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BREGMAN BEST PROXIMITY POINTS FOR BREGMAN ASYMPTOTIC CYCLIC CONTRACTION MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we first introduce the notion of the BUC property, which extends the notion of UC property introduced and studied in [T. Suzuki, M. Kikawa, C. Vetro, The existence of best proximity points in metric spaces with the property UC, Nonlinear Anal. 71 (2009), 2918-2926] for Banach spaces. We then prove, without continuity assumption on mappings, the existence and convergence of Bregman best proximity points for Bregman asymptotic cyclic contractions in Banach spaces with the property BUC and Bregman asymptotic proximal pointwise contractions in uniformly convex Banach spaces. Moreover, we consider a Bregman generalized cyclic contraction mapping and prove the existence of Bregman best proximity points for such a mapping in Banach spaces which do not necessarily satisfy the geometric property BUC.
Keywords. Bregman best proximity point; Fixed point; Bregman proximal pointwise contraction; Cyclic mapping; Property BUC.

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1. Introduction

Let E be a Banach space with norm $\|.\|$. Let E^* be the dual space of E. The value of a functional $x^* \in E^*$ at $x \in E$ is denoted by $\langle x, x^* \rangle$. For a sequence $\{x_n\}_{n\in\mathbb{N}}$ in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \to \infty$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$, respectively. A Banach space E is said to be uniformly convex if

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\} > 0 \quad (0 \le \varepsilon \le 2).$$

Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if, for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists. In this case, E is said to be smooth. If limit (1.1) is attained uniformly for all $x, y \in S_E$, then E is said to be uniformly smooth. The Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \ne y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth (for more details, see [12]).

Let A and Q be nonempty and closed subsets of E and let T be a cyclic mapping on $A \cup Q$, that is, a map on $A \cup Q$ such that $T(A) \subset Q$ and $T(Q) \subset A$. Eldred and Veeramani [3] introduced a notion called

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cyclic contraction and gave sufficient conditions ([3], Theorem 3.10) for the existence of a point $x \in A$ such that

$$\|x - Tx\| = \mathrm{dist}(A,Q) := \inf\{\|u - v\| : u \in A,\ v \in Q\}$$

for such a mapping T in the setting of uniformly convex Banach spaces. Such a point is called the best proximity point of T.

Let us recall the definition of the Bregman distances. Let E be a Banach space and let $g : E \to (-\infty, +\infty]$ be a strictly convex and Gâteaux differentiable function. The Bregman distance [7] (see also [4, 6]) corresponding to g is the function $D_g : \mathrm{dome}(g) \times \mathrm{Int\,dom}(g) \to \mathbb{R}$ defined by

$$D_g(x,y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \quad \forall x,y \in E. \tag{1.2}$$

It is clear that $D_g(x,y) \geq 0$ for all x,y in E. However, D_g is not symmetric and D_g does not satisfy the triangular inequality. When E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all x in E, we have that $\nabla g(x) = 2Jx$, where J is the normalized duality mapping from E into E^*, for all x in E. Hence, $D_g(\cdot,\cdot)$ reduces to the usual map $\phi(\cdot,\cdot)$ as

$$D_g(x,y) = \phi(x,y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E. \tag{1.3}$$

Let E be a Banach space and let $g : E \to (-\infty, +\infty]$ be a strictly convex and Gâteaux differentiable function. From the Bregman distance D_g, the following Bregman three-point identity holds for every Banach space E [7]:

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x - y, \nabla g(y) - \nabla g(z)\rangle, \quad \forall x,y,z \in E. \tag{1.4}$$

In particular,

$$D_g(x,y) = -D_g(y,x) + \langle y - x, \nabla g(y) - \nabla g(x)\rangle, \quad \forall x,y \in E. \tag{1.5}$$

Bregman distance D_g motivates us to introduce the notion of Bregman best proximity points and develop the approximation theory, the fixed point theorems and convergence of best proximity points in general Banach spaces. For a pair of subsets (A,Q), let

$$A_0 = \{x \in A : D_g(x,y) = \mathrm{Bdist}(A,Q), \quad \text{for some } y \text{ in } Q\};$$

$$Q_0 = \{y \in Q : D_g(x,y) = \mathrm{Bdist}(A,Q), \quad \text{for some } x \text{ in } A\}.$$

Definition 1.1 ([13])**.** Let E be a Banach space and let $g : E \to (-\infty, +\infty]$ be a strictly convex and Gâteaux differentiable function. Let A and Q be monempty subsets of E and let $T : A \cup Q \to A \cup Q$ be a cyclic mapping. A point $x \in A$ is called a Bregman best proximity point, if $D_g(x,Tx) = \mathrm{Bdist}(A,Q)$.

Definition 1.2 ([13])**.** Let E be a Banach space and let $g : E \to (-\infty, +\infty]$ be a strictly convex and Gâteaux differentiable function. A cyclic map T on $A \cup Q$ is said to be a Bregman cyclic contraction if and only if one of the following two alternatives is satisfied:

(i) There exists α in $[0,1)$ such that

$$D_g(Ty,Tx) \leq \alpha D_g(x,y) + (1-\alpha)\mathrm{Bdist}(A,Q)$$

for $x \in A$ and $y \in Q$;

(ii) There exists β in $[0,1)$ such that

$$D_g(Tx,Ty) \leq \beta D_g(y,x) + (1-\beta)\mathrm{Bdist}(Q,A)$$

for $x \in A$ and $y \in Q$.

Let A be a nonempty subset of a Banach space E. Let $T : A \to E$ be a map. We denote by $F(T) = \{x \in A : Tx = x\}$ the set of fixed points of T.

Eldred and Veeramani [3] studied the existence and convergence of best proximity points in a uniformly convex Banach space E and proved the following interesting result.

Theorem 1.1. *Let A and Q be nonempty, closed and convex subsets of a uniformly convex Banach space E and let $T : A \cup Q \to A \cup Q$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $\|x - Tx\| = \operatorname{dist}(A, Q) = \inf\{\|u - v\| : u \in A,\ v \in Q\}$.*

Remark 1.1. Although the proposed methods mentioned above work for the existence of best proximity points of nonlinear mappings, the continuity of mapping T plays an important role in all the results above. So, the following question arises naturally in a Banach space setting.

Question 1.1. Let A and Q be nonempty, closed and convex subsets of a reflexive Banach space E. Is it possible to obtain a best proximity point theorem if T is a Bregman asymptotic contraction not necessarily continuous on A?

