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SOME APPLICATIONS OF HIGHER-ORDER DERIVATIVES INVOLVING CERTAIN SUBCLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS

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Abstract. In this paper, we introduce and study a new class of analytic and multivalent (p-valent) functions involving higher-order derivatives. For this multivalent function class, we derive several interesting properties including coefficient inequalities, distortion theorems and extreme points. Several applications involving an integral operator are also considered. Finally, we obtain some results for the modified Hadamard product of functions belonging to the multivalent function class introduced in this paper. Relevant connections of the results presented in this paper with those in a number of other related works on this subject are also pointed out.
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1. Introduction and Definitions

Let $\mathscr{A}(p)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p} \qquad (p\in\mathbb{N}=\{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk given by

$$\mathbb{E}=\{z:z\in\mathbb{C} \text{ and } |z|<1\}.$$

In particular, we write

$$\mathscr{A}(1)=\mathscr{A}.$$

Furthermore, by $\mathscr{S}\subset\mathscr{A}$, we denote the class of all functions which are univalent in $\mathbb{E}$.

The familiar class of p-valently starlike functions in $\mathbb{E}$ will be denoted by $\mathscr{S}^*(p)$ which consists of functions $f\in\mathscr{A}(p)$ that satisfy the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0 \qquad (\forall\, z\in\mathbb{E}).$$

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It is easily seen that

$$\mathscr{S}^*(1) = \mathscr{S}^*,$$

where $\mathscr{S}^*$ is the well-known class of starlike functions in $\mathbb{E}$.

The class of p-valently convex functions in the open unit disk $\mathbb{E}$ will be denoted by $\mathscr{C}(p)$ which consists of functions $f \in \mathscr{A}(p)$ that satisfy the following inequality:

$$\Re\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{E}).$$

We observe that

$$\mathscr{C}(1) = \mathscr{C},$$

where $\mathscr{C}$ is the well-known class of convex functions in $\mathbb{E}$.

Next, for a function $f \in \mathscr{A}\,(p)$ given by (1.1) and another function $g \in \mathscr{A}\,(p)$ given by

$$g(z) = z^p + \sum_{n=2}^{\infty} b_{n+p} z^{n+p} \qquad (\forall\, z \in \mathbb{E}),$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{n=2}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

Moreover, the subclass of $\mathscr{A}$ consisting of all analytic functions which have a positive real part in $\mathbb{E}$ is denoted by $\mathscr{P}$. An analytic description of a function $h(z)$ in the class $\mathscr{P}$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad (\forall\, z \in \mathbb{E}).$$

Recently, Kanas and Wiśniowska (see [3] and [4]) defined the conic domain Ω_k $(k \geqq 0)$ as follows (see also [2]):

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\} \tag{1.2}$$

and, subjected to this conic domain, they also introduced and studied the corresponding classes k-$\mathscr{ST}$ and k-$\mathscr{UCV}$ of k-uniformly starlike functions and k-uniformly convex functions (see Definition 1.1 and Definition 1.2 below). We note that, for fixed k, Ω_k represents the conic region bounded successively by the imaginary axis $(k = 0)$, by a parabola $(k = 1)$, by the right branch of a hyperbola $(0 < k < 1)$, and by an ellipse $(k > 1)$. For these conic regions, the following functions play the role of extremal functions

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots & (k = 0), \\[2ex] 1 + \dfrac{2}{\pi^2}\left[\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2 & (k = 1), \\[2ex] 1 + \dfrac{2}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\arctan\left(h\sqrt{z}\right)\right] & (0 \leqq k < 1), \\[2ex] 1 + \dfrac{1}{k^2-1}\left[1 + \sin\left(\dfrac{\pi}{2K(\kappa)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{\kappa}}} \dfrac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}\right)\right] & (k > 1), \end{cases} \tag{1.3}$$

where

$$u(z)=\frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa}z}\qquad(\forall\, z\in\mathbb{E})$$

and $\kappa\in(0,1)$ is chosen such that

$$k=\cosh\left(\frac{\pi K'(\kappa)}{4K(\kappa)}\right).$$

Here $K(\kappa)$ is Legendre's complete elliptic integral of the first kind and

$$K'(\kappa)=K(\sqrt{1-\kappa^2}),$$

that is, $K'(\kappa)$ is the complementary integral of $K(\kappa)$ (see, for example, [18, p. 326, Eq. 9.4 (209)]).

