p, $L^p(G)$ and Sobolev spaces $W_m^p(G)$, $p \in (1,\infty)$, $p^{-1} + q^{-1} = 1$.

Remark 4.4. Corollaries 3.3 and 3.5 are the main results of the paper of Chidume, Bello and Usman[5].

5. Conclusion

In this paper, a Krasnoselskii-type algorithm that approximates the unique zero of a Hölder continuous and strongly monotone map is constructed. Strong convergence of the sequence generated by this algorithm is established in strictly convex, reflexive and smooth real Banach spaces. Results obtained in this paper are also applied to approximate a minimizer of some convex functionals and a solution of a Hammerstein integral equation.

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