Let $T : A \to A$, where A is a nonempty subset of a metric space, a normed linear space or a topological vector space X, be a mapping. One of the most important problems in nonlinear analysis is to solve the equation $Tx = x$. Fixed point theory is one of the most fundamental tools for solving this problem, however, as a non-self mapping, $T : A \to Q$ does not necessarily have a fixed point. It is reasonable to determine an element x in A, which is, in some sense, closest to Tx. Best approximation theory guarantees the existence of such approximate solutions, however, the results need not yield optimal solutions. Best proximity point theorems provide the existence of approximate solutions which are optimal as well. Though there has been a large volume of literature studying the best approximation problems and the best proximity point results, most of them are geared towards exploiting the symmetry property or the triangle inequality of the standard metrics (see, for example, [2, 15, 16]). In this paper, using Bregman functions and Bregman distances, we first prove the existence of Bregman best proximity points in a reflexive Banach space. We then prove convergence results of Bregman best proximity points for Bregman asymptotic contraction mappings in the setting of Banach spaces. It is well known that the Bregman distance does not satisfy either the symmetry property or the triangle inequality which are required for standard distances. So, Bregman distances enable us to provide affirmative answers to question 1.1 in a reflexive Banach space. This could be done in the absence of either the symmetry property or the triangle inequality which are required for standard distances. Our results improve and generalize many known results in the current literature; see, for example, [1, 2, 3, 17].

2. Preliminaries

First, we list some definitions, notations and conclusions which will be needed in the sequel.

Let E be a Banach space. A function $g : E \to (-\infty, +\infty]$ is said to be proper if the interior of its domain

$$\operatorname{dom} g = \{x \in E : g(x) < +\infty\}$$

is nonempty. For any $x \in \operatorname{int} \operatorname{dom} g$ and any $y \in E$, we denote by $g^o(x,y)$ the right-hand derivative of g at x in the direction y, that is,

$$g^o(x,y) = \lim_{t \downarrow 0} \frac{g(x+ty) - g(x)}{t}.$$

The function g is said to be Gâteaux differentiable at x if $\lim_{t \to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case $g^o(x,y)$ coincides with $\nabla g(x)$, the value of the gradient ∇g of g at x. The function g is said to be Gâteaux differentiable at x if $\nabla g(x)$ is well-defined, and g is Gâteaux differentiable if it is Gâteaux differentiable everywhere in the interior of $\operatorname{dom} g$. We call g Fréchet differentiable at x (see, for example, [5, p. 13] or [8, p. 508]) if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$| g(y) - g(x) - \langle y - x, \nabla g(x) \rangle | \leq \varepsilon \|y - x\| \quad \text{whenever } \|y - x\| \leq \delta.$$

The function g is said to be Fréchet differentiable if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to (-\infty, +\infty]$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [5, Proposition 1.1.10]). If g is also Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [8, p. 508]).

For any $r > 0$, let $B_r := \{x \in E : \|x\| \leq r\}$. A function $g : E \to (-\infty, +\infty]$ is said to be

- strongly coercive if
$$\lim_{\|x_n\| \to +\infty} \frac{g(x_n)}{\|x_n\|} = +\infty;$$
- locally bounded if $g(B_r)$ is bounded for all $r > 0$;
- locally uniformly convex on E (or uniformly convex on bounded subsets of E ([21, pp. 203, 221])) if the gauge $\rho_r : [0, +\infty) \to [0, +\infty]$ of uniform convexity of g, defined by
$$\rho_r(t) = \inf_{x,y \in B_r, \|x-y\|=t, \gamma \in (0,1)} \frac{\gamma g(x) + (1-\gamma) g(y) - g(\gamma x + (1-\gamma) y)}{\gamma(1-\gamma)},$$
satisfies
$$\rho_r(t) > 0, \quad \forall r, t > 0;$$
- locally uniformly smooth on E ([21, pp. 207, 221]) if the function $\sigma_r : [0, +\infty) \to [0, +\infty]$, defined by
$$\sigma_r(t) = \sup_{x \in B_r, y \in S_E, \gamma \in (0,1)} \frac{\gamma g(x + (1-\gamma) t y) + (1-\gamma) g(x - \gamma t y) - g(x)}{\gamma(1-\gamma)},$$
satisfies
$$\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0, \quad \forall r > 0;$$

For a locally uniformly convex map $g : E \to (-\infty, +\infty]$, we have

$$g(\gamma x + (1-\gamma) y) \leq \gamma g(x) + (1-\gamma) g(y) - \gamma(1-\gamma) \rho_r(\|x - y\|), \tag{2.1}$$

for all x, y in B_r and for all γ in $(0,1)$.

Lemma 2.1 ([10]). *Let E be a Banach space and let $g : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function which is locally uniformly convex on E. Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent.*

(1) $\lim_{n \to \infty} D_g(x_n, y_n) = 0$.

(2) $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

We say that a function $g : E \to (-\infty, +\infty]$ is said to be lower semicontinuous if $\{x \in E : g(x) \le r\}$ is closed for all r in $\mathbb{R}$. For a proper lower semicontinuous convex function $g : E \to (-\infty, +\infty]$, the subdifferential ∂g of g is defined by

$$\partial g(x) = \{x^* \in E^* : g(x) + \langle y - x, x^* \rangle \le g(y), \quad \forall y \in E\}$$

for all x in E. It is well known that $\partial g \subset E \times E^*$ is maximal monotone [11]. For any proper lower semicontinuous convex function $g : E \to (-\infty, +\infty]$, the conjugate function g^* of g is defined by

$$g^*(x^*) = \sup_{x \in E}\{\langle x, x^* \rangle - g(x)\}, \quad \forall x^* \in E^*.$$

It is well known that

$$g(x) + g^*(x^*) \ge \langle x, x^* \rangle, \quad \forall (x, x^*) \in E \times E^*,$$

and

$$(x, x^*) \in \partial g \quad \text{is equivalent to} \quad g(x) + g^*(x^*) = \langle x, x^* \rangle. \tag{2.2}$$

We also know that if $g : E \to (-\infty, +\infty]$ is a proper lower semicontinuous convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous convex function.

Definition 2.1 ([5])**.** Let E be a Banach space. The function $g : E \to (-\infty, +\infty]$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D_g(x, y) \le r\}$ is bounded for all x in E and $r > 0$.

The following lemma follows from Butnariu and Iusem [5] and Zălinscu [21].

Lemma 2.2. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Then*

(1) $\nabla g : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;
(2) $\langle x - y, \nabla g(x) - \nabla g(y) \rangle = 0$ if and only if $x = y$;
(3) $\{x \in E : D_g(x, y) \le r\}$ is bounded for all y in E and $r > 0$;
(4) $\operatorname{dom} g^* = E^*$, g^* is Gâteaux differentiable and $\nabla g^* = (\nabla g)^{-1}$.

The following two results follow from [21, Proposition 3.6.4].

Proposition 2.1. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a locally bounded convex function. The following assertions are equivalent.*

(1) g is strongly coercive and locally uniformly convex on E;
(2) $\operatorname{dom} g^* = E^*$, g^* is locally bounded and locally uniformly smooth on E;
(3) $\operatorname{dom} g^* = E^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*.

Proposition 2.2. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a continuous convex function which is strongly coercive. The following assertions are equivalent.*

(1) g is locally bounded and locally uniformly smooth on E;
(2) g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E;
(3) $\operatorname{dom} g^* = E^*$, g^* is strongly coercive and locally uniformly convex on E.

