J. Nonlinear Var. Anal. 3 (2019), No. 1, pp. 45-52
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.3.2019.1.05

MINIMAX THEOREMS IN A FULLY NON-CONVEX SETTING

BIAGIO RICCERI

Department of Mathematics and Informatics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy

Dedicated to Professor Wataru Takahashi, with esteem and friendship, on his 75th birthday

Abstract. In this paper, we establish two minimax theorems for a function $f : X \times I \to \mathbf{R}$, where I is a real interval, without assuming that $f(x,\cdot)$ is quasi-concave. Also, some related applications are presented.

Keywords. Minimax theorem; Connectedness; Real interval; Global extremum.

2010 Mathematics Subject Classification. 49J35, 49K35, 49K27, 90C47.

1. Statements of the Main Results and Preliminaries

The most known minimax theorem ([7]) ensures the occurrence of the equality

$$\sup_Y \inf_X f = \inf_X \sup_Y f$$

for a function $f : X \times Y \to \mathbf{R}$ under the following assumptions: X, Y are convex sets in Hausdorff topological vector spaces, one of them is compact, and f is lower semicontinuous and quasi-convex in X, and upper semicontinuous and quasi-concave in Y.

In the past years, we provided some contributions to the subject where, keeping the assumption of quasi-concavity on $f(x,\cdot)$, we proposed alternative hypotheses on $f(\cdot,y)$. Precisely, in [2], we assumed the inf-connectedness of $f(\cdot,y)$ and, the same time, that Y is a real interval, while, in [5], we assumed the inf-compactness and uniqueness of the global minimum of $f(\cdot,y)$.

In the present paper, we offer a new contribution where the hypothesis that $f(x,\cdot)$ is quasi-concave is no longer assumed.

Let T be a topological space. A function $g : T \to [-\infty,+\infty[$ is said to be relatively inf-compact if, for each $r \in \mathbf{R}$, there exists a compact set $K \subseteq T$ such that $g^{-1}(]-\infty,r[) \subseteq K$. Moreover, g is said to be inf-connected if, for each $r \in \mathbf{R}$, the set $g^{-1}(]-\infty,r[)$ is connected. For the basic notions on multifunctions, we refer to [1].

Our main results are as follows.

Theorem 1.1. *Let X be a topological space. Let I be a real interval and let $f : X \times I \to \mathbf{R}$ be a continuous function such that, for each $\lambda \in I$, the set of all global minima of the function $f(\cdot,\lambda)$ is connected. Moreover, assume that there exists a non-decreasing sequence of compact intervals, $\{I_n\}$, with $I = \cup_{n\in\mathbf{N}} I_n$, such that, for each $n \in \mathbf{N}$, the following conditions are satisfied:*

E-mail address: ricceri@dmi.unict.it.

Received December 17, 2018; Accepted February 10, 2019.

(a_1) *the function* $\inf_{\lambda\in I_n} f(\cdot,\lambda)$ *is relatively inf-compact ;*
(b_1) *for each* $x\in X$*, the set of all global maxima of the restriction of the function* $f(x,\cdot)$ *to* I_n *is connected.*

Then, one has

$$\sup_Y \inf_X = \inf_X \sup_Y f \ .$$

Theorem 1.2. *Let* X *be a topological space. Let* I *be a compact real interval and let* $f : X\times I \to \mathbf{R}$ *be an upper semicontinuous function such that* $f(\cdot,\lambda)$ *is continuous for all* $\lambda \in I$*. Assume that:*
(a_2) *there exists a set* $D\subseteq I$*, dense in* I*, such that the function* $f(\cdot,\lambda)$ *is inf-connected for all* $\lambda\in D$ *;*
(b_2) *for each* $x\in X$*, the set of all global maxima of the function* $f(x,\cdot)$ *is connected.*