The classes k-$\mathscr{ST}$ and k-$\mathscr{UCV}$ are defined as follows (see, for details, [2, 3, 4, 11, 13, 19]).

Definition 1.1. A function $f\in\mathscr{A}$ is said to be in the class k-$\mathscr{ST}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)\geqq k\left|\frac{zf'(z)}{f(z)}-1\right|\qquad(\forall\, z\in\mathbb{E};\ k\geqq 0).$$

Definition 1.2. A function $f\in\mathscr{A}$ is said to be in the class k-$\mathscr{UCV}$ if and only if

$$\Re\left(\frac{(zf'(z))'}{f'(z)}\right)\geqq k\left|\frac{(zf'(z))'}{f'(z)}-1\right|\qquad(\forall\, z\in\mathbb{E};\ k\geqq 0).$$

Definition 1.3. A function $f\in\mathscr{A}(p)$ is said to be in the class $\mathscr{UST}(p,\alpha,\beta)$ of p-valent β-uniformly starlike functions of order α if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}-\alpha\right)\geqq \beta\left|\frac{zf'(z)}{f(z)}-p\right|\qquad(\forall\, z\in\mathbb{E};\ -p\leqq\alpha<p;\ \beta\geqq 0).\tag{1.4}$$

Definition 1.4. A function $f\in\mathscr{A}(p)$ is said to be in the class $\mathscr{UCV}(p,\alpha,\beta)$ of p-valent β-uniformly convex functions of order α if and only if

$$\Re\left(\frac{(zf'(z))'}{f'(z)}-\alpha\right)\geqq \beta\left|\frac{(zf'(z))'}{f'(z)}-p\right|\qquad(\forall\, z\in\mathbb{E};\ -p\leqq\alpha<p;\ \beta\geqq 0).\tag{1.5}$$

The above-defined function classes $\mathscr{UST}(p,\alpha,\beta)$ and $\mathscr{UCV}(p,\alpha,\beta)$ were introduced recently by Khairnar and More [5]. Various analogous classes of analytic and multivalent functions were defined and studied by many other authors (see, for example, [1, 6, 7, 9]).

We notice from inequalities (1.4) and (1.5) that

$$f(z)\in\mathscr{UCV}(p,\alpha,\beta)\iff\frac{zf'(z)}{p}\in\mathscr{UST}(p,\alpha,\beta).$$

Now, for each $f(z)\in\mathscr{A}(p)$, it is easily seen upon differentiating both sides of (1.1) q times with respect to z that

$$f^{(q)}(z)=\delta(p,q)\,z^{p-q}+\sum_{n=p+1}^{\infty}\delta(p,q)\,a_n z^{n-p}\qquad(q\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\ p>q),\tag{1.6}$$

where, and in what follows, $\delta(p,q)$ denotes the q permutations of p objects $(p\geqq q\geqq 0)$, that is,

$$\delta(p,q)=\frac{p!}{(p-q)!}=\begin{cases}p(p-1)\cdots(p-q+1) & (q\neq 0),\\[2ex] 1 & (q=0),\end{cases}$$

which may also be identified with the notation $\{p\}_q$ for the *descending* factorial.

In recent years, several interesting subclasses of analytic functions were introduced and investigated from different viewpoints (see, for example, [2, 10, 14, 15, 16, 20, 21]). Motivated and inspired by these recent as well as the ongoing researches and by the above-mentioned works, we here introduce and investigate a new subclass of analytic and p-valent functions as follows.