Let E be a Banach space and let A be a nonempty convex subset of E. Let $g : E \to (-\infty, +\infty]$ be a strictly convex and Gâteaux differentiable function. Then, we know, from [9] that, for x in E and x_0 in A,

$$D_g(x_0, x) = \min_{y \in A} D_g(y, x) \quad \text{if and only if} \quad \langle y - x_0, \nabla g(x) - \nabla g(x_0) \rangle \le 0, \ \forall y \in A. \tag{2.3}$$

Further, if A is a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ is a strongly coercive Bregman function, then, for each x in E, there exists a unique x_0 in A such that

$$D_g(x_0, x) = \min_{y \in A} D_g(y, x).$$

The Bregman projection proj_A^g from E onto A defined by $\text{proj}_A^g(x) = x_0$ has the following property:

$$D_g(y, \text{proj}_A^g x) + D_g(\text{proj}_A^g x, x) \le D_g(y, x), \quad \forall y \in A, \forall x \in E. \tag{2.4}$$

See [5] for more details.

Let us prove the following simple lemma which plays a crucial role in the sequel.

Lemma 2.3. *Let E be a Banach space and let $g : E \to (-\infty, +\infty]$ be a convex function which is uniformly convex on bounded subsets of E. Let $s > 0$ be a constant, $B_s := \{z \in E : \|z\| \le s\}$ and let ρ_s be the gauge of uniform convexity of g. Then, for any $x, y \in B_s$, $z \in E$ and $\gamma \in (0, 1)$*

$$D_g(\gamma x + (1-\gamma)y, z) \le \gamma D_g(x, z) + (1-\gamma) D_g(y, z) - \gamma(1-\gamma)\rho_s(\|x - y\|).$$

Proof. In view of (2.1), for any $x, y \in B_s$ and $\gamma \in (0, 1)$, we obtain

$$\begin{aligned} D_g(\gamma x + (1-\gamma)y, z) &= g(\gamma x + (1-\gamma)y) - g(z) - \langle \gamma x + (1-\gamma)y - z, \nabla g(z) \rangle \\ &\le \gamma g(x) + (1-\gamma) g(y) - \gamma(1-\gamma)\rho_s(\|x - y\|) \\ &\quad - g(z) - \gamma \langle x - z, \nabla g(z) \rangle - (1-\gamma)\langle y - z, \nabla g(z) \rangle \\ &= \gamma[g(x) - g(z) - \langle x - z, g(z) \rangle] \\ &\quad + (1-\gamma)[g(y) - g(z) - \langle y - z, \nabla g(z) \rangle] \\ &\quad - \gamma(1-\gamma)\rho_s(\|x - y\|) \\ &= \gamma D_g(x, z) + (1-\gamma) D_g(y, z) - \gamma(1-\gamma)\rho_s(\|x - y\|). \end{aligned}$$

This completes the proof. □

Definition 2.2 ([13]). Let E be a reflexive Banach space and let $g : E \to (-\infty. + \infty]$ be a convex function which is also strongly coercive. A sequence $\{x_n\}_{n \in \mathbb{N}} \subset E$ is said to be Bregman Cauchy if the following condition is satisfied:

$$\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N}, \ \forall m, n \ge N_0 \Longrightarrow D_g(x_m, x_n) < \varepsilon.$$

Lemma 2.4 ([14]). *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function which is uniformly convex on bounded subsets of E. Let A be a nonempty, closed and convex subset and Q be a nonempty and closed subset of E. Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{z_n\}_{n \in \mathbb{N}}$ be sequences in A and $\{y_n\}_{n \in \mathbb{N}}$ be a sequence in Q satisfying:*

(i) $D_g(z_n, y_n) \to \text{Bdist}(A, Q)$ as $n \to \infty$.
(ii) For every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $m > n \ge N_0$, $D_g(x_m, y_n) \le \text{Bdist}(A, Q) + \varepsilon$.
Then, for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $m > n \ge N_1$, $D_g(x_m, z_n) \le \varepsilon$.

Lemma 2.5 ([14]). *Let E be a reflexive Banach space and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function which is uniformly convex on bounded subsets of E. Let A be a nonempty, closed and convex subset and Q be a nonempty and closed subset of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ be sequences in A and $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in Q satisfying:*
(i) $D_g(x_n, y_n) \to \mathrm{Bdist}(A, Q)$ as $n \to \infty$.
(ii) $D_g(z_n, y_n) \to \mathrm{Bdist}(A, Q)$ as $n \to \infty$.
Then, $D_g(x_n, z_n) \to 0$ as $n \to \infty$.

Corollary 2.1. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function which is uniformly convex on bounded subsets of E. Let A be a nonempty, closed and convex subset and Q be a nonempty and closed subset of E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in A and $y_0 \in Q$ such that $D_g(x_n, y_0) \to Bdist(A, Q)$ as $n \to \infty$. Then, $x_n \to proj_A^g(y_0)$ as $n \to \infty$.*

Proof. Since $\mathrm{Bdist}(A,Q) \le D_g(\mathrm{proj}_A^g(y_0), y_0) \le D_g(x_n, y_0)$, we obtain that $D_g(\mathrm{proj}_A^g(y_0), y_0) = \mathrm{Bdist}(A,Q)$. Now, setting $y_n = y_0$ and $z_n = \mathrm{proj}_A^g(y_0)$, we get the desired conclusion. □

Definition 2.3 ([13]). Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let A and Q be nonempty sunsets of E. A sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ in $A \cup Q$, with $x_{2n} \in A$ and $x_{2n+1} \in Q$ for $n \ge 0$, is said to be a Bregman cyclical Cauchy sequence if and only if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $D_g(x_n, x_m) < \mathrm{Bdist}(A,Q) + \varepsilon$ whenever n is even, m is add and $n, m \ge N$.

Notice that if $\mathrm{Bdist}(A,Q) = 0$, then Bregman cyclical Cauchy sequences in $A \cup Q$ are Cauchy sequences. So,

(1) $\lim_{n\to\infty} D_g(x_n, y_n) = 0$.
(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Definition 2.4 ([13]). Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. A pair (A, Q) of subsets of E is Bregman proximally complete if and only if, for every Bregman cyclical Cauchy sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ in $A \cup Q$, the sequences $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{x_{2n+1}\}_{n\in\mathbb{N}\cup\{0\}}$ have convergent subsequences in A and Q, respectively.

It is obvious that, if A and Q are closed subsets of a Banach space E with $\mathrm{Bdist}(A,Q) = 0$, then (A, Q) is a Bregman proximally complete pair. The following results will be useful in subsequent results.