Then, one has

$$\sup_Y \inf_X = \inf_X \sup_Y f.$$

Remark 1.1. In both Theorems 1.1 and 1.2, it is essential that I is a real interval. To see this, consider the following example. Take

$$X = I = \{(t,s)\in \mathbf{R}^2 : t^2+s^2 = 1\}$$

and define $f : X\times I\to \mathbf{R}$ by

$$f(t,s,u,v) = tu+sv$$

for all $(t,s),(u,v)\in X$. Clearly, f is continuous, $f(\cdot,\cdot,u,v)$ is inf-connected and has a unique global minimum, and $f(t,s,\cdot,\cdot)$ has a unique global maximum. However, we have

$$\sup_X \inf_I f = -1 < 1 = \inf_X \sup_I f.$$

The common key tool in our proofs of Theorems 1.1 and 1.2 is provided by the following general principle.

Theorem A. [2, Theorem 2.2] *Let* X *be a topological space. Let* I *be a compact real interval and let* $S\subseteq X\times I$ *be a connected set whose projection on* I *is the whole of* I*. Then, for every upper semicontinuous multifunction* $\Phi : X\to 2^I$*, with non-empty, closed and connected values, the graph of* Φ *intersects* S*.*

Another known proposition which is used in the proof of Theorem 1.1 is as follows.

Proposition A. [5, Proposition 2.1] *Let* X *be a topological space and let* Y *be a non-empty set. Let* $y_0\in Y$ *and let* $f : X\times Y\to \mathbf{R}$ *be a function such that* $f(\cdot,y)$ *is lower semicontinuous for all* $y\in Y$ *and relatively inf-compact for* $y = y_0$*. Assume also that there is a non-decreasing sequence of sets* $\{Y_n\}$*, with* $Y = \cup_{n\in\mathbf{N}} Y_n$*, such that*

$$\sup_{Y_n} \inf_X f = \inf_X \sup_{Y_n} f$$

for all $n\in \mathbf{N}$*.*

Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f \ .$$

A further result which is used in the proofs of Theorems 1.1 and 1.2 is provided by the following proposition which, in the given generality, is new.

Proposition 1.1. *Let X,Y be two topological spaces and let $f : X\times Y \to \mathbf{R}$ be a lower semicontinuous function such that $f(x,\cdot)$ is continuous for all $x\in X$. Moreover, assume that, for each $y\in Y$, there exists a neighbourhood V of y such that the function $\inf_{v\in V} f(\cdot,v)$ is relatively inf-compact. For each $y\in Y$, set*

$$F(y)=\left\{u\in X : f(u,y)=\inf_{x\in X} f(x,y)\right\}.$$

Then, the multifunction F is upper semicontinuous.

Proof. Let $C\subseteq X$ be a closed set. We have to prove that $F^-(C)$ is closed. So, let $\{y_\alpha\}_{\alpha\in D}$ be a net in $F^-(C)$ converging to some $\tilde{y}\in Y$. For each $\alpha\in D$, pick

$$u_\alpha\in F(y_\alpha)\cap C.$$

By assumption, there is a neighbourhood V of $\tilde{y}$ such that the function $\inf_{v\in V} f(\cdot,v)$ is relatively inf-compact. Since the function $\inf_{x\in X} f(x,\cdot)$ is upper semicontinuous, we can assume that it is bounded above on V. Fix $\rho>\sup_V \inf_X f$. Then, there is a compact set $K\subseteq X$ such that

$$\left\{x\in X : \inf_{v\in V} f(x,v)<\rho\right\}\subseteq K\ .$$

But

$$\left\{x\in X : \inf_{v\in V} f(x,v)<\rho\right\}=\bigcup_{v\in V}\{x\in X : f(x,v)<\rho\}\,.$$

