STRONG CONVERGENCE FOR A MODIFIED FORWARD-BACKWARD SPLITTING METHOD IN BANACH SPACES

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Dedicated to Professor Wataru Takahashi on the occasion of his 75th birthday

Abstract. We propose a modified forward-backward splitting method and prove a new strong convergence theorem of solutions to a zero problem of the sum of a monotone operator and an inverse-strongly-monotone operator in a real 2-uniformly convex and uniformly smooth Banach space. Some new results for variational inequality problems and monotone inclusions are obtained.

Keywords. Sum of maximal monotone operators; Forward-backward splitting method; Variational inequality; Inverse strongly monotone operators; 2-uniformly convex Banach space.

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1. INTRODUCTION

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ be the dual of $E$. For $x \in E$ and $x^* \in E^*$, let $\langle x, x^* \rangle$ be the value of $x^*$ at $x$. Let $A \subset E \times E^*$ and $B \subset E \times E^*$ be maximal monotone operators such that $A + B$ is maximal monotone and $(A + B)^{-1}0 \neq \emptyset$. Finding an element of $(A + B)^{-1}0$ is so general that it concludes a number of important problems such as convex minimization problems, variational inequality problems, complementary problems, and others. In a real Hilbert space $H$, Passty [31] and Lions and Mercier [20] introduced the following forward-backward splitting method as one of the methods of finding an element of $(A + B)^{-1}0$:

$$x_1 = x \in D(B), \quad x_{n+1} = J_{\lambda_n}^A (x_n - \lambda_n w_n)$$

for every $n \in \mathbb{N}$, where $D(B) \subset H$ is the domain of $B$, $w_n \in Bx_n$, $\{\lambda_n\} \subset (0, \infty)$, and $J_{\lambda_n}^A$ is the resolvent of $A$. Later, the splitting method was widely studied by Gabay [12], Chen and Rockafellar [9], Moudafi and Théra [26] and Tseng [41].

Let $\alpha > 0$. A single valued operator $B : H \to H$ is said to be $\alpha$-inverse-strongly-monotone if

$$\langle x - y, Bx - By \rangle \geq \alpha \| Bx - By \|^2$$

for all $x, y \in H$; see [5, 11, 21, 45]. If $\alpha = 1$, $B$ is called a firmly nonexpansive mapping. Gabay [12] proved that the sequence $\{x_n\}$ generated by (1.1) converges weakly to some $z \in (A + B)^{-1}0$ when $B$ is

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\( \alpha \)-inverse-strongly-monotone and \( \lambda_n = \bar{\lambda} \) (constant) with \( 0 < \bar{\lambda} < 2\alpha \). Later, many researchers studied the weak convergence in a real Hilbert space; see \([4, 27, 28, 37]\) and references therein. Nakajo, Shimoji and Takahashi \([30]\) considered the following Halpern’s type iteration \([13]\):

\[
x_1 = x \in H, \quad x_{n+1} = y_n + (1 - \gamma_n)J_{\lambda_n}^A(x_n - \lambda_n Bx_n)
\]

for all \( n \in \mathbb{N}, \) where \( B \) is \( \alpha \)-inverse-strongly-monotone, \( \{\lambda_n\} \subset [a, 2\alpha] \) for some \( a \in (0, 2\alpha) \) with \( \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty \) and \( \{\gamma_n\} \subset [0, 1) \) such that \( \lim_{n \to \infty} \gamma_n = 0, \) \( \sum_{n=1}^{\infty} (1 - \gamma_n) = 0 \) and \( \sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty. \)

They proved \( \{x_n\} \) converges strongly to \( P_{(A+B)^{-1}0}x, \) where \( P_{(A+B)^{-1}0} \) is the metric projection of \( H \) onto \( (A + B)^{-1}0. \) Later, strong convergence by the viscosity approximation \([25]\) which extends that by Halpern’s type iteration was studied by many researchers \(([7, 10, 32, 36]\) and references therein) in a real Hilbert space. On the other hand, strong convergence by the hybrid method \([14]\) and shrinking projection method \([40]\) were researched by several authors \(([28, 29, 43]\) and references therein) in a real Hilbert space.

In this paper, we consider a new iteration scheme and study its strong convergence in a real Banach space. Let \( C \) be a nonempty closed convex subset of a real \( 2 \)-uniformly convex and uniformly smooth Banach space \( E. \) Let \( A \subset E \times E^* \) be a maximal monotone operator, and let \( B \) be an inverse-strongly-monotone operator of \( C \) into \( E^*, \) that is, there exists \( \alpha > 0 \) such that \( \langle x - y, Bx - By \rangle \geq \alpha \|Bx - By\|^2 \)
holds for every \( x, y \in C. \) Suppose that \( F = (A + B)^{-1}0 \neq \emptyset. \) We propose the following modified forward-backward splitting method:

\[
x_1 \in C, \quad x_{n+1} = \Pi_{E}J_{\lambda_n}^AJ^{-1}(\gamma_nJu + (1 - \gamma_n)Jx_n - \lambda_n Bx_n)
\]

for each \( n \in \mathbb{N}, \) where \( u \in E, \) \( \Pi_{E} \) is the generalized projection of \( E \) onto \( C, \) \( \{\lambda_n\} \subset (0, \infty), \) \( J_{\lambda_n}^A \) is the resolvent of \( A, J \) is the duality mapping of \( E, \) and \( \{\gamma_n\} \subset (0, 1] \) such that \( \gamma_n \to 0 \) and \( \sum_{n=1}^{\infty} \gamma_n = \infty. \) We prove that \( \{x_n\} \) converges strongly to \( \Pi_{F}u \) under some conditions on \( \{\lambda_n\}. \) Further, we obtain new results for variational inequality problems and monotone inclusions.