Definition 1.5. Let

$$-\delta(p-q,m) \leqq \alpha < -\delta(p-q,m), \ \beta,k,\gamma,\xi \geqq 0 \ \text{ and } \ p > q+m \quad (p \in \mathbb{N};\ q,m \in \mathbb{N}_0).$$

Also let $f \in \mathscr{A}(p)$. We then say that $f \in \mathscr{U}\mathscr{S}_m(p,q,\alpha,k,\xi,\beta)$ if and only if

$$\Re\big(\mathscr{H}(p,q,\alpha,k,\xi,\beta)-\alpha\big) \geqq k\left|\mathscr{H}(p,q,\alpha,k,\xi,\beta)-\delta(p-q,m)\right| \qquad (\forall\, z \in \mathbb{E}),$$

where

$$\mathscr{H}(p,q,\alpha,k,\xi,\beta) = \frac{1-\xi}{1-\beta}\left(\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)}-\beta\right)+\frac{\xi}{1-\gamma}\left(1+\frac{z^m f^{(q+1+m)}(z)}{f^{(q+1)}(z)}-\gamma\right). \tag{1.7}$$

We also denote by $\mathscr{T}(p)$ the subclass of $\mathscr{A}(p)$ consisting of functions of the following form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \qquad \big(a_k > 0;\ p \in \mathbb{N}\big). \tag{1.8}$$

Further, we define the class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$ as follows:

$$\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta) = \mathscr{U}\mathscr{S}_m(p,q,\alpha,k,\xi,\beta) \cap \mathscr{T}(p). \tag{1.9}$$

Remark 1.1. If, in Definition 1.5, we put $\xi = 0 = \beta$, we are led to the well-known function class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k)$, which was introduced and studied recently by Srivastava *et al.* [17].

In this paper, we obtain several properties (including the coefficient inequalities, distortion theorems and extreme points) of the general class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$. We also consider some applications involving an integral operator. Finally, we obtain some results for the *modified* Hadamard product of functions in the class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$.

2. Coefficient Estimates

Unless otherwise mentioned, we assume throughout this paper that

$$-\delta(p-q,m) \leqq \alpha < -\delta(p-q,m), \ \beta,k,\gamma,\xi \geqq 0 \text{ and } p > q+m \quad (p \in \mathbb{N};\ q,m \in \mathbb{N}_0),$$

$$\chi = (1-\xi)\delta(n-q,m)-(1-\beta)\delta(p-q,m) \tag{2.1}$$

and

$$\Lambda = (1-\beta)\left[\delta(p-q,m)-\alpha\right]. \tag{2.2}$$

Our first result (Theorem 2.1 below) provides the coefficient inequalities for functions in the class $\mathscr{U}\mathscr{S}_m(p,q,\alpha,k,\xi,\beta)$.

Theorem 2.1. *A function f of the form (1.1) is in class $\mathscr{U}\mathscr{S}_m(p,q,\alpha,k,\xi,\beta)$ if*

$$\sum_{n=p+1}^{\infty}\left[(1+k)\chi+\Lambda\right]\delta(n,q)a_n \leqq \Lambda\delta(p,q), \tag{2.3}$$

where χ and Λ are given by (2.1) and (2.2), respectively.

Proof. It is easy to show that

$$\Re\big(\mathscr{H}(p,q,\alpha,k,\xi,\beta)-\alpha\big) \geqq k\left|\mathscr{H}(p,q,\alpha,k,\xi,\beta)-\delta(p-q,m)\right| \qquad (\forall\, z\in\mathbb{E}), \tag{2.4}$$

where

$$\mathscr{H}(p,q,\alpha,k,\xi,\beta)=\frac{1-\xi}{1-\beta}\left(\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)}-\beta\right)+\frac{\xi}{1-\gamma}\left(1+\frac{z^m f^{(q+1+m)}(z)}{f^{(q+1)}(z)}-\gamma\right).$$

The condition (2.4) implies (2.3) asserted by Theorem 2.1. □

Theorem 2.2. *A necessary and sufficient condition for a function f of the form (1.8) to be in the class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$ is that*

$$\sum_{n=p+1}^{\infty}\left[(1+k)\chi+\Lambda\right]\delta(n,q)a_n \leqq \Lambda\delta(p,q), \tag{2.5}$$

where χ and Λ are given by (2.1) and (2.2), respectively.

Proof. In view of Theorem 2.1, we need only to prove the necessity part of Theorem 2.2. If $f(z)\in\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$ and z is a real number, then

$$\mathscr{H}(p,q,\alpha,k,\xi,\beta)-\alpha \geqq k\left|\mathscr{H}(p,q,\alpha,k,\xi,\beta)-\delta(p-q,m)\right|,$$

where $\mathscr{H}(p,q,\alpha,k,\xi,\beta)$ is given by (1.7).