It follows that

$$\bigcup_{v\in V}\{x\in X : f(x,v)<\rho\}\subseteq K. \tag{1.1}$$

Let $\alpha_1\in D$ be such that $y_\alpha\in V$ for all $\alpha\geq\alpha_1$. Consequently, by (1.1), $u_\alpha\in K$ for all $\alpha\geq\alpha_1$. By compactness, the net $\{u_\alpha\}_{\alpha\in D}$ has a cluster point $\tilde{u}\in K$. Clearly, $(\tilde{u},\tilde{y})$ is a cluster point in $X\times Y$ of the net $\{(u_\alpha,y_\alpha)\}_{\alpha\in D}$. We claim that

$$f(\tilde{u},\tilde{y})\leq\limsup_\alpha f(u_\alpha,y_\alpha)\ .$$

Arguing by contradiction, assume the contrary and fix r so that

$$\limsup_\alpha f(u_\alpha,y_\alpha)<r<f(\tilde{u},\tilde{y})\ .$$

Then, there would be $\alpha_2\in D$ such that

$$f(u_\alpha,y_\alpha)<r$$

for all $\alpha\geq\alpha_2$. On the other hand, since the set $f^{-1}(]r,+\infty[)$ is open, there would be $\alpha_3\geq\alpha_2$ such that

$$r<f(u_{\alpha_3},y_{\alpha_3})$$

which gives a contradiction.

Now, fix $x\in X$. Since $u_\alpha\in F(y_\alpha)$, we have

$$f(\tilde{u},\tilde{y})\leq\limsup_\alpha f(u_\alpha,y_\alpha)\leq\lim_\alpha f(x,y_\alpha)=f(x,\tilde{y}).$$

That is, $\tilde{u}\in F(\tilde{y})$. Since C is closed, $\tilde{u}\in C$. Hence, $\tilde{y}\in F^-(C)$ and this ends the proof. □

2. Proofs and Applications of the Main Results

We now can prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Fix $n \in \mathbf{N}$. Let us prove that

$$\sup_{I_n} \inf_X f = \inf_X \sup_{I_n} f. \tag{2.1}$$

Consider the multifunction $F : I_n \to 2^X$ defined by

$$F(\lambda) = \left\{ u \in X : f(u,\lambda) = \inf_{x \in X} f(x,\lambda) \right\}$$

for all $\lambda \in I_n$. Thanks to Proposition 1.1, F is upper semicontinuous and, by assumption, its values are non-empty, compact and connected. As a consequence, by [1, Theorem 7.4.4], the graph of F is connected. Let S denote the graph of the inverse of F. So, S is connected as it is homeomorphic to the graph of F. Now, consider the multifunction $\Phi : X \to 2^{I_n}$ defined by

$$\Phi(x) = \left\{ \mu \in I_n : f(x,\mu) = \sup_{\lambda \in I_n} f(x,\lambda) \right\}$$

for all $x \in X$. By Proposition 1.1 again, the multifunction Φ is upper semicontinuous and, by assumption, its values are non-empty, closed and connected. After noticing that the projection of S on I_n is the whole of I_n, we can apply Theorem A. Therefore, there exists $(\tilde{x},\tilde{\lambda}) \in S$ such that $\tilde{\lambda} \in \Phi(\tilde{x})$, that is,

$$f(\tilde{x},\tilde{\lambda}) = \inf_{x \in X} f(x,\tilde{\lambda}) = \sup_{\lambda \in I_n} f(\tilde{x},\lambda) \ . \tag{2.2}$$

Clearly, (2.1) follows from (2.2). Now, the conclusion is a direct consequence of Proposition A. This ends the proof.

Proof of Theorem 1.2. Arguing by contradiction, assume the contrary and fix a constant r such that

$$\sup_I \inf_X f < r < \inf_X \sup_I f \ .$$

Let $G : I \to 2^X$ be the multifunction defined by

$$G(\lambda) = \{ x \in X : f(x,\lambda) < r \}$$

for all $\lambda \in I$. Notice that $G(\lambda)$ is non-empty for all $\lambda \in I$ and connected for all $\lambda \in D$. Moreover, the graph of G is open in $X \times I$ and so G is lower semicontinuous. Then, by [3, Proposition 5.7], the graph of G is connected and so the graph of the inverse of G, say S, is connected too. Consider the multifunction $\Phi : X \to 2^I$ defined by