Proposition 2.3 ([13]). *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let a pair (A, Q) of subsets of E be Bregman proximally complete. Then, A_0 is nonempty if and only if there exists a Bregman cyclical Cauchy sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ in $A \cup Q$.*

Definition 2.5 ([13]). Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function which is uniformly convex on bounded subsets of E. A pair (A, Q) of nonempty subsets of E is said to be Bregman sharp proximinal if and only if, for each x in A and y in Q, there exist a unique x' in Q and y' in A such that $D_g(x, x') = D_g(y', y) = \mathrm{Bdist}(A,Q) := \inf\{D_g(u, v) : u \in A,\ v \in D\}$. The pair (A, Q) is said to be a Bregman semi-sharp proximinal if and only if, for each x in A and y in Q, there exist at most one point x' in Q and, at most, one point y' in A such that $D_g(x, x') = D_g(y', y) = \mathrm{Bdist}(A,Q) := \inf\{D_g(u, v) : u \in A,\ v \in Q\}$.

Theorem 2.1 ([13])**.** *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let (A, Q) be a Bregman semi-sharp proximinal pair in E. If $\{x_{2n}\}_{n\in\mathbb{N}}$ is a Bregman cyclical Cauchy sequence in $A \cup Q$, then $x_{2n} \to x \in A$, and $x_{2n+1} \to y \in Q$ as $n \to \infty$. Moreover, $D_g(x, y) = \mathrm{Bdist}(A, Q)$.*

Theorem 2.2 ([12])**.** *Let E be a reflexive Banach space. Let C be a closed convex subset of E and let $g : E \to (-\infty, +\infty]$ be a strongly coercive, lowersemicontimuous and convex function. Then there exists $x_0 \in C_0 \subset C$ such that*

$$f(x_0) = \min_{x\in C_0} f(x) = \min_{x\in C} f(x).$$

Working on the Bregman distance D_g, the following Bregman Opial-like inequality holds, for every Banach space E,

$$\limsup_{n\to\infty} D_g(x_n, x) < \limsup_{n\to\infty} D_g(x_n, y),$$

whenever $x_n \rightharpoonup x \neq y$.

The following Bregman Opial-like inequality has been proved in [19, 20].

Lemma 2.6 ([19, 20])**.** *Let E be a Banach space and let $g : E \to (-\infty, +\infty]$ be a strictly convex and Gâteaux differentiable function. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in E such that $x_n \rightharpoonup x$ for some x in E. Then*

$$\limsup_{n\to\infty} D_g(x_n, x) < \limsup_{n\to\infty} D_g(x_n, y),$$

for all y in the interior of $\mathrm{dom}\, g$ *with $y \neq x$.*

3. The BUC property of subsets of a Banach space

In this section, we introduce the notion of the BUC, which extends the notion of the UC introduced and studied by Suzuki, Kikawa and Vetro [17], a kind of geometric property for subsets of a metric space (X, d).

Definition 3.1. Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let A and Q be nonempty subsets of E. Then, (A, Q) is said to satisfy the property BUC if the following holds: If $\{x_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ are sequences in A and $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Q such that $\lim_{n\to\infty} D_g(x_n, y_n) = \mathrm{Bdist}(A, Q)$ and $\lim_{n\to\infty} D_g(z_n, y_n) = \mathrm{Bdist}(A, Q)$, then we have $\lim_{n\to\infty} D_g(x_n, z_n) = 0$.

The following two lemmas play an important role in the next sections.

Lemma 3.1. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let A and Q be subsets of E. Assume that (A, Q) has the property BUC. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in A and Q, respectively, such that either of the following holds:*

$$\lim_{m\longrightarrow\infty} \sup_{n\geq m} D_g(x_m, y_n) = \mathrm{Bdist}(A, Q) \text{ or } \lim_{n\longrightarrow\infty} \sup_{m\geq n} D_g(x_m, y_n) = \mathrm{Bdist}(A, Q).$$

Then $\{x_n\}_{n\in\mathbb{N}}$ is a Bregman Cauchy sequence.

Proof. Assume, on the contrary, that $\{x_n\}_{n\in\mathbb{N}}$ is not a Bregman Cauchy sequence. Then there exist a positive real number ε and two subsequences $\{\ell_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $\ell_k < m_k$ and $D_g(x_{\ell_k}, x_{m_k}) > \varepsilon$ for all $k \in \mathbb{N}$. In the case $\lim_{m\to\infty}\sup_{n\geq m} D_g(x_m, y_n) = \text{Bdist}(A,Q)$, we conclude that

$$\lim_{k\longrightarrow\infty} D_g(x_{\ell_k}, y_{m_k}) = \text{Bdist}(A,Q) \text{ and } \lim_{k\longrightarrow\infty} D_g(x_{m_k}, y_{m_k}) = \text{Bdist}(A,Q).$$

In the case $\lim_{n\to\infty}\sup_{m\geq n} D_g(x_m, y_n) = \text{Bdist}(A,Q)$, we arrive at

$$\lim_{k\longrightarrow\infty} D_g(x_{\ell_k}, y_{\ell_k}) = \text{Bdist}(A,Q) \text{ and } \lim_{k\longrightarrow\infty} D_g(x_{m_k}, y_{\ell_k}) = \text{Bdist}(A,Q).$$

Since (A,Q) has the property BUC, we obtain $\lim_{k\to\infty} D_g(x_{\ell_k}, x_{m_k}) = 0$ in both cases. This contradicts $D_g(x_{\ell_k}, x_{m_k}) > \varepsilon$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ is a Bregman Cauchy sequence. □

Throughout this paper, (A,Q) stands for a nonempty pair in a Banach space E. When we say that a pair (A,Q) satisfies a specific property, we mean that both A and Q satisfy the mentioned property.

Lemma 3.2. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let A and Q be nonempty subsets of E such that (A,Q) satisfies the property BUC. Let T be a cyclic mapping on $A\cup Q$ such that*

$$D_g(T^2x, Tx) \leq D_g(x, Tx) \textit{ for all } x \in A \tag{3.1}$$

and

$$D_g(T^2x, Tx) < D_g(x, Tx) \textit{ for all } x \in A \textit{ with Bdist}(A,Q) < D_g(x, Tx). \tag{3.2}$$

Let $z \in A$. Then the following are equivalent

(i) z is a Bregman best proximity point of T.
(ii) z is a fixed point of T^2.

In this case, Tz is a Bregman best proximity point in Q.

Proof. We first prove that (i) implies (ii). We assume that z is a Bregman best proximity point of T. Then, in view of (3.1), we find that

$$D_g(T^2z, Tz) \leq D_g(z, Tz) = \text{Bdist}(A,Q).$$

Thus, Tz is a Bregman best proximity point in Q. It follows from $D_g(z,Tz) = D_g(T^2z,Tz) = \text{Bdist}(A,Q)$ and the property BUC that $T^2z = z$ holds. We next show that (ii) implies (i). In order to do this, we assume that z is a fixed point of T^2 and z is not a Bregman best proximity point. Then

$$D_g(z, Tz) = D_g(T^2z, Tz) < D_g(z, Tz)$$

by (3.2), which yields a contradiction. This completes the proof. □

Corollary 3.1. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let A and Q be nonempty subsets of E such that (A,Q) satisfies the property BUC. Let T be a cyclic mapping on $A\cup Q$ such that*

$$D_g(T^2x, Tx) \leq D_g(x, Tx) \textit{ for all } x \in A; \tag{3.3}$$

and

$$D_g(T^2z, Tx) < D_g(z, Tx) \textit{ for all } x \in A \textit{ and } z \in A \textit{ with Bdist}(A,Q) < D_g(z, Tx). \tag{3.4}$$

Let $z \in A$. Then the following are equivalent:

(i) z is a Bregman best proximity point of T.
(ii) z is a fixed point of T^2.