2. Preliminaries

Throughout this paper, we denote by \( \mathbb{N} \) and by \( \mathbb{R} \) the set of all positive integers and the set of all real numbers, respectively. We use \( x_n \to x \) to indicate that a sequence \( \{x_n\} \) converges weakly to \( x \) and \( x_n \to x \) will symbolize strong convergence. We define the modulus of convexity \( \delta_E \) of \( E \) as follows: \( \delta_E \) is a function of \([0, 2]\) into \([0, 1]\) such that

\[
\delta_E(\varepsilon) = \inf \{1 - \|x + y\|/2 : x, y \in E, \|x\| = 1, \|y\| = 1, \|x - y\| \geq \varepsilon\}
\]

for every \( \varepsilon \in [0, 2]. \) \( E \) is said to be uniformly convex if \( \delta_E(\varepsilon) > 0 \) for each \( \varepsilon > 0. \) For \( p > 1, \) we say \( E \) is \( p \)-uniformly convex if there exists a constant \( c > 0 \) such that \( \delta_E(\varepsilon) \geq c\varepsilon^p \) for every \( \varepsilon \in [0, 2]. \) It is obvious that a \( p \)-uniformly convex Banach space is uniformly convex. \( E \) is said to be strictly convex if \( \|x + y\|^2 < 2 \|x\|^2 + 2 \|y\|^2 \) for all \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( x \neq y. \) We know that a uniformly convex Banach space is strictly convex and reflexive. The duality mapping \( J : E \to 2^{E^*} \) of \( E \) is defined by

\[
J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}
\]
for every \( x \in E \). It is also known that if \( E \) is strictly convex and reflexive, then, the duality mapping \( J \) of \( E \) is bijective, and \( J^{-1} : E^* \to 2^E \) is the duality mapping of \( E^* \). \( E \) is said to be smooth if the limit
\[
\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \quad (2.1)
\]
exists for every \( x, y \in S(E) \), where \( S(E) = \{ x \in E : \|x\| = 1 \} \). \( E \) is said to be uniformly smooth if limit (2.1) is attained uniformly for \((x,y)\in S(E)\times S(E)\). It is known that \( E \) is uniformly smooth if and only if \( E^* \) is uniformly convex. We know that the duality mapping \( J \) of \( E \) is single-valued if and only if \( E \) is smooth. We also know that if \( E \) is uniformly smooth, then the duality mapping \( J \) of \( E \) is uniformly continuous on bounded subsets of \( E \); see [38, 39] for more details.

The following was proved by Xu [42]; see also [44].

**Theorem 2.1.** Let \( E \) be a smooth Banach space. Then, \( E \) is 2-uniformly convex if and only if there exists a constant \( c > 0 \) such that for each \( x, y \in E \),
\[
\|x + y\|^2 \geq \|x\|^2 + 2\langle y, Jx \rangle + c\|y\|^2 \quad \text{holds.}
\]

**Remark 2.1.** In a real Hilbert space, we can choose \( c = 1 \).

Let \( E \) be a smooth Banach space. The function \( \phi : E \times E \to \mathbb{R} \) is defined by
\[
\phi(y,x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2
\]
for every \( x, y \in E \). It is obvious that \((\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \) for each \( x, y \in E \) and \( \phi(z, x) + \phi(x, y) = \phi(z, y) + 2\langle x-z, Jx-Jy \rangle \) for all \( x, y, z \in E \). We also know that if \( E \) is strictly convex and smooth, then, for \( x, y \in E \), \( \phi(y,x) = 0 \) if and only if \( x = y \); see [23].

We have the following result by Theorem 2.1; see also [17].

**Lemma 2.1.** Let \( E \) be a 2-uniformly convex and smooth Banach space. Then, for each \( x, y \in E \), \( \phi(x,y) \geq c\|x-y\|^2 \) holds, where \( c \) is the constant in Theorem 2.1.

Let \( C \) be a nonempty closed convex subset of a strictly convex, reflexive and smooth Banach space \( E \) and let \( x \in E \). Then, there exists a unique element \( x_0 \in C \) such that
\[
\phi(x_0, x) = \inf_{y \in C} \phi(y, x).
\]
We denote \( x_0 \) by \( \Pi_Cx \) and call \( \Pi_C \) the generalized projection of \( E \) onto \( C \); see [1, 2, 16]. We have the following well-known result [1, 2, 16] for the generalized projection.

**Lemma 2.2.** Let \( C \) be a nonempty convex subset of a smooth Banach space \( E \), \( x \in E \) and \( x_0 \in C \). Then, \( \phi(x_0, x) = \inf_{y \in C} \phi(y, x) \) if and only if \( \langle x_0 - z, Jx - Jx_0 \rangle \geq 0 \) for every \( z \in C \), or equivalently, \( \phi(z, x) \geq \phi(x_0, x) + \phi(x_0, z) \) for all \( z \in C \).

An operator \( A \subset E \times E^* \) is said to be monotone if \( \langle x-y, x^*-y^* \rangle \geq 0 \) for every \((x,x^*),(y,y^*) \in A \). A monotone operator \( A \) is said to be maximal if the graph of \( A \) is not properly contained in the graph of any other monotone operator. We know that a monotone operator \( A \) is maximal if and only if for \((u,u^*) \in E \times E^* \), \( \langle x-u, x^*-u^* \rangle \geq 0 \) for every \((x,x^*) \in A \) implies \((u,u^*) \in A \). Rockafellar [35] proved the following result; see also [8].

**Theorem 2.2.** Let \( E \) be a strictly convex, reflexive and smooth Banach space and let \( A \subset E \times E^* \) be a monotone operator. Then, \( A \) is maximal if and only if \( R(J+rA) = E^* \), for all \( r > 0 \), where \( R(J+rA) \) is the range of \( J+rA \).
Let $E$ be a strictly convex, reflexive and smooth Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. By Theorem 2.2 and strict convexity of $E$, for any $x \in E$ and $r > 0$, there exists a unique element $x_r \in D(A)$ such that

$$J(x) \in J(x_r) + rAx_r,$$

where $D(A)$ is the domain of $A$. We define $J_r$ by $J_rx = x_r$ for every $x \in E$ and $r > 0$ and such $J_r$ is called the resolvent of $A$; see [6, 39] for more details.

Let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Then, it is known that the subdifferential $\partial f$ of $f$ defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle y-x, x^* \rangle, \forall y \in E\}$$

for $x \in E$ is a maximal monotone operator; see [33, 34].

A function $\tau : \mathbb{N} \to \mathbb{N}$ is said to be eventually increasing if $\lim_{n \to \infty} \tau(n) = \infty$ and $\tau(n) \leq \tau(n+1)$ for all $n \in \mathbb{N}$. The following was proved by Aoyama, Kimura and Kohsaka [3, Lemma 3.4]; see also [22, Lemma 3.1].