$$\Phi(x) = \left\{ \mu \in I : f(x,\mu) = \sup_{\lambda \in I} f(x,\lambda) \right\}$$

for all $x \in X$. Notice that $\Phi(x)$ is non-empty, closed and connected, in view of (b_2). By Proposition 1.1, the multifunction Φ is upper semicontinuous. Now, we can apply Theorem A. So, there exists $(\hat{x},\hat{\lambda}) \in S$ such that $\hat{\lambda} \in \Phi(\hat{x})$. This implies that

$$f(\hat{x},\hat{\lambda}) < r < \inf_X \sup_I f \leq \sup_{\lambda \in I} f(\hat{x},\lambda) = f(\hat{x},\hat{\lambda})$$

which is absurd. This completes the proof.

Here is an application of Theorem 1.1.

Theorem 2.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a real inner product space. Let $K \subset H$ be a compact and convex set, with $0 \notin K$, and let $f : X \to K$ be a continuous function, where*

$$X = \bigcup_{\lambda \in \mathbf{R}} \lambda K \ .$$

Assume that there are two numbers α, c, with

$$\inf_{x \in X} \|f(x)\| < c < \|f(0)\| \ ,$$

such that:
(a) $\{x \in X : \langle x, f(x)\rangle = \alpha\} \subset \{x \in X : \|f(x)\| < c\}$;
(b) $\{x \in X : c^2\langle x, f(x)\rangle = \alpha\|f(x)\|^2\} \subset \{x \in X : \|f(x)\| \geq c\}$.

Then, there exists $\tilde{\lambda} \in \mathbf{R}$ such that the set

$$\{x \in X : x = \tilde{\lambda} f(x)\}$$

is disconnected.

Proof. Consider the function $\varphi : X \times \mathbf{R} \to \mathbf{R}$ defined by

$$\varphi(x,\lambda) = \|x - \lambda f(x)\|^2 - c^2\lambda^2 + 2\alpha\lambda$$

for all $(x,\lambda) \in X \times \mathbf{R}$. Notice that

$$\varphi(x,\lambda) = \|x\|^2 + (\|f(x)\|^2 - c^2)\lambda^2 - 2(\langle x, f(x)\rangle - \alpha)\lambda \ .$$

Further, observe that, when $\|f(x)\| \geq c$, in view of (a), we have

$$\sup_{\lambda \in \mathbf{R}} \varphi(x,\lambda) = +\infty \tag{2.3}$$

as well as

$$\varphi(x,-\lambda) \neq \varphi(x,\lambda) \tag{2.4}$$

for all $\lambda > 0$. When $\|f(x)\| \geq c$ again, the function $\varphi(x,\cdot)$ is convex and so, by (2.4), for each $\lambda > 0$, its restriction to $[-\lambda,\lambda]$ it has a unique global maximum. Clearly, $\varphi(x,\cdot)$ has the same uniqueness property also when $\|f(x)\| < c$. Now, observe that, for each $\lambda \in \mathbf{R}$, the function λf has a fixed point in X, in view of the Schauder Theorem. Hence, we have

$$\sup_{\lambda \in \mathbf{R}} \inf_{x \in X} \varphi(x,\lambda) = \sup_{\lambda \in \mathbf{R}} (-c^2\lambda^2 + 2\alpha\lambda) = \frac{\alpha^2}{c^2}. \tag{2.5}$$

We claim that

$$\frac{\alpha^2}{c^2} < \inf_{x \in X} \sup_{\lambda \in \mathbf{R}} \varphi(x,\lambda). \tag{2.6}$$