In this case, Tz is a Bregman best proximity point in Q.

Theorem 3.1. *Let E be a reflexive Banach space and leet $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let (A, Q) be a pair of weakly closed subsets of E which is Bregman semi-sharp proximinal $T : A \cup Q \to A \cup Q$ be a Bregman cyclic mapping which satisfies the conditions (3.1) and (3.2) of Lemma 3.2. Then T has a unique Bregman best proximity point in A. Moreover, if $x_0 \in A$ and $x_{n+1} = Tx_n$, for each $n \geq 0$, then $\{x_{2n}\}_{n\in\mathbb{N}}$ converges to the Bregman best proximity point of T. Also there exists a unique $x \in A$ such that $x_{2n} \to x$ as $n \to \infty$, $T^2x = x$ and $D_g(x, Tx) = \mathrm{Bdist}(A, Q)$.*

Proof. In view of Lemma 3.2, $\{x_{2n}\}_{n\in\mathbb{N}}$ is a Bregman Cauchy sequence and hence there exists $x \in A$ such that $D_g(x_{2n}, x) \to 0$ as $n \to \infty$. Employing Lemma 2.1, we see that $\|x_{2n} - x\| \to 0$ as $n \to \infty$. In the light of Theorem 2.1, we get that $D_g(x, Tx) = \mathrm{Bdist}(A, Q)$. To prove that such x is unique, we assume that there exists $z \in A$ such that $D_g(z, Tz) = \mathrm{Bdist}(A, Q)$ with $T^2z = z$. In view of Lemma 3.2 (ii), we obtain that

$$D_g(z, Tx) \leq D_g(x, Tz) \quad \text{and} \quad D_g(x, Tz) \leq D_g(z, Tx).$$

Therefore, $D_g(z, Tx) = D_g(x, Tz)$. In addition, $D_g(z, Tx) = \mathrm{Bdist}(A, Q)$ since otherwise $D_g(z, Tx) > \mathrm{Bdist}(A, Q)$. Hence the Bregman property of T assures that

$$D_g(z, Tx) = D_g(T^2z, Tx) \quad < \quad D_g(z, Tx),$$

which is a contradiction. On the other hand, by Lemma 2.3 and the fact that $D_g(z, Tx) = \mathrm{Bdist}(A, Q) = D_g(x, Tx)$, we infer that

$$D_g\Big(\frac{x+z}{2}, Tx\Big) \leq \frac{1}{2}D_g(x, Tx) + \frac{1}{2}D_g(z, Tx) - \frac{1}{4}\rho_{s_1}(\|x - z\|) < \mathrm{Bdist}(A, Q),$$

where $s_1 = \max\{\|x\|, \|z\|\}$. This is a contradiction, which entails to $x = z$. This completes the proof. $\square$

4. Bregman ϕ-asymptotic cyclic contraction maps

In this section, we introduce the notion of a Bregman ϕ-asymptotic contraction mapping and prove existence theorems of Bregman best proximity points in a Banach space E.

Definition 4.1. Let E be a reflexive Banach space and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. A mapping $T : E \longrightarrow E$ is said to be a Bregman ϕ-asymptotic contraction if

$$D_g(T^i(x), T^i(y)) \leq \phi_i(D_g(x, y)) \; \forall x, y \in E,$$

where $\phi_i : [0, \infty) \longrightarrow [0, \infty)$ and $\phi_i \longrightarrow \varphi$ uniformly on the rang of D_g in which $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is continuous and $\varphi(s) < s$ for $s > 0$.

In the following theorem, we state and prove the analogue of Kirk's theorem for cyclic mappings, and find a Bregman best proximity point.

Theorem 4.1. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let (A, Q) be a Bregman semi-sharp proximinal pair in E, which is a Bregman proximally complete pair in E. Let $T : A \cup Q \longrightarrow A \cup Q$ be a cyclic mapping which satisfies the conditions (3.1) and (3.2) of Lemma 3.2. Assume that $\{\phi_i\}_{i\in\mathbb{N}}$ is a sequence of continuous functions such that $\varphi_i : [0,\infty) \longrightarrow [0,\infty)$ and for each $x \in A,\ y \in Q, D_g{}^*(T^{2i}(x), T^{2i}(y)) \leq \phi_i(D_g{}^*(x,y))$, where $D_g{}^*(x,y) := D_g(x,y) - \mathrm{Bdist}(A,Q)$. Assume also there exists a function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that, for any $r > 0$, $\varphi(r) < r$, $\varphi(0) = 0$ and $\phi_i \to \varphi$ uniformly on cl(ran)$D_{g_{|A\cup Q}}$ as $n \to \infty$. If there exists $x \in A$ such that the orbit of T at x is bounded, then T has a unique Bregman best proximity point in A. Moreover if $x_0 \in A$ and $x_{n+1} = Tx_n$ for each $n \geq 0$, then $\{x_{2n}\}_{n\in\mathbb{N}}$ converges to the Bregman best proximity point of T.*

Proof. Since $\phi_i \longrightarrow \varphi$ uniformly, and φ is continuous. Now for any $(x,y) \in A \times Q$, we get

$$\limsup_{i\to\infty} D_g{}^*(T^{2i}x, T^{2i}y) \leq \limsup_{i\to\infty} \phi_i(D_g{}^*(x,y)) = \varphi(D_g{}^*(x,y)) < D_g{}^*(x,y). \tag{4.1}$$

We claim that $\lim_{i\to\infty} D_g{}^*(T^{2i}x, T^{2i}y) = 0$ for all $(x,y) \in A \times Q$. Let $\varepsilon > 0$ and $(x,y) \in A \times Q$ be such that $\limsup_{i\to\infty} D_g{}^*(T^{2i}x, T^{2i}y) = \varepsilon$. This implies that

$$\limsup_{i\to\infty} \varphi(D_g{}^*(T^{2i}x, T^{2i}y)) = \varphi(\limsup_{i\to\infty} D_g{}^*(T^{2i}x, T^{2i}y)) = \varphi(\varepsilon) < \varepsilon.$$