**Lemma 2.3.** Let $\{\xi_n\}$ be a sequence of nonnegative real numbers which is not convergent. Then there exist $n_0 \in \mathbb{N}$ and an eventually increasing function $\tau$ such that $\xi_{\tau(n)} \leq \xi_{\tau(n)+1}$ for all $n \in \mathbb{N}$ and $\xi_n \leq \xi_{\tau(n)+1}$ for every $n \geq n_0$.

3. MAIN RESULTS

We first prove the following important lemmas.

**Lemma 3.1.** Let $C$ be a nonempty closed convex subset of a strictly convex, reflexive and smooth Banach space $E$, $A \subset E \times E^*$ a maximal monotone operator and $B : C \to E^*$ such that $(A+B)^{-1}0 \neq 0$. Suppose that there exists a real number $\alpha > 0$ with $\langle x-z, Bx-Bz \rangle \geq \alpha\|Bx-Bz\|^2$ for all $x \in C$ and $z \in (A+B)^{-1}0$. Let $T_\lambda x = J_\lambda^A J^{-1}_\lambda (Jx - \lambda Bx)$ for $\lambda > 0$ and $x \in C$, where $J_\lambda^A$ is the resolvent of $A$. Then, the following hold:

(i) $F(T_\lambda) = (A+B)^{-1}0$ for all $\lambda > 0$, where $F(T_\lambda)$ is the set of all fixed points of $T_\lambda$;

(ii) if $E$ is 2-uniformly convex, $\phi(z, T_\lambda x) \leq \phi(z, x) - (c - \lambda \beta)\|x - T_\lambda x\|^2 - \lambda(2\alpha - 1/\beta)\|Bx-Bz\|^2$ holds for every $\lambda, \beta > 0, x \in C$ and $z \in (A+B)^{-1}0$, where $c$ is the constant in Theorem 2.1;

(iii) if $E$ is 2-uniformly convex, $(A+B)^{-1}0$ is closed and convex.

**Proof.** (i) Let $z \in (A+B)^{-1}0$. Then we have $-Bz \in Az$ and it follows that

$$Jz - \lambda Bz \in Jz + \lambda Az,$$

which is equivalent to $J_\lambda^A J^{-1}_\lambda (Jz - \lambda Bz) = z$. Thus we have $z \in F(T_\lambda)$. The implication of the opposite direction is also straightforward. Hence we have $F(T_\lambda) = (A+B)^{-1}0$.

(ii) Let $\lambda, \beta > 0, x \in C, z \in (A+B)^{-1}0$ and $y = T_\lambda x$. We have

$$\phi(z, y) = \phi(z, x) - \phi(y, x) + 2\langle y - z, Jy - Jx \rangle.$$

Since $z \in (A+B)^{-1}0$ and $y = J_\lambda^A J^{-1}_\lambda (Jx - \lambda Bx)$, we get

$$\langle y - z, \frac{1}{\lambda}(Jx - Jy) - Bx + Bz \rangle \geq 0,$$
which implies
\[ \langle y - z, Jx - Jy \rangle \geq \lambda \langle y - z, Bx - Bz \rangle. \]
So, we obtain
\[
\begin{align*}
\phi(z, y) &\leq \phi(z, x) - \phi(y, x) - 2\lambda \langle y - z, Bx - Bz \rangle \\
&= \phi(z, x) - \phi(y, x) - 2\lambda \langle y - x, Bx - Bz \rangle - 2\lambda \langle x - z, Bx - Bz \rangle \\
&\leq \phi(z, x) - \phi(y, x) - 2\lambda \langle y - x, Bx - Bz \rangle - 2\lambda \alpha \|Bx - Bz\|^2.
\end{align*}
\]
By Lemma 2.1,
\[
\phi(z, y) \leq \phi(z, x) - c\|y - x\|^2 + 2\lambda \|y - x\|\|Bx - Bz\| - 2\lambda \alpha \|Bx - Bz\|^2.
\]
Since \(\|y - x\|\|Bx - Bz\| \leq \frac{\beta}{2} \|y - x\|^2 + \frac{1}{2\beta} \|Bx - Bz\|^2\), we have
\[
\phi(z, y) \leq \phi(z, x) - (c - \lambda \beta) \|y - x\|^2 - \lambda \left(2\alpha - \frac{1}{\beta}\right) \|Bx - Bz\|^2.
\]

(iii) For \(\alpha > 0\), there exists \(\lambda_0 > 0\) with \(c/\lambda_0 > 1/(2\alpha)\), where \(c\) is the constant in Theorem 2.1. So, we can select \(\beta > 0\) such that \(c/\lambda_0 > \beta > 1/(2\alpha)\). By (ii), we get \(\phi(z, T_{\lambda_0}x) \leq \phi(z, x)\) for every \(x \in C\) and \(z \in (A + B)^{-1}0\). From (i), \(F(T_{\lambda_0}) = (A + B)^{-1}0\). By the result in [23, 24], we have \(F(T_{\lambda_0})\) is closed and convex. Indeed, let \(\{z_n\} \subset F(T_{\lambda_0})\) such that \(z_n \to z\). We have
\[
\phi(z_n, T_{\lambda_0}z) \leq \phi(z_n, z)
\]
for all \(n \in \mathbb{N}\), which implies
\[
\phi(z, T_{\lambda_0}z) \leq 0.
\]
So, we obtain \(z \in F(T_{\lambda_0})\), that is, \(F(T_{\lambda_0})\) is closed. Next, let \(z_1, z_2 \in F(T_{\lambda_0}), 0 \leq \gamma \leq 1\) and \(x = \gamma z_1 + (1 - \gamma)z_2\). It follows that
\[
\begin{align*}
\phi(x, T_{\lambda_0}x) &= \|x\|^2 - 2\langle x, J(T_{\lambda_0}x) \rangle + \|T_{\lambda_0}x\|^2 \\
&= \|x\|^2 - 2\langle \gamma z_1 + (1 - \gamma)z_2, J(T_{\lambda_0}x) \rangle + \|T_{\lambda_0}x\|^2 \\
&= \|x\|^2 + \gamma \phi(z_1, T_{\lambda_0}x) + (1 - \gamma)\phi(z_2, T_{\lambda_0}x) - \gamma\|z_1\|^2 - (1 - \gamma)\|z_2\|^2 \\
&\leq \|x\|^2 + \gamma \phi(z_1, x) + (1 - \gamma)\phi(z_2, x) - \gamma\|z_1\|^2 - (1 - \gamma)\|z_2\|^2 \\
&= 2\|x\|^2 - 2\langle \gamma z_1 + (1 - \gamma)z_2, Jx \rangle \\
&= 2\|x\|^2 - 2\|x\|^2 = 0.
\end{align*}
\]
Hence \(x = T_{\lambda_0}x\), which implies \(F(T_{\lambda_0})\) is convex. \(\square\)