First, observe that, since $0 \notin K$, every closed and bounded subset of X is compact. This easily implies that, for each $\mu > 0$, the function $x \to \inf_{|\lambda| \leq \mu} \varphi(x,\lambda)$ is relatively inf-compact. Consequently, the sublevel sets of the function $x \to \sup_{\lambda \in \mathbf{R}} \varphi(x,\lambda)$ (which is finite if $\|f(x)\| < c$) are compact. Therefore, there exists $\tilde{x} \in X$ such that

$$\sup_{\lambda \in \mathbf{R}} \varphi(\tilde{x},\lambda) = \inf_{x \in X} \sup_{\lambda \in \mathbf{R}} \varphi(x,\lambda). \tag{2.7}$$

So, by (2.3), one has $\|f(\tilde{x})\| < c$. Clearly, we also have

$$\sup_{\lambda \in \mathbf{R}} \varphi(\tilde{x},\lambda) = \|\tilde{x}\|^2 + \frac{|\langle \tilde{x}, f(\tilde{x})\rangle - \alpha|^2}{c^2 - \|f(\tilde{x})\|^2} \ . \tag{2.8}$$

Let us prove that

$$\|\tilde{x}\|^2 + \frac{|\langle \tilde{x}, f(\tilde{x})\rangle - \alpha|^2}{c^2 - \|f(\tilde{x})\|^2} > \frac{\alpha^2}{c^2} . \tag{2.9}$$

After some manipulations, one realizes that (2.9) is equivalent to

$$\frac{1}{c^2 - \|f(\tilde{x})\|^2}\left(2\alpha\langle \tilde{x}, f(\tilde{x})\rangle - |\langle \tilde{x}, f(\tilde{x})\rangle|^2 - \frac{\alpha^2}{c^2}\|f(\tilde{x})\|^2\right) < \|\tilde{x}\|^2 . \tag{2.10}$$

Now, for each $y \in X \setminus \{0\}$, $t \in \mathbf{R}$, set

$$I(y,t) = \{x \in H : \langle x, y\rangle = t\} .$$

Consider the inequality

$$\frac{1}{c^2 - \|y\|^2}\left(2\alpha t - t^2 - \frac{\alpha^2}{c^2}\|y\|^2\right) < \frac{t^2}{\|y\|^2} . \tag{2.11}$$

After some manipulations, one realizes that (2.11) is equivalent to

$$(\alpha\|y\|^2 - tc^2)^2 > 0 .$$

So, (2.11) is satisfied if and only if

$$\alpha\|y\|^2 \neq tc^2 . \tag{2.12}$$

Observe that

$$\frac{|t|}{\|y\|} = \mathrm{dist}(0, I(y,t)) \leq \mathrm{dist}(0, I(y,t) \cap X) . \tag{2.13}$$

Therefore, if (2.12) is satisfied, for each $x \in I(y,t) \cap X$, we conclude from (2.11) and (2.13) that

$$\frac{1}{c^2 - \|y\|^2}\left(2\alpha\langle x, y\rangle - |\langle x, y\rangle|^2 - \frac{\alpha^2}{c^2}\|y\|^2\right) < \|x\|^2 . \tag{2.14}$$

At this point, taking into account that $c^2\langle \tilde{x}, f(\tilde{x})\rangle \neq \alpha\|f(\tilde{x})\|^2$ (by (b)), we draw (2.10) from (2.14) since $\tilde{x} \in I(f(\tilde{x}), \langle \tilde{x}, f(\tilde{x})\rangle)$. Summarizing: taking $I = \mathbf{R}$ and $I_n = [-n, n]$ ($n \in \mathbf{N}$), the continuous function φ satisfies (a_1) and (b_1) of Theorem 1.1, but, in view of (2.5)-(2.9), it does not satisfy the conclusion of that theorem. As a consequence, there exists $\tilde{\lambda} \in \mathbf{R}$ such that the set of all global minima of $\varphi(\cdot, \tilde{\lambda})$ is disconnected. But such a set agrees with the set of all solutions of the equation $x = \tilde{\lambda} f(x)$, and the proof is complete. □

Remark 2.1. We do not know whether Theorem 2.1 is still true when $0 \in K$ and (b) is (necessarily) changed in

$$\{x \in X : f(x) \neq 0,\ c^2\langle x, f(x)\rangle = \alpha\|f(x)\|^2\} \subset \{x \in X : \|f(x)\| \geq c\} .$$

However, the proof of Theorem 2.1 shows that the following is true.