Hence, there exists k in $\mathbb{N}$ such that $\varphi(D_g{}^*(T^{2k}x, T^{2k}y)) < \varepsilon$, which, in turn, implies that

$$\begin{aligned} \varepsilon &= \limsup_{i\to\infty} D_g{}^*(T^{2i}x, T^{2i}y) = \limsup_{i\to\infty} D_g{}^*(T^{2i}(T^{2k}x), T^{2i}(T^{2k}y)) \\ &\leq \limsup_{i\to\infty} \phi_i(D_g{}^*(T^{2k}x, T^{2k}y)) = \varphi(D_g{}^*(T^{2k}x, T^{2k}y)) < \varepsilon. \end{aligned}$$

This is a contradiction. So,

$$\lim_{i\to\infty} D_g{}^*(T^{2i}x, T^{2i}y) = 0, \text{ for all } (x,y) \in A \times Q. \tag{4.2}$$

We note that

$$\lim_{m\longrightarrow\infty} \sup_{n\geq m} D_g^*(x_{2m}, x_{2n+1}) = 0.$$

In fact, if $m \in \mathbb{N}$ and $n \geq m$, then there exists a positive integer k such that $n = m + k$ and

$$D_g^*(x_{2m}, x_{2n+1}) = D_g^*(T^{2m}x_0, T^{2n+1}x_0) = D_g^*(T^{2m}x_0, T^{2m}(T^{2k+1}x_0)).$$

Now if $m \longrightarrow \infty$, by (4.2) we have

$$\lim_{m\longrightarrow\infty} D_g{}^*(x_{2m}, x_{2n+1}) = 0 \text{ for all } n \geq m. \tag{4.3}$$

Since (4.3) holds for all $n \geq m$, it follows that

$$\lim_{m\longrightarrow\infty} \sup_{n\geq m} D_g^*(x_{2m}, x_{2n+1}) = 0.$$

Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is a Bregman cyclical Cauchy sequence in E. Hence, in the light of Theorem 2.1, $x_{2n} \to x \in A$ and $x_{2n+1} \to y \in Q$ as $n \to \infty$ and $D_g(x,y) = \mathrm{Bdist}(A,Q)$. On the other hand, we have

$$\begin{aligned} D_g{}^*(x,Tx) &= \limsup_{n\to\infty} D_g{}^*(x_{2n},Tx) \\ &= \limsup_{n\to\infty} D_g{}^*(Tx_{2n-1},Tx) \\ &\leq \limsup_{n\to\infty} \phi_1(D_g{}^*(x,x_{2n-1})) \\ &= \phi_1(\limsup_{n\to\infty} D_g{}^*(x,x_{2n-1})) \\ &= \phi_1(D_g{}^*(x,y)) \\ &\leq D_g{}^*(x,y) \\ &= 0, \end{aligned}$$

which means that $D_g(x,Tx) = \mathrm{Bdist}(A,Q)$. This completes the proof. □

5. Bregman asymptotic proximal pointwise contraction mappings

In this section, we prove the existence of Bregman best proximity points for a new class of Bregman asymptotic proximal pointwise contraction mappings. We first mention the notion of Bregman asymptotic pointwise contraction mapping which was introduced in [18].

Definition 5.1. Let E be a reflexive Banach space and let $g : E \to (-\infty,+\infty]$ be a strongly coercive Bregman function. A mapping $T : E \longrightarrow E$ is said to be a Bregman asymptotic contraction if there exists a sequence of functions $\alpha_n : E \longrightarrow \mathbb{R}^+$ such that $\alpha_n \longrightarrow \alpha$ pointwise on E and for each integer $n \geq 1$,

$$D_g(T^n x, T^n y) \leq \alpha_n(x) D_g(x,y), \ \forall x,y \in E.$$

It was announced in [18] that any Bregman asymptotic pointwise contraction defined on a bounded closed convex subset of a reflexive Banach space has a fixed point.

The main purpose of this section is the approximation of Bregman best proximity points in uniformly convex Banach spaces by the Picard iteration. To begin, we define the new notion of a Bregman asymptotic proximal pointwise contraction mapping as follows.

Definition 5.2. Let E be a reflexive Banach space and let $g : E \to (-\infty,+\infty]$ be a strongly coercive Bregman function. Let (A,Q) be a nonempty pair in a Banach space E. A mapping $T : A\cup Q \longrightarrow A\cup Q$ is said to be Bregman relatively nonexpansive if T is cyclic and for all $(x,y) \in A\times Q$,

$$D_g(Ty,Tx) \leq D_g(x,y).$$

Definition 5.3. Let E be a reflexive Banach space and let $g : E \to (-\infty,+\infty]$ be a strongly coercive Bregman function. Let (A,Q) be a nonempty pair in E. A mapping $T : A\cup Q \longrightarrow A\cup Q$ is said to be a Bregman asymptotic proximal pointwise contraction if T is cyclic and there exists a function $\alpha : A\cup Q \longrightarrow [0,1)$ such that for any integer $n \geq 0$ and $(x,y) \in A\times Q$,

$$D_g(T^{2n}x, T^{2n}y) \leq \max\{\alpha_n(x) D_g(x,y), \mathrm{Bdist}(A,Q)\} \text{ for all } y \in Q,$$
$$D_g(T^{2n}x, T^{2n}y) \leq \max\{\alpha_n(y) D_g(x,y), \mathrm{Bdist}(A,Q)\} \text{ for all } x \in A,$$

where $\alpha_n \longrightarrow \alpha$ pointwise on $A\cup Q$ as $n \to \infty$.

The following theorem provides an affirmative answer to Question 1.1 in a Banach space setting.

Theorem 5.1. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let (A, Q) be a nonempty bounded closed convex pair in E and let $T : A \cup Q \longrightarrow A \cup Q$ be a Bregman asymptotic proximal pointwise contraction mapping which is also Bregman relatively mapping, that is,*

$$D_g(Tx, Ty) \leq D_g(y, x),\ x \in A,\ y \in Q.$$

Then there exits a unique pair $(v^*, u^*) \in A \times Q$ such that

$$D_g(v^*, Tv^*) = D_g(Tu^*, u^*) = Bdist(A, Q).$$

Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$ for each $n \geq 0$, then $\{x_{2n}\}_{n\in\mathbb{N}}$ converges in norm to v^* and $\{x_{2n+1}\}_{n\in\mathbb{N}}$ converges in norm to u^*.