**Lemma 3.2.** Let \(C\) be a nonempty closed convex subset of a strictly convex, reflexive and uniformly smooth Banach space \(E\). Let \(A \subseteq E \times E^*\) be a maximal monotone operator, \(\alpha > 0\) and \(B : C \to E^*\) an \(\alpha\)-inverse-strongly-monotone operator such that \(F = (A + B)^{-1}0 \neq \emptyset\). Let \(\{x_n\}\) be a bounded sequence in \(C\), \(u \in E\) and \(y_n = J_{\lambda_n}^A J_{\lambda_n}^{-1}(\gamma_n J + (1 - \gamma_n)Jx_n - \lambda_n Bx_n)\), where \(\{\lambda_n\} \subset (0, \infty)\) with \(\inf_{n \in \mathbb{N}} \lambda_n > 0\), \(J_{\lambda_n}^A\) is the resolvent of \(A\), \(J\) is the duality mapping of \(E\) and \(\{\gamma_n\} \subset (0, 1]\) such that \(\gamma_n \to 0\). If \(\|x_n - y_n\| \to 0\) and \(\|Bx_n - Bz\| \to 0\) for some \(z \in F\), then \(\omega_n(\{x_n\}) \subset F\), where \(\omega_n(\{x_n\})\) is the set of all weak cluster points of \(\{x_n\}\).
Proof. Let \((x, x^*) \in A\) and \(\{x_n\} \subset \{x\}\) such that \(x_n \rightharpoonup v\). Using
\[
\langle x_n - v, Bx_n - Bv \rangle \geq \alpha \|Bx_n - Bv\|^2
\]
and \(\|Bx_n - Bz\| \to 0\), we have \(\|Bx_n - Bv\| \to 0\). Since
\[
\frac{1}{\lambda_n} (Jx_n - Jy_n) - Bx_n - \frac{\gamma_n}{\lambda_n} (Jx_n - Ju) \in Ay_n,
\]
we have
\[
\langle x - y_n, x^* - Bx_n \rangle \geq \frac{1}{\lambda_n} \langle x - y_n, Jx_n - Jy_n \rangle - \frac{\gamma_n}{\lambda_n} \langle x - y_n, Jx_n - Ju \rangle
\]
which implies
\[
\langle x - y_n, x^* + Bx_n \rangle \geq \frac{1}{\lambda_n} \langle x - y_n, Jx_n - Jy_n \rangle - \frac{\gamma_n}{\lambda_n} \langle x - y_n, Jx_n - Ju \rangle - \frac{\gamma_n}{\lambda_n} \|x - y_n\| \|x_n - Jy_n\| - \frac{\gamma_n}{\lambda_n} \|x - y_n\| \|x_n - Jx_n - Ju\|
\]
for all \(i \in \mathbb{N}\). Since \(\|y_n - x_n\| \to 0\) and \(E\) is uniformly smooth, we have \(\|Jy_n - Jx_n\| \to 0\). By \(y_n \rightharpoonup v\), \(\gamma_n \to 0\) and \(\|Bx_n - Bv\| \to 0\), we obtain
\[
\langle x - v, x^* + Bv \rangle \geq 0.
\]
Since \(A\) is maximal monotone, we have \(-Bv \in Ax\), that is, \(v \in (A + B)^{-1}0 = F\). \(\square\)

Now, we prove our main result.

Theorem 3.1. Let \(C\) be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space \(E\). Let \(A\) be a maximal monotone operator in \(E \times E^*\), \(\alpha > 0\) and \(B\) an \(\alpha\)-inverse-strongly-monotone operator of \(C\) into \(E^*\) such that \(F = (A + B)^{-1}0 \neq \emptyset\). Let \(u \in E\) and \(\{x_n\}\) be a sequence generated by
\[
x_1 \in C, \quad x_{n+1} = \Pi_C F_{\lambda_n} J^{-1} (\gamma_n Ju + (1 - \gamma_n)Jx_n - \lambda_n Bx_n)
\]
for each \(n \in \mathbb{N}\), where \(\Pi_C\) is the generalized projection of \(E\) onto \(C\), \(\{\lambda_n\} \subset (0, \infty)\) and \(\{\gamma_n\} \subset (0, 1]\) such that \(\gamma_n \to 0\) and \(\sum_{n=1}^{\infty} \gamma_n = \infty\). Then, if \(0 < \inf_{n \in \mathbb{N}} \lambda_n \leq \sup_{n \in \mathbb{N}} \lambda_n < 2\alpha\), where \(c\) is the constant in Theorem 2.1, then \(\{x_n\}\) converges strongly to \(\Pi_F u\).