Theorem 2.2. *Let $(X, \langle\cdot,\cdot\rangle)$ be a finite-dimensional real Hilbert space and let $f : X \to X$ be a continuous function with bounded range. Assume that there are two numbers α, c, with*

$$\inf_{x \in X}\|f(x)\| < c < \|f(0)\| ,$$

such that:

(a') $\{x \in X : \langle x, f(x)\rangle = \alpha\} \subset \{x \in X : \|f(x)\| < c\}$;

(b') $\{x \in X : f(x) \neq 0,\ c^2\langle x, f(x)\rangle = \alpha\|f(x)\|^2\} \subset \{x \in X : \|f(x)\| \geq c\}$.

Then, there exists $\tilde{\lambda} \in \mathbf{R}$ *such that the set*

$$\{x \in X : x = \tilde{\lambda} f(x)\}$$

is disconnected.

Finally, we present two applications of Theorem 1.2.

Theorem 2.3. *Let X be a Banach space. Let* $\varphi \in X^* \setminus \{0\}$ *and let* $\psi : X \to \mathbf{R}$ *be a Lipschitzian functional whose Lipschitz constant is equal to* $\|\varphi\|_{X^*}$. *Moreover, let* $[a,b]$ *be a compact real interval,* $\gamma : [a,b] \to [-1,1]$ *a convex (resp. concave) and continuous function, with* $\mathrm{int}(\gamma^{-1}(\{-1,1\})) = \emptyset$*, and* $c \in \mathbf{R}$. *Assume that*

$$\gamma(a)\psi(x) + ca \neq \gamma(b)\psi(x) + cb$$

for all $x \in X$ *such that* $\psi(x) > 0$ *(resp.* $\psi(x) < 0$*).*

Then (with the convention $\sup \emptyset = -\infty$*), one has*

$$\sup_{\lambda \in \gamma^{-1}(\{-1,1\})} \inf_{x \in X} (\varphi(x) + \gamma(\lambda)\psi(x) + c\lambda) = \inf_{x \in X} \sup_{\lambda \in [a,b]} (\varphi(x) + \gamma(\lambda)\psi(x) + c\lambda) .$$

Proof. Consider the continuous function $f : X \times [a,b] \to \mathbf{R}$ defined by

$$f(x,\lambda) = \varphi(x) + \gamma(\lambda)\psi(x) + c\lambda$$

for all $(x,\lambda) \in X \times [a,b]$. By [4, Theorem 2], for each $\lambda \in \gamma^{-1}(]-1,1[)$, the function $f(\cdot,\lambda)$ is inf-connected and unbounded below. Also, notice that $\gamma^{-1}(]-1,1[)$, by assumption, is dense in $[a,b]$. Now fix $x \in X$. If $\psi(x) > 0$ (resp. $\psi(x) < 0$) the function $f(x,\cdot)$ is convex and, by assumption, $f(x,a) \neq f(x,b)$. As a consequence, the unique global maximum of this function is either a or b. If $\psi(x) \leq 0$, the function is concave and so, obviously, the set of all its global maxima is connected. Now, the conclusion follows directly from Theorem 1.2. □

Let $(T,\mathscr{F},\mu)$ be a σ-finite measure space, E a real Banach space and $p \geq 1$.