Now, after some straightforward simplification and upon letting $z\to 1-$ along the real axis, we get the desired result as asserted by Theorem 2.2. □

Remark 2.1. In its special case when $\xi=0=\beta$, Theorem 2.2 would yield the following known result due to Srivastava *et al.* [17].

Corollary 2.1. (see [17]) *A necessary and sufficient condition for a function f of the form (1.8) to be in the class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k)$ is that*

$$\sum_{n=p+1}^{\infty}\big[(1+k)\left[\delta(n-q,m)-\delta(p-q,m)\right]+\left[\delta(p-q,m)-\alpha\right]\big]\delta(n,q)a_n \leqq \left[\delta(p-q,m)-\alpha\right]\delta(p,q).$$

3. Distortion Theorems

Theorem 3.1. *Let f defined by (1.8) be in the class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$. Then, for $|z|=r<1$, it is asserted that*

$$|f(z)| \geqq r^p-\frac{\Lambda\delta(p,q)}{\left[(1+k)\chi+\Lambda\right]\delta(n,q)}r^{p+1} \tag{3.1}$$

and

$$|f(z)| \leqq r^p+\frac{\Lambda\delta(p,q)}{\left[(1+k)\chi+\Lambda\right]\delta(n,q)}r^{p+1}, \tag{3.2}$$

where χ and Λ are given by (2.1) and (2.2), respectively. The equalities in (3.1) and (3.2) are attained for f given by

$$f(z)=z^p-\frac{\Lambda\delta(p,q)}{\left[(1+k)\chi+\Lambda\right]\delta(n,q)}z^{p+1} \tag{3.3}$$

at $z=r$ and $z=r\mathrm{e}^{\mathrm{i}(2s+1)\pi}$ $(s\in\mathbb{Z})$.

Proof. For $n \geqq p+1$, we have

$$[(1+k)\chi+\Lambda]\delta(p+1,q) \leqq [(1+k)\chi+\Lambda]\delta(n,q).$$

Now, using the hypothesis of Theorem 2.2, we find that

$$\sum_{n=p+1}^{\infty} a_n \leqq \frac{\Lambda\delta(p,q)}{[(1+k)\chi+\Lambda]\delta(n,q)}.$$

By making use of the form (1.8) of f, the proof of Theorem 3.1 is completed. □

Theorem 3.2. *Let the function f defined by (1.8) be in the class $\mathcal{U}\mathcal{M}_m(p,q,\alpha,k,\xi,\beta)$. Then, for $|z|=r<1$, it is asserted that*

$$\left|f'(z)\right| \geqq pr^{p-1} - \frac{(p+1)\Lambda\delta(p,q)}{[(1+k)\chi+\Lambda]\delta(n,q)}r^p \tag{3.4}$$

and

$$\left|f'(z)\right| \leqq pr^{p-1} - \frac{(p+1)\Lambda\delta(p,q)}{[(1+k)\chi+\Lambda]\delta(n,q)}r^p, \tag{3.5}$$

where χ and Λ are given by (2.1) and (2.2), respectively. The result is sharp for the function $f(z)$ given by (3.3).

Proof. By using similar techniques as in our demonstration of Theorem 3.1, we have

$$\sum_{n=p+1}^{\infty} na_n \leqq \frac{(p+1)\Lambda\delta(p,q)}{[(1+k)\chi+\Lambda]\delta(n,q)},$$

which leads to the completion of the proof of Theorem 3.2. □

Remark 3.1. First, if we put $\xi=0=\beta$ in Theorem 3.1 and Theorem 3.2 above, we obtain the results which are precisely the same to those proved earlier by Srivastava *et al.* [17]. Second, by putting $\xi=k=\beta=0$, we find those results which were obtained by Liu and Liu [8].