Proof. By Lemma 3.1 (iii), \(F\) is closed and convex. Thus \(\Pi_F u\) is well defined. Let \(z \in F\) and
\[
y_n = J_{\lambda_n}^A J^{-1} (\gamma_n Ju + (1 - \gamma_n)Jx_n - \lambda_n Bx_n).
\]
We have
\[
\Phi(z, y_n) = \Phi(z, x_n) - \Phi(y_n, x_n) + 2 \langle y_n - z, Jy_n - Jx_n \rangle.
\]
Since \(-Bz \in Az\) and
\[
\frac{1}{\lambda_n} \{Jx_n - Jy_n - \lambda_n Bx_n - \gamma_n (Jx_n - Ju)\} \in Ay_n,
\]
by the monotonicity of \(A\) we get
\[
\langle y_n - z, \frac{1}{\lambda_n} (Jx_n - Jy_n) - (Bx_n - Bz) - \frac{\gamma_n}{\lambda_n} (Jx_n - Ju) \rangle \geq 0,
\]
which implies
\[
\langle y_n - z, Jx_n - Jy_n \rangle \geq \lambda_n \langle y_n - z, Bx_n - Bz \rangle + \gamma_n \langle y_n - z, Jx_n - Ju \rangle
\]
for all \( n \in \mathbb{N} \). Thus we have
\[
\phi(z, y_n) \leq \phi(z, x_n) - \phi(y_n, x_n) - 2\lambda_n \langle y_n - z, Bx_n - Bz \rangle
- 2\gamma_n \langle y_n - z, Jx_n - Ju \rangle \tag{3.1}
\]
for each \( n \in \mathbb{N} \). Similarly in the calculation of Lemma 3.1 (ii), we get
\[
\phi(z, y_n) \leq \phi(z, x_n) - (c - \lambda_n\beta) \| x_n - y_n \|^2
- \lambda_n(2\alpha - 1/\beta) \| Bx_n - Bz \|^2 - 2\gamma_n \langle y_n - z, Jx_n - Ju \rangle \tag{3.2}
\]
for every \( \beta > 0 \) and \( n \in \mathbb{N} \). Using \( x_{n+1} = \Pi_{C} y_n \) and Lemma 2.2, we obtain
\[
\phi(x_{n+1}, y_n) + \phi(z, x_{n+1}) \leq \phi(z, y_n). \tag{3.3}
\]
Therefore we have
\[
\phi(z, x_{n+1}) \leq \phi(z, x_n) - (c - \lambda_n\beta) \| x_n - y_n \|^2
- \lambda_n(2\alpha - 1/\beta) \| Bx_n - Bz \|^2 - 2\gamma_n \langle y_n - z, Jx_n - Ju \rangle \tag{3.4}
\]
for each \( \beta > 0 \) and \( n \in \mathbb{N} \). Since
\[
0 < \inf_{n \in \mathbb{N}} \lambda_n \leq \sup_{n \in \mathbb{N}} \lambda_n < 2c\alpha,
\]
there exists \( \beta_0 > 0 \) such that \( \inf_{n \in \mathbb{N}} (c - \lambda_n\beta_0) > 0 \) and \( \inf_{n \in \mathbb{N}} \lambda_n(2\alpha - 1/\beta_0) > 0 \). Using
\[
2\langle y_n - z, Jx_n - Ju \rangle = \phi(y_n, u) + \phi(z, x_n) - \phi(y_n, x_n) - \phi(z, u)
\]
with (3.1), we have
\[
\phi(z, x_{n+1}) \leq \phi(z, x_n) - (1 - \gamma_n) \phi(y_n, x_n) - 2\lambda_n \langle y_n - z, Bx_n - Bz \rangle
- \gamma_n \{ \phi(y_n, u) + \phi(z, x_n) - \phi(z, u) \}. \tag{3.5}
\]
Again, since \( 0 < \inf_{n \in \mathbb{N}} \lambda_n \leq \sup_{n \in \mathbb{N}} \lambda_n < 2c\alpha \) and \( \gamma_n \to 0 \), there exist \( N \in \mathbb{N} \) and \( \delta_0 > 0 \) such that \( \inf_{n \geq N} \{(1 - \gamma_n)c - \lambda_n\delta_0\} > 0 \) and \( \inf_{n \geq N} \lambda_n(2\alpha - 1/\delta_0) > 0 \). As in the calculation of Lemma 3.1 (ii), we obtain
\[
\phi(z, x_{n+1}) \leq \phi(z, x_n) - \{(1 - \gamma_n)c - \lambda_n\delta_0\} \| y_n - x_n \|^2
- \lambda_n(2\alpha - 1/\delta_0) \| Bx_n - Bz \|^2
- \gamma_n \{ \phi(y_n, u) + \phi(z, x_n) - \phi(z, u) \} \tag{3.5}
\]
for all \( n \in \mathbb{N} \). We prove \( \{x_n\} \) is bounded. If \( \{\phi(z, x_n)\} \) converges, it is trivial. If not so, by Lemma 2.3, there exist \( n_0 \in \mathbb{N} \) and an eventually increasing function \( \tau \) such that \( \phi(z, x_{\tau(n)}) \leq \phi(z, x_{\tau(n)+1}) \) for every \( n \in \mathbb{N} \) and \( \phi(z, x_n) \leq \phi(z, x_{\tau(n)+1}) \) for each \( n \geq n_0 \). Since
\[
\inf_{\tau(n) \geq N} \{(1 - \gamma_{\tau(n)})c - \lambda_{\tau(n)}\delta_0\} > 0,
\inf_{n \in \mathbb{N}} \lambda_{\tau(n)}(2\alpha - 1/\delta_0) > 0,
\phi(z, x_{\tau(n)}) \leq \phi(z, x_{\tau(n)+1}), \text{ and } \gamma_{\tau(n)} > 0 \]
and follows from (3.5) that
\[
\phi(y_{\tau(n)}, u) + \phi(z, x_{\tau(n)}) - \phi(z, u) < 0
\]
for all \( n \) with \( \tau(n) \geq N \), which implies that \( \{x_{\tau(n)}\} \) and \( \{y_{\tau(n)}\} \) are bounded. Further, we have
\[
\phi(z, x_{\tau(n)+1}) \leq \phi(z, x_{\tau(n)}) - \gamma_{\tau(n)} \{ \phi(y_{\tau(n)}, u) + \phi(z, x_{\tau(n)}) - \phi(z, u) \}
\]
for every $n$ with $\tau(n) \geq N$. Thus we have $\{x_{\tau(n)+1}\}$ is bounded. Since $\phi(z,x_n) \leq \phi(z,x_{\tau(n)+1})$ for each $n \geq n_0$, it follows that $\{x_n\}$ is bounded. From (3.2) with $\beta = \beta_0$, we obtain

$$\phi(z,y_n) \leq \phi(z,x_n) - 2\gamma_n \langle y_n - z, Jx_n - Ju \rangle,$$

which implies

$$(\|z\| - \|y_n\|)^2 \leq (\|z\| + \|x_n\|)^2 + 2\gamma_n(\|y_n\| + \|z\|)(\|x_n\| + \|u\|)$$

for each $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, so is $\{y_n\}$.