As usual, $L^p(T,E)$ denotes the space of all (equivalence classes of) strongly μ-measurable functions $u : T \to E$ such that $\int_T \| u(t) \|^p \, d\mu < +\infty$, equipped with the norm

$$\| u \|_{L^p(T,E)} = \left(\int_T \| u(t) \|^p \, d\mu \right)^{\frac{1}{p}} .$$

A set $D \subseteq L^p(T,E)$ is said to be decomposable if, for every $u,v \in D$ and every $A \in \mathscr{F}$, the function

$$t \to \chi_A(t)u(t) + (1-\chi_A(t))v(t)$$

belongs to D, where χ_A denotes the characteristic function of A.

A real-valued function on $T \times E$ is said to be a Caratéodory function if it is measurable in T and continuous in E.

Theorem 2.4. *Let* $(T,\mathscr{F},\mu)$ *be a* σ*-finite non-atomic measure space, E a real Banach space,* $p \in [1,+\infty[$, $X \subseteq L^p(T,E)$ *a decomposable set,* $[a,b]$ *a compact real interval, and* $\gamma : [a,b] \to \mathbf{R}$ *a convex (resp. concave) and continuous function. Moreover, let* $\varphi,\psi,\omega : T \times E \to \mathbf{R}$ *be three Carathéodory functions such that, for some* $M \in L^1(T)$, $k \in \mathbf{R}$*, one has*

$$\max\{|\varphi(t,x)|, |\psi(t,x)|, |\omega(t,x)|\} \leq M(t) + k\|x\|^p$$

for all $(t,x) \in T \times E$ *and*

$$\gamma(a)\int_T \psi(t,u(t))d\mu + a\int_T \omega(t,u(t))d\mu \neq \gamma(b)\int_T \psi(t,u(t))d\mu + b\int_T \omega(t,u(t))d\mu$$

for all $u \in X$ *such that* $\int_T \psi(t,u(t))d\mu > 0$ *(resp.* $\int_T \psi(t,u(t))d\mu < 0$*).*

Then, one has

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T (\varphi(t,u(t)) + \gamma(\lambda)\psi(t,u(t))) + \lambda\omega(t,u(t)))d\mu\right) =$$

$$\inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\int_T (\varphi(t,u(t)) + \gamma(\lambda)\psi(t,u(t))) + \lambda\omega(t,u(t))d\mu\right).$$

Proof. The proof goes on exactly as that of Theorem 2.3. So, one considers the function $f : X \times [a,b] \to \mathbf{R}$ defined by

$$f(u,\lambda) = \int_T (\varphi(t,u(t)) + \gamma(\lambda)\psi(t,u(t))) + \lambda\omega(t,u(t)))d\mu$$

for all $(u,\lambda) \in X \times [a,b]$, and realizes that it satisfies the hypotheses of Theorem 1.2. In particular, for each $\lambda \in [a,b]$, the inf-connectedness of the function $f(\cdot,\lambda)$ is due to [6], Théorème 7. □

Acknowledgments

The author was supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by the Università degli Studi di Catania, "Piano della Ricerca 2016/2018 Linea di intervento 2".

REFERENCES

[1] E. Klein, A.C. Thompson, Theory of correspondences, Wiley, New York, (1984)

[2] B. Ricceri, Some topological mini-max theorems via an alternative principle for multifunctions, Arch. Math. (Basel), 60 (1993), 367-377.

[3] B. Ricceri, Nonlinear eigenvalue problems, in "Handbook of Nonconvex Analysis and Applications" D. Y. Gao and D. Motreanu eds., 543-595, International Press, 2010.

[4] B. Ricceri, On the infimum of certain functionals, in "Essays in Mathematics and its Applications - In Honor of Vladimir Arnold", Th. M. Rassias and P. M. Pardalos eds., 361-367, Springer, 2016.

[5] B. Ricceri, On a minimax theorem: an improvement, a new proof and an overview of its applications, Minimax Theory Appl. 2 (2017), 99-152.

[6] J. Saint Raymond, Connexité des sous-niveaux des fonctionnelles intégrales, Rend. Circ. Mat. Palermo 44 (1995), 162-168.

[7] M. Sion, On general minimax theorems, Pacific J. Math. 8 (1958), 171-176.