4. Convex Linear Combinations

Theorem 4.1. *Let $\mu_\nu \geqq 0$ $(\nu=1,\cdots,l)$ and*

$$\sum_{\nu=1}^{l}\mu_\nu \leqq 1.$$

If the functions $f_\nu(z)$ $(\nu=1,\cdots,l)$ defined by

$$f_\nu(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,\nu}z^n \qquad \left(a_{n,\nu} \geqq 0;\ \nu=1,\cdots,l\right) \tag{4.1}$$

are in the class $\mathcal{U}\mathcal{M}_m(p,q,\alpha,k,\xi,\beta)$ for every $\nu=1,\cdots,l$, then the function $f(z)$ given by

$$f(z) = z^p - \sum_{n=p+1}^{\infty}\left(\sum_{\nu=1}^{l}\mu_\nu a_{n,\nu}\right)z^n \tag{4.2}$$

is also in the class $\mathcal{U}\mathcal{M}_m(p,q,\alpha,k,\xi,\beta)$.

Proof. From Theorem 2.2, we can find the desired conclusion easily. □

Theorem 4.2. *Let*

$$f_p(z) = z^p.$$

Also, for $n \geqq p+1$, *suppose that*

$$f_n(z) = z^p - \frac{\Lambda\delta(p,q)}{[(1+k)\chi+\Lambda]\delta(n,q)}\, z^n. \tag{4.3}$$

Then the function $f(z)$ *is in the class* $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$ *if and only if it can be expressed in the following form*:

$$f(z) = \sum_{n=p}^{\infty} \mu_n f_n(z), \tag{4.4}$$

where

$$\mu_n \geqq 0 \quad (n \geqq p) \qquad \text{and} \qquad \sum_{n=p}^{\infty} \mu_n = 1,$$

and χ *and* Λ *are given by* (2.1) *and* (2.2), *respectively.*

Proof. From Theorem 2.2, we can obtain the sufficiency part of Theorem 4.2 easily. For the necessity part, we can easily see that f can be expressed in the form (4.4), if we set

$$\mu_n = \frac{[(1+k)\chi+\Lambda]\delta(n,q)\,a_n}{\Lambda\delta(p,q)}$$

for $n \geqq p+1$ and

$$\mu_p = 1 - \sum_{n=p+1}^{\infty} \mu_n$$

such that $\mu_p \geqq 0$. □

Remark 4.1. If we set $\xi = 0 = \beta$ in Theorem 4.1 and Theorem 4.2, we are led to the results similar to those given by Srivastava *et al.* [17].

5. Integral Operators

In view of Theorem 2.2, we see that the following function.

$$z^p - \sum_{n=p+1}^{\infty} e_n z^n$$

is in the class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$ as long as $0 < e_n < a_n$ for all $n \geqq p+1$, where a_n is the coefficient of corresponding function in the class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$. We are thus led to Theorem 5.1 below.

Theorem 5.1. *Let the function* f *defined by* (1.8) *be in the class* $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$. *Also let* c *be a real number such that* $c > -p$. *Then the function* $\mathfrak{F}(z)$ *defined by*

$$\mathfrak{F}(z) = \frac{c+p}{z}\int_0^z t^{c-1} f(t)\mathrm{d}t \qquad (c > -p) \tag{5.1}$$

also belongs to the class $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$.

Proof. From the representation (5.1) of $\mathfrak{F}(z)$, it follows that

$$\mathfrak{F}(z) = z^p - \sum_{n=p+1}^{\infty} e_n z^n,$$

where

$$e_n = \left(\frac{c+p}{c+n}\right) a_n \leqq a_n \qquad (n \geqq p+1).$$

This completes the proof of Theorem 5.1. □

Remark 5.1. The converse of Theorem 5.1 is not true. This observation leads to the following result involving the radius of p-valence.

Theorem 5.2. *Let the function $F(z)$ given by*

$$F(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \qquad \left(a_n \geqq 0;\ \forall\, z \in \mathbb{E}\right)$$

be in the class $\mathcal{UM}_m(p,q,\alpha,k,\xi,\beta)$. Also let c be a real number such that $c > -p$. Then the function f given by (5.1) is p-valent in $|z| < r_p^*$, where

$$r_p^* = \inf_{n \geqq p+1} \left[\frac{[(1+k)\chi + \Lambda]\delta(n,q)}{\Lambda\delta(p,q)} \left(\frac{p(c+p)}{n(c+n)}\right)\right]^{\frac{1}{n-p}}, \tag{5.2}$$

and χ and Λ are given by (2.1) and (2.2), respectively. The result is sharp.