Suppose that $\{\phi(Pu,x_n)\}$ is not convergent. By Lemma 2.3, there exist $n_0 \in \mathbb{N}$ and an eventually increasing function $\tau$ such that $\phi(Pu,x_{\tau(n)}) \leq \phi(Pu,x_{\tau(n)+1})$ for all $n \in \mathbb{N}$ and $\phi(Pu,x_n) \leq \phi(Pu,x_{\tau(n)+1})$ for every $n \geq n_0$. Since $\{x_n\}$ and $\{y_n\}$ are bounded, and $\gamma_0 \to 0$, using (3.4) with $\beta = \beta_0$, we get

$$\|y_{\tau(n)} - x_{\tau(n)}\| \to 0 \quad \text{and} \quad \|Bx_{\tau(n)} - BPu\| \to 0. \quad (3.6)$$

By Lemma 3.2, we get that $F$ includes $\omega_0(x_{\tau(n)})$. By (3.4) with $\beta = \beta_0$, we have

$$0 \leq (c - \lambda_{\tau(n)}\beta_0)\|x_{\tau(n)} - y_{\tau(n)}\|^2 + \lambda_{\tau(n)}(2\alpha - 1/\beta_0)\|Bx_{\tau(n)} - BPu\|^2$$

$$\leq -2\gamma_{\tau(n)}\langle y_{\tau(n)} - Pu, Jx_{\tau(n)} - Ju \rangle$$

for every $n \in \mathbb{N}$. Since $\gamma_{\tau(n)} > 0$,

$$\langle y_{\tau(n)} - Pu, Jx_{\tau(n)} - Ju \rangle \leq 0 \quad (3.7)$$

for each $n \in \mathbb{N}$. On the other hand,

$$\langle y_{\tau(n)} - Pu, Jx_{\tau(n)} - Ju \rangle = \langle y_{\tau(n)} - Pu, Jx_{\tau(n)} - Jy_{\tau(n)} \rangle$$

$$+ \langle y_{\tau(n)} - Pu, Jy_{\tau(n)} - JPu \rangle$$

$$+ \langle y_{\tau(n)} - Pu, JPu - Ju \rangle$$

$$\geq \langle y_{\tau(n)} - Pu, Jx_{\tau(n)} - Jy_{\tau(n)} \rangle$$

$$+ \frac{1}{2}\phi(Pu,y_{\tau(n)}) + \langle y_{\tau(n)} - Pu, JPu - Ju \rangle.$$

By (3.7), we get

$$-\frac{1}{2}\phi(Pu,y_{\tau(n)}) \leq \langle y_{\tau(n)} - Pu, Jx_{\tau(n)} - Jy_{\tau(n)} \rangle + \langle y_{\tau(n)} - Pu, JPu - Ju \rangle$$

for all $n \in \mathbb{N}$. So,

$$-\frac{1}{2}\limsup_{n \to \infty}\phi(Pu,y_{\tau(n)}) \geq \liminf_{n \to \infty}\langle y_{\tau(n)} - Pu, Jx_{\tau(n)} - Jy_{\tau(n)} \rangle$$

$$+ \liminf_{n \to \infty}\langle y_{\tau(n)} - Pu, JPu - Ju \rangle. \quad (3.8)$$

Since $J$ is uniformly continuous on bounded subsets of $E$ and $\|x_{\tau(n)} - y_{\tau(n)}\| \to 0$ in (3.6), we obtain

$$\|Jx_{\tau(n)} - Jy_{\tau(n)}\| \to 0,$$

which implies

$$\|\langle y_{\tau(n)} - Pu, Jx_{\tau(n)} - Jy_{\tau(n)} \rangle \leq \|y_{\tau(n)} - Pu\| \cdot \|Jx_{\tau(n)} - Jy_{\tau(n)}\| \to 0.$$
There exists a subsequence \( \{y_{n_i}\} \) of \( \{y_{n}\} \) such that \( y_{n_i} \to w \in F \) and
\[
\liminf_{n \to \infty}\langle y_{\tau(n)} - \Pi_F u, J\Pi_F u - Ju \rangle = \lim_{n \to \infty} \langle y_{n_i} - \Pi_F u, J\Pi_F u - Ju \rangle = \langle w - \Pi_F u, J\Pi_F u - Ju \rangle \geq 0
\]
by Lemma 2.2. So, from (3.8), we get
\[
\limsup_{n \to \infty} \phi(\Pi_F u, y_{\tau(n)}) = 0.
\]
Since \( \phi(\Pi_F u, x_n) \leq \phi(\Pi_F u, x_{\tau(n)+1}) \) for every \( n \geq n_0 \), by (3.3) we obtain
\[
\lim_{n \to \infty} \phi(\Pi_F u, x_n) = 0.
\]
This is a contradiction. So, \( \{\phi(\Pi_F u, x_n)\} \) is convergent. Since \( \{x_n\} \) and \( \{y_n\} \) are bounded and \( \gamma_n \to 0 \), by (3.4) with \( \beta = \beta_0 \), we have
\[
||x_n - y_n|| \to 0 \quad \text{and} \quad ||Bx_n - B\Pi_F u|| \to 0.
\]
By Lemma 3.2, we get \( \omega_n(\{x_n\}) \subset F \). We show that
\[
\limsup_{n \to \infty} \langle \Pi_F u - y_n, Jx_n - Ju \rangle \geq 0. \tag{3.9}
\]
Suppose that \( \limsup_{n \to \infty} \langle \Pi_F u - y_n, Jx_n - Ju \rangle = l < 0 \). There exists \( n_1 \in \mathbb{N} \) such that
\[
\langle \Pi_F u - y_n, Jx_n - Ju \rangle \leq \frac{1}{2} l
\]
for every \( n \geq n_1 \). From (3.4) with \( \beta = \beta_0 \), we have
\[
|l| \gamma_n \leq 2 \gamma_n \langle y_n - \Pi_F u, Jx_n - Ju \rangle \leq \phi(\Pi_F u, x_n) - \phi(\Pi_F u, x_{n+1})
\]
for each \( n \geq n_1 \), which implies
\[
\sum_{n=n_1}^{\infty} |l| \gamma_n \leq \phi(\Pi_F u, x_{n_1}) < \infty.
\]
Since \( \sum_{n=1}^{\infty} \gamma_n = \infty \), this is a contradiction. So, we get (3.9). Next, we have
\[
\langle \Pi_F u - y_n, Jx_n - Ju \rangle = \langle \Pi_F u - y_n, Jx_n - Jy_n \rangle + \langle \Pi_F u - y_n, Jy_n - J\Pi_F u \rangle \\
+ \langle \Pi_F u - y_n, J\Pi_F u - Ju \rangle \leq ||\Pi_F u - y_n|| \cdot ||Jx_n - Jy_n|| - \frac{1}{2} \phi(\Pi_F u, y_n) \\
+ \langle \Pi_F u - y_n, J\Pi_F u - Ju \rangle
\]
for all \( n \in \mathbb{N} \). Since \( J \) is uniformly continuous on bounded subsets of \( E \) and \( ||x_n - y_n|| \to 0 \), we have
\[
||Jx_n - Jy_n|| \to 0.
\]
So, we obtain
\[
0 \leq \limsup_{n \to \infty} \langle \Pi_F u - y_n, Jx_n - Ju \rangle \\
\leq - \frac{1}{2} \liminf_{n \to \infty} \phi(\Pi_F u, y_n) + \limsup_{n \to \infty} \langle \Pi_F u - y_n, J\Pi_F u - Ju \rangle.
\]
There exists a subsequence \( \{y_{n_j}\} \) of \( \{y_n\} \) such that \( y_{n_j} \rightharpoonup w \in F \) and
\[
\limsup_{n \to \infty} (\Pi_F u - y_n, J\Pi_F u - Ju) = \lim_{j \to \infty} (\Pi_F u - y_{n_j}, J\Pi_F u - Ju) = (\Pi_F u - w, J\Pi_F u - Ju) \leq 0
\]
from Lemma 2.2. So, we have
\[
\liminf_{n \to \infty} \phi(\Pi_F u, y_n) = 0.
\]
By (3.3), we get \( \liminf_{n \to \infty} \phi(\Pi_F u, x_{n+1}) = 0 \), that is, \( \{x_n\} \) converges strongly to \( \Pi_F u \) from Lemma 2.1. □