Proof. Let $x_0 \in A$ be fixed. We define a function $f : Q \longrightarrow [0, \infty)$ by $f(u) = \limsup_{n\longrightarrow\infty} D_g(T^{2n}x_0, u)$. By the Bregman-Opial-like property of E, we conclude that f has a unique minimizer, say u^*. We note that, for all integers $m \geq 1$ and $u \in Q$,

$$\begin{aligned} f(T^{2m}u) &= \limsup_{n\longrightarrow\infty} D_g(T^{2n}x_0, T^{2m}u) = \limsup_{n\longrightarrow\infty} D_g(T^{2n+2m}x_0, T^{2m}u) \\ &= \limsup_{n\longrightarrow\infty} D_g(T^{2m}(T^{2n}x_0), T^{2m}u) \leq \limsup_{n\longrightarrow\infty} \max\{\alpha_m(u)D_g(T^{2n}x_0, u), \text{Bdist}(A, Q)\} \\ &= \max\{\alpha_m(u)f(u), \text{Bdist}(A, Q)\}. \end{aligned}$$

Since u^* is the minimum of f, we have

$$f(u^*) \leq f(T^{2m}u^*) \leq \max\{\alpha_m(u^*)f(u^*), \text{Bdist}(A, Q)\},\ \text{for all } m \geq 1. \tag{5.1}$$

Since $\alpha_m(u^*) \longrightarrow \alpha(u^*) < 1$, we get

$$f(u^*) \leq \max\{\alpha(u^*)f(u^*), \text{Bdist}(A, Q)\}.$$

This proves that $f(u^*) = \text{Bdist}(A, Q)$. On the other hand,

$$f(T^2u^*) = \limsup_{n\longrightarrow\infty} D_g(T^{2n}x_0, T^2u^*) \leq \limsup_{n\longrightarrow\infty} D_g(T^{2n-2}x_0, u^*) = f(u^*),$$

which shows that $T^2u^* = u^*$, by the uniqueness of minimum of f. Then u^* is a fixed point of T^2 in Q. We also note that

$$\lim_{m\longrightarrow\infty} \sup_{n\geq m} D_g(T^{2m}x_0, T^{2n}u^*) = \lim_{m\longrightarrow\infty} D_g(T^{2m}x_0, u^*) = f(u^*) = \text{Bdist}(A, Q).$$

Since (A, Q) has the property BUC, it follows from Lemma 3.1 that $\{T^{2n}x_0\}_{n\in\mathbb{N}}$ is a Bregman Cauchy sequence, so that there exists $\tilde{x} \in A$ such that $x_{2n} \longrightarrow \tilde{x}$. For any fixed $y_0 \in Q$, let us define a function $h : C \to \mathbb{R}$ by

$$h(v) = \limsup_{n\to\infty} D_g(v, T^{2n}y_0),\ v \in C.$$

Employing Lemma 2.3 , it is easy to see that h is a convex function. Below we show that h is coerceive. Indeed,

$$\begin{aligned}
h(v_k) &= \limsup_{n\to\infty} D_g(v_k, T^{2n}y_0)\\
&= \limsup_{n\to\infty}[g(v_k)-g(T^{2n}y_0)-\langle v_k-T^{2n}y_0, \nabla g(T^{2n}y_0)\rangle]\\
&\geq \limsup_{n\to\infty}[g(v_k)-g(T^{2n}y_0)-\|v_k-T^{2n}y_0\|\|\nabla g(T^{2n}y_0)\|]\\
&\geq \limsup_{n\to\infty}[g(v_k)-g(T^{2n}y_0)-(\|v_k\|+\|T^{2n}y_0\|)\|\nabla g(T^{2n}y_0)\|]\\
&\geq \limsup_{n\to\infty}[g(v_k)-g(T^{2n}y_0)-\|v_k\|\|\nabla g(T^{2n}y_0)\|-\|T^{2n}y_0\|\|\nabla g(T^{2n}y_0)\|],\ \forall k\in\mathbb{N}.
\end{aligned}$$

Without loss of generality, we may assume that $\|v_k\|\neq 0$ for each $k\in\mathbb{N}$. This implies that

$$\begin{aligned}
\frac{h(v_k)}{\|v_k\|} &\geq \limsup_{n\to\infty}\Big[\frac{g(v_k)}{\|v_k\|}-\frac{g(T^{2n}y_0)}{\|v_k\|}-\|\nabla g(T^{2n}y_0)\|-\frac{\|T^{2n}y_0\|\|\nabla g(T^{2n}y_0)\|}{\|v_k\|}\Big]\\
&= \frac{g(v_k)}{\|v_k\|}-\limsup_{n\to\infty}\frac{g(T^{2n}y_0)}{\|v_k\|}-\limsup_{n\to\infty}\|\nabla g(T^{2n}y_0)\|-\limsup_{n\to\infty}\frac{\|T^{2n}y_0\|\|\nabla g(T^{2n}y_0)\|}{\|v_k\|},
\end{aligned}$$

for all k in $\mathbb{N}$. Since g is strongly coercive, we conclude that

$$\lim_{\|v_k\|\to\infty}\frac{h(v_k)}{\|v_k\|}=\infty.$$

By Theorem 2.2 , h takes its minimum in exactly one point, v^*, which is a fixed point of T^2 in A, also $T^{2n}y_0 \longrightarrow \tilde{y}\in Q$. Hence we obtain $u^*=T^{2n}u^*\longrightarrow \tilde{y}$ and $v^*=T^{2n}v^*\longrightarrow \tilde{x}$. This shows that $(v^*,u^*)=(\tilde{x},\tilde{y})$, and $T^{2n}x_0\longrightarrow v^*, T^{2n}y_0\longrightarrow u^*$. Further

$$D_g(v^*,u^*)=D_g(T^{2n}v^*,T^{2n}u^*)\leq \max\{\alpha_n(v^*)D_g(v^*,u^*),\mathrm{Bdist}(A,Q)\}.$$

Now if $n\longrightarrow\infty$, then $D_g(v^*,u^*)=\mathrm{Bdist}(A,Q)$. Let us show that v^* is the unique element of A which satisfies that $D_g(v^*,u^*)=\mathrm{Bdist}(A,Q)$. If not, then there exists $w^*\in A$ such that $D_g(w^*,u^*)=\mathrm{Bdist}(A,Q)$. In view of Lemma 2.3 and the fact that $D_g(v^*,u^*)=\mathrm{Bdist}(A,Q)=D_g(w^*,u^*)$ we infer that

$$D_g\Big(\frac{v^*+w^*}{2},Tx\Big)\leq\frac{1}{2}D_g(v^*,u^*)+\frac{1}{2}D_g(w^*,u^*)-\frac{1}{4}\rho_{s_2}(\|v^*-w^*\|)<\mathrm{Bdist}(A,Q),$$

where $s_2=\max\{\|v^*\|,\|w^*\|\}$. This is a contradiction, which entails to $v^*=w^*$. From the definition of T, we infer that

$$D_g(Tv^*,Tu^*)\leq D_g(v^*,u^*)=\mathrm{Bdist}(A,Q).$$

Therefore, $Tv^*=u^*$ and $Tu^*=v^*$. This implies that

$$D_g(v^*,Tv^*)=D_g(Tu^*,u^*)=\mathrm{Bdist}(A,Q).$$

This completes the proof. $\square$

6. Existence of Bregman best proximity points without the property BUC

In this section, based on Zorn's lemma, we prove the existence of Bregman best proximity points without using the BUC property in Banach space E. We shall use some notations and definitions as follows. Let (A_1,Q_1) be a nonempty pair in a Banach space $(E,\|.\|)$. We define $(A_1,Q_1)\subseteq(A_2,Q_2)\Longleftrightarrow A-1\subseteq Q_1$ and $A_2\subseteq Q_2$. We shall also use the following notations:

$$\delta_g(x,A_1)=\sup\{D_g(x,y):y\in A_1\} \text{ for all} x\in E,$$
$$\delta_g(A_1,Q_1)=\sup\{D_g(x,y):x\in A_1,y\in Q_1\}.$$