Proof. From the representation (5.1) of the function $F(z)$, we have

$$f(z) = \frac{z^{1-c}\,|z^c\,F(z)|'}{c+p} = z^p - \sum_{n=p+1}^{\infty} \frac{n+c}{c+p} a_n z^n.$$

In order to get the result asserted by Theorem 5.2, it is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leqq p \qquad \text{for} \qquad |z| < r_p^*,$$

where r_p^* is given by (5.2). By applying Theorem 2.2, we find that the required inequality in (5.2) is true if

$$|z| \leqq \left(\frac{[(1+k)\chi + \Lambda]\,\delta(n,q)}{\Lambda\delta(p,q)} \left(\frac{p(c+p)}{n(c+n)}\right)\right)^{\frac{1}{n-p}}. \tag{5.3}$$

The assertion of Theorem 5.2 is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(c+n)\Lambda\delta(p,q)}{(c+p)\,[(1+k)\chi + \Lambda]\delta(n,q)} z^{p+1}$$

where Λ is given by (2.2). □

Remark 5.2. In its special case when $\xi = 0 = \beta$, Theorem 5.2 reduces to a result which was proved earlier by Srivastava *et al.* [17].

6. Modified Hadamard Products

Let the functions $f_j(z)$ $(j=1,2)$ be defined by (4.1). Then the *modified* Hadamard product of $f_1(z)$ and $f_2(z)$ is defined, in this section, by

$$(f_1 \bullet f_2)(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,1}a_{n,2}z^n = (f_2 \bullet f_1)(z).$$

Theorem 6.1. *Let the functions $f_j(z)$ $(j=1,2)$ defined by (4.1) be in the class $\mathcal{UM}_m(p,q,\alpha,k,\xi,\beta)$. Then*

$$(f_1 \bullet f_2) \in \mathcal{UM}_m(p,q,\eta,k,\xi,\beta),$$

where

$$\eta = \delta(p-q,m) - \frac{\Lambda^2(1+k)\chi\delta(p,q)}{[(1+k)\chi+\Lambda]^2\delta(n,q) - \Lambda^2\delta(p,q)}, \tag{6.1}$$

and χ and Λ are given by (2.1) and (2.2), respectively. The result is sharp when

$$f_1(z) = f_2(z) = f(z),$$

where $f(z)$ is given by

$$f(z) = z^p - \frac{\Lambda\delta(p,q)}{[(1+k)\chi+\Lambda]\delta(n,q)}z^{p+1}. \tag{6.2}$$

Proof. Employing the technique used earlier by Schlid and Silverman [12], we need to find the largest η such that

$$\sum_{n=p+1}^{\infty} \frac{[(1+k)\chi+\Lambda]\delta(n,q)}{(1-\beta)[\delta(p-q,m)-\eta]\delta(p,q)}a_{n,1}a_{n,2} \leqq 1. \tag{6.3}$$

By using (2.3) for the function in the class $\mathcal{UM}_m(p,q,\eta,k,\xi,\beta)$ and by applying the familiar Cauchy Schwarz inequality, it is suffices to show that

$$\eta = \delta(p-q,m) - \frac{\Lambda^2(1+k)\chi\delta(p,q)}{[(1+k)\chi+\Lambda]^2\delta(n,q) - \Lambda^2\delta(p,q)}. \tag{6.4}$$

Now, if we define the function $X(n)$ by

$$X(n) = \delta(p-q,m) - \frac{\Lambda^2(1+k)\chi\delta(p,q)}{[(1+k)\chi+\Lambda]^2\delta(n,q) - \Lambda^2\delta(p,q)}, \tag{6.5}$$

we see that $X(n)$ is an increasing function of n $(n > p+1)$. This completes the proof of Theorem 6.1. □

By using the arguments similar to those used in the proof of Theorem 6.1, we can prove Theorem 6.2 below.