By Theorem 3.1, we have the following new result in real Hilbert spaces. This holds under weaker conditions than the result in [30].

**Theorem 3.2.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( A \) be a maximal monotone operator in \( H \times H \), \( \alpha > 0 \), \( B \) an \( \alpha \)-inverse-strongly-monotone operator of \( C \) into \( H \) such that \( F = (A + B)^{-1}0 \neq 0 \). Let \( u \in H \) and \( \{x_n\} \) be a sequence generated by
\[
x_1 \in C, \quad x_{n+1} = P_C J_{\lambda_n}^A (\gamma_n u + (1 - \gamma_n)x_n - \lambda_n Bx_n)
\]
for each \( n \in \mathbb{N} \), where \( P_C \) is the metric projection of \( H \) onto \( C \), \( \{\lambda_n\} \subset (0, \infty) \) and \( \{\gamma_n\} \subset (0, 1] \) such that \( \gamma_n \to 0 \) and \( \sum_{n=1}^{\infty} \gamma_n = \infty \). Then, if \( 0 < \inf_{n \in \mathbb{N}} \lambda_n \leq \sup_{n \in \mathbb{N}} \lambda_n < 2\alpha \), then \( \{x_n\} \) converges strongly to \( P_C u \).

**4. DEDUCED RESULTS**

Let \( C \) be a nonempty closed convex subset of \( E \) and \( A \) a single valued monotone operator of \( C \) into \( E^* \), that is, \( \langle x - y, Ax - Ay \rangle \geq 0 \) for all \( x, y \in C \). We consider the variational inequality problem [19] for \( A \), that is, the problem of finding an element \( z \in C \) such that
\[
\langle x - z, Az \rangle \geq 0 \quad \text{for all} \quad x \in C.
\]
The set of all solutions of the variational inequality problem for \( A \) is denoted by \( VI(C,A) \). By Theorem 3.1, we obtain a new result for the variational inequality problem of an inverse-strongly-monotone operator in a 2-uniformly convex and uniformly smooth Banach space \( E \).

**Theorem 4.1.** Let \( C \) be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space \( E \). For \( \alpha > 0 \), let \( B \) be an \( \alpha \)-inverse-strongly-monotone operator of \( C \) into \( E^* \) with \( VI(C,B) \neq \emptyset \). Let \( u \in E \) and \( \{x_n\} \) a sequence generated by \( x_1 \in C \) and
\[
x_{n+1} = \Pi_{VI(C,B)} (\gamma_n u + (1 - \gamma_n)Jx_n - \lambda_n Bx_n)
\]
for every \( n \in \mathbb{N} \), where \( \{\lambda_n\} \subset (0, \infty) \) and \( \{\gamma_n\} \subset (0, 1] \) are real sequences. Suppose that \( 0 < \inf_{n \in \mathbb{N}} \lambda_n \leq \sup_{n \in \mathbb{N}} \lambda_n < 2\alpha c \), where \( c \) is the constant in Theorem 2.1, \( \gamma_n \to 0 \), and \( \sum_{n=1}^{\infty} \gamma_n = \infty \). Then, \( \{x_n\} \) converges strongly to \( \Pi_{VI(C,B)} u \).