Theorem 6.1. *Let E be a reflexive Banach space and let $g:E\to(-\infty,+\infty]$ be a strongly coercive Bregman function. Let (A,Q) be a nonempty weakly compact convex pair in E. Assume that $T:A\cup Q\longrightarrow A\cup Q$ is a cyclic map such that*

$$D_g(Ty,Tx)\le r\max\{D_g(x,y),D_g(x,Tx),D_g(Ty,y)\}+(1-r)Bdist(A,Q) \tag{6.1}$$

for some $r\in[0,1)$ and for all $x\in A,y\in Q$. If $BBPP(T)$ denotes the set of all Bregman best proximity points of T, then both $BBPP(T)\cap A$ and $BBPP(T)\cap Q$ are nonempty weakly compact convex subsets of E.

Proof. Let $\sum$ denote the collection of all nonempty weakly compact convex pairs (A_1,Q_1), which are sunsets of (A,Q) and such that T is cyclic on $A_1\cup Q_1$. Then $\sum$ is nonempty, since $(A,Q)\in\sum$, which $\sum$ is partially ordered by revers inclusion, that is, $(A,Q)\le(A_1,Q_1)\Longleftrightarrow(A_1,Q_1)\subseteq(A,Q)$. It is easy to check that every increasing chain in $\sum$ is bounded above. Hence, by Zorn's lemma, we can get a minimal element, say $(K_1,K_2)\in\sum$. It follows that

$$(\overline{co}(T(K_2)),\overline{co}(T(K_1)))\subseteq(K_1,K_2).$$

Moreover

$$T(\overline{co}(T(K_1)))\subseteq T(K_1)\subseteq\overline{co}(T(K_1)),$$

and

$$T(\overline{co}(T(K_1)))\subseteq\overline{co}(T(K_2)).$$

Now by the minimality of (K_1,K_2), we have $\overline{co}(T(K_2))=K_1\overline{co}(T(K_1))=K_2$. Letting $a\in K_1$, we find $K_2\subseteq B_g(a;\delta_g(a,K_2))$. If $y\in K_2$, then

$$\begin{aligned}
D_g(Ta,Ty) &\le r\max\{D_g(a,y),D_g(a,Ta),D_g(Ty,y)\}+(1-r)\text{Bdist}(A,Q)\\
&\le r\max\{\delta_g(a,K_2),\delta_g(a,K_2),\delta_g(K_1,K_2)\}+(1-r)\text{Bdist}(A,Q)\\
&= r\delta_g(K_1,K_2)+(1-r)\text{Bdist}(A,Q).
\end{aligned}$$

Therefore, for all $y\in K_2$, we have

$$Ty\in B_g(Ta;r\delta_g(K_1,K_2)+(1-r)\text{Bdist}(A,Q)).$$

So

$$T(K_2)\subseteq B_g(Ta;r\delta_g(K_1,K_2)+(1-r)\text{Bdist}(A,Q)).$$

It follows that

$$K_1 = \overline{co}(T(K_2)) \subseteq B_g(Ta; r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q)).$$

This implies that

$$D_g(x, Ta) \leq r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q) \text{ for all } x \in K_1.$$

Thus

$$\delta_g(Ta, K_1) \leq r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q). \tag{6.2}$$

Also if $b \in K_2$, we have

$$\delta_g(Tb, K_2) \leq r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q). \tag{6.3}$$

Put

$$E_1 := \{x \in K_1 : \delta_g(x, K_2) \leq r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q)\},$$
$$E_2 := \{y \in K_2 : \delta_g(y, K_1) \leq r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q)\}.$$

Then (E_1, E_2) is nonempty. Indeed, by (6.2) and (6.3), we have $T(K_2) \subseteq E_1$ and $T(K_1) \subseteq E_2$. On the other hand, we have

$$E_1 = \bigcap_{y \in K_2} B_g(y; r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q)) \cap K_1,$$
$$E_2 = \bigcap_{x \in K_1} B_g(x; r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q)) \cap K_2.$$

Moreover if $a \in E_1$, then by (6.2) $Ta \in E_2$, i.e., $T(E_1) \subseteq E_2$ and also by (6.3) $T(E_2) \subseteq E_1$. Therefore T is cyclic on $E_1 \cup E_2$. Now by the minimality of (K_1, K_2) we must have $E_1 = K_1$ and $E_2 = K_2$. Thus we obtain

$$\delta_g(x, K_2) \leq r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q), \text{ for all } x \in K_1,$$

which implies that

$$\delta_g(K_1, K_2) = \sup_{x \in K_1} \delta_g(x, K_2) \leq r\delta_g(K_1, K_2) + (1-r)\text{Bdist}(A, Q).$$

Therefore

$$(1-r)\delta_g(K_1, K_2) \leq (1-r)\text{Bdist}(A, Q),$$

or

$$\delta_g(K_1, K_2) = \text{Bdist}(A, Q).$$

Now we have

$$D_g(x, Tx) \leq \delta_g(x, K_2) = \text{Bdist}(A, Q) \text{ for all } x \in K_1.$$

This shows that each point in K_1 is a Bregman best proximity point and hence $BBPP(T) \cap C = K_1$. By a similar argument, we see that $BBPP(T) \cap Q = K_2$. The proof is complete. □

Corollary 6.1. *Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let (A, Q) be a nonempty bounded closed convex pair in E. Assume that $T : A \cup Q \longrightarrow A \cup Q$ is a cyclic mapping such that*

$$D_g(Ty, Tx) \leq r \max\{D_g(x, y), D_g(x, Tx), D_g(Ty, y)\} + (1 - r)Bdist(A, Q)$$

for some $r \in [0, 1)$ and for all $x \in A, y \in Q$. Then both $BBPP(T) \cap A$ and $BBPP(T) \cap Q$ are nonempty bounded closed convex subsets of E.

Remark 6.1. The main result of [1, Theorem 2.3] gave a best proximity point theorem for a continuous cyclic mapping on a closed convex set A, while the main result (Theorem 4.1) of the present paper gives a Bregman best proximity point theorem for a Bregman cyclic (not necessarily continuous) mapping on a closed convex subset A of a reflexive Banach space E. We note that the proof of Theorem 2.3 of [1] (the authors used the continuity assumption of mapping T) is not valid in our results. So we extend and improve the corresponding results of [1].

We also improve the main result of [1] in the following aspects.

(1) For the structure of Banach spaces, we extend the duality mapping to more general case, that is, a convex, continuous and strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets.

(2) For the mappings, we extend the mapping from cyclic mappings to Bregman cyclic (not necessarily continuous) mappings.

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