Theorem 6.2. *Let the function $f_1(z)$ defined by (4.1) be in the class $\mathcal{UM}_m(p,q,\alpha,k,\xi,\beta)$. Suppose also that the function $f_2(z)$ defined by (4.1) is in the class $\mathcal{UM}_m(p,q,\phi,k,\xi,\beta)$. Then*

$$(f_1 \bullet f_2) \in \mathcal{UM}_m(p,q,\varsigma,k,\xi,\beta),$$

where

$$\varsigma = \delta(p-q,m) - \frac{\Lambda^2[\delta(p-q,m)-\phi](1+k)\chi\delta(p,q)}{[(1+k)\chi+\Lambda]\{(1+k)\chi+(1-\beta)[\delta(p-q,m)-\phi]\}\delta(n,q)-\Omega}, \tag{6.6}$$

with

$$\Omega=(1-\beta)^2\left[\delta(p-q,m)-\alpha\right]\left[\delta(p-q,m)-\phi\right]\delta(p,q),$$

and χ *is given by* (2.1). *The result is sharp for the functions* $f_j(z)$ $(j=1,2)$ *given by*

$$f_1(z)=z^p-\frac{\Lambda\delta(p,q)}{[(1+k)\chi+\Lambda]\delta(n,q)}z^{p+1} \tag{6.7}$$

and

$$f_2(z)=z^p-\frac{(1-\beta)\left[\delta(p-q,m)-\phi\right]\delta(p,q)}{\left[(1+k)\chi+(1-\beta)\left[\delta(p-q,m)-\phi\right]\right]\delta(n,q)}z^{p+1}. \tag{6.8}$$

Theorem 6.3. *Let the function* $f_\nu(z)$ *defined by* (4.1) *be in the class* $\mathscr{U}\mathscr{M}_m(p,q,\alpha,k,\xi,\beta)$. *Then the function* $h(z)$ *given by*

$$h(z)=z^p-\sum_{n=p+1}^{\infty}(a_{n,1+}^2a_{n,2}^2)z^n \tag{6.9}$$

belongs to the class $\mathscr{U}\mathscr{M}_m(p,q,\tau,k,\xi,\beta)$, *where*

$$\tau=\delta(p-q,m)-\frac{2\Lambda^2(1+k)\chi\delta(p,q)}{[(1+k)\chi+\Lambda]^2\delta(n,q)-2\Lambda^2\delta(p,q)}. \tag{6.10}$$

The result is sharp when

$$f_1(z)=f_2(z)=f(z),$$

where $f(z)$, Λ *and* χ *are given by* (6.2), (2.2) *and* (2.1), *respectively.*

Proof. If we combine the assertion of Theorem 2.2 for both functions $f_1(z)$ and $f_2(z)$, we find that

$$\sum_{n=p+1}^{\infty}\frac{1}{2}\left(\frac{[(1+k)\chi+\Lambda]\delta(n,q)}{\Lambda\delta(p,q)}\right)^2(a_{n,1+}^2a_{n,2}^2)\leqq 1. \tag{6.11}$$

We need to find the largest $\tau=\tau(p,q,\alpha,k)$ such that

$$\frac{[(1+k)\chi+\Lambda]\delta(n,q)}{\Lambda\delta(p,q)}\leqq\frac{1}{2}\left(\frac{[(1+k)\chi+\Lambda]\delta(n,q)}{\Lambda\delta(p,q)}\right)^2. \tag{6.12}$$

Since $M(n)$ given by

$$M(n)=\delta(p-q,m)-\frac{2\Lambda^2(1+k)\chi\delta(p,q)}{[(1+k)\chi+\Lambda]^2\delta(n,q)-2\Lambda^2\delta(p,q)} \tag{6.13}$$

is an increasing function of n $(n\geqq p+1)$, we have

$$\tau\leqq M(p+1).$$

This completes the proof of Theorem 6.3. □

7. Concluding Remarks and Observations

The main object in this paper is to introduce and study a new class of analytic and p-valent functions involving higher-order derivatives. For this p-valent function class, we have derived such interesting properties as (for example) coefficient inequalities, distortion theorems and extreme points. We have considered several applications involving an integral operator. We have also obtained some results for the modified Hadamard product of functions belonging to the p-valent function class introduced in this paper. We have chosen to point out relevant connections of the results presented in this paper with those in a number of other related works on the subject of our present investigation.

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