**Proof.** Let \( i_C : E \to (\infty, \infty] \) be the indicator function of \( C \). We know that \( i_C \) is proper lower semicontinuous and convex, and hence its subdifferential \( \partial i_C \) is a maximal monotone operator. Let \( A = \partial i_C \). Then, it is easy to see that \( J_\lambda x = \Pi_{C}x \) for every \( \lambda > 0 \) and \( x \in E \), where \( J_\lambda \) is the resolvent of \( A \). Further, we also get \( (A + B)^{-1}0 = VI(C,B) \). Hence the proof is complete. □
Remark 4.1. Using Theorem 4.1, we get a result for the variational inequality problem for a finite family of inverse-strongly-monotone operators as follows: Let $C$ be a nonempty closed convex subset of $E$, $r \in \mathbb{N}$ and $B_i$ a $\beta_i$-inverse-strongly-monotone operator of $C$ into $E^*$ for $i = 1, 2, \ldots, r$. Suppose $\bigcap_{i=1}^{r} VI(C, B_i) \neq \emptyset$ and let $Ax = \frac{1}{r}(B_1x + B_2x + \cdots + B_rx)$ for every $x \in C$. Then, we have $A$ is also inverse-strongly-monotone and $VI(C, A) = \bigcap_{i=1}^{r} VI(C, B_i)$. Indeed, for $x, y \in C$ and $\alpha = \min\{\beta_1, \beta_2, \ldots, \beta_r\}$, we get

\[
\langle x - y, Ax - Ay \rangle \\
= \frac{1}{r} \left( \langle x - y, B_1x - B_1y \rangle + \langle x - y, B_2x - B_2y \rangle + \cdots + \langle x - y, B_rx - B_ry \rangle \right) \\
\geq \frac{1}{r} \left( \beta_1 \|B_1x - B_1y\|^2 + \beta_2 \|B_2x - B_2y\|^2 + \cdots + \beta_r \|B_rx - B_ry\|^2 \right) \\
\geq \alpha \left( \frac{1}{r} \|B_1x - B_1y\|^2 + \frac{1}{r} \|B_2x - B_2y\|^2 + \cdots + \frac{1}{r} \|B_rx - B_ry\|^2 \right) \\
\geq \alpha \left( \frac{1}{r} \|B_1x - B_1y\| + \frac{1}{r} \|B_2x - B_2y\| + \cdots + \frac{1}{r} \|B_rx - B_ry\| \right) \\
= \alpha \|Ax - Ay\|^2.
\]

Thus $A$ is an $\alpha$-inverse strongly monotone operator. Next, we show

\[
VI(C, A) = \bigcap_{i=1}^{r} VI(C, B_i).
\]

The inclusion $VI(C, A) \supset \bigcap_{i=1}^{r} VI(C, B_i)$ is trivial. Let $u \in VI(C, A)$ and $z \in \bigcap_{i=1}^{r} VI(C, B_i)$. We have $\langle z - u, Au \rangle \geq 0$ and $\langle u - z, B_i z \rangle \geq 0$ for all $i = 1, 2, \ldots, r$, we also get

\[
\langle u - z, Az \rangle = \frac{1}{r} \sum_{i=1}^{r} \langle u - z, B_i z \rangle \geq 0.
\]

It follows that

\[
\langle z - u, Au - Az \rangle = \langle z - u, Au \rangle + \langle u - z, Az \rangle \geq \frac{1}{r} \sum_{i=1}^{r} \langle u - z, B_i z \rangle \geq 0.
\]

On the other hand, we have

\[
\langle z - u, Au - Az \rangle \leq -\frac{\alpha}{r} \sum_{i=1}^{r} \|B_i z - B_i u\|^2 \leq 0.
\]

Therefore, we obtain

\[
B_i z = B_i u \text{ and } \langle u - z, B_i z \rangle = 0 \text{ for all } i = 1, 2, \ldots, r.
\]

Hence we have

\[
\langle x - u, B_i u \rangle = \langle x, B_i z \rangle - \langle u, B_i z \rangle = \langle x, B_i z \rangle - \langle z, B_i z \rangle = \langle x - z, B_i z \rangle \geq 0
\]

for every $i = 1, 2, \ldots, r$ and $x \in C$, that is, $u \in VI(C, B_i)$ for all $i = 1, 2, \ldots, r$. Therefore, $VI(C, A) \subset \bigcap_{i=1}^{r} VI(C, B_i)$.

Remark 4.2. In the result of Iiduka and Takahashi [15], under the assumption that (i) $\|By\| \leq \|By - Bu\|$ for every $y \in C$ and $u \in VI(C, A)$, and (ii) $J$ is weakly sequentially continuous, they proved the weak convergence of the generated sequence to an element of $VI(C, B)$, whereas we get the strong convergence to an element of $VI(C, B)$ in Theorem 4.1, without the assumptions (i) and (ii); see also [18].
Remark 4.3. In Lemma 3.2, if \( A = 0 \), then, \( J^A_\lambda = I \) for every \( \lambda > 0 \). Moreover, we can get the inclusion \( \omega_w(\{x_n\}) \subseteq B^{-1}0 \) from the assumption \( \|Bx_n - Bz\| \to 0 \) for \( z \in B^{-1}0 \) only. In fact, suppose \( x_n \rightharpoonup w \in C \). Since

\[
\langle x_n - w, Bx_n - Bw \rangle \geq \alpha \|Bx_n - Bw\|^2
\]

for every \( i \in \mathbb{N} \), we have \( Bw = Bz = 0 \), that is, \( w \in B^{-1}0 \). Therefore, instead of uniform smoothness of \( E \), smoothness of \( E \) guarantees that Lemma 3.2 holds. Hence we obtain the following strong convergence theorem for monotone inclusions by Theorem 3.1. This is a result of strong convergence under weaker conditions than that of weak convergence in [15].

Theorem 4.2. Let \( C \) be a nonempty closed convex subset of a 2-uniformly convex and smooth Banach space \( E \). For \( \alpha > 0 \), let \( B \) be an \( \alpha \)-inverse-strongly-monotone operator of \( C \) into \( E^* \) with \( B^{-1}0 \neq \emptyset \). Let \( u \in E \) and \( \{x_n\} \) a sequence generated by \( x_1 \in C \) and

\[
x_{n+1} = \Pi_C J^{-1}(\gamma_n Ju + (1 - \gamma_n)Jx_n - \lambda_n Bx_n)
\]

for every \( n \in \mathbb{N} \), where \( \{\lambda_n\} \subseteq (0, \infty) \) and \( \{\gamma_n\} \subseteq (0, 1] \) are real sequences. Suppose that \( 0 < \inf_{n \in \mathbb{N}} \lambda_n \leq \sup_{n \in \mathbb{N}} \lambda_n < 2\alpha c \), where \( c \) is the constant in Theorem 2.1, \( \gamma_n \to 0 \), and \( \sum_{n=1}^{\infty} \gamma_n = \infty \). Then, \( \{x_n\} \) converges strongly to \( \Pi_{B^{-1}0} u \).

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