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ON A WEAKLY C-ε-VECTOR SADDLE POINT APPROACH IN WEAK VECTOR PROBLEMS

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Dedicated to Professor Wataru Takahashi on the occasion of his 75th birthday with respect and wishes

Abstract. This paper focuses on the study of optimality conditions (both necessary and sufficient) for a weakly C-ε-efficient solution to a weak vector problem by means of a weakly C-ε-vector saddle point approach.

Keywords. Weak vector problems; Weakly C-ε-efficient solutions; Weakly C-ε-vector saddle points; C-convexity.

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1. INTRODUCTION

Let $C \subset \mathbb{R}^p$ be a given cone. In the following, it will be assumed that C is closed, convex and pointed with apex at the origin and with $\operatorname{int} C \neq \emptyset$. Note that cone C is said to be pointed if $x \in C,\ -x \in C \Rightarrow x = 0$, or $C \cap (-C) = \{0\}$, see [14] and the references therein.

In this paper, we are interested in the following weak vector problem [10]:

$$\min\nolimits_{\operatorname{int} C} f(x) \quad \text{subject to} \quad x \in F, \tag{VP}$$

where $\min_{\operatorname{int} C}$ means the vector minimum with respect to cone $\operatorname{int} C$, $f : \mathbb{R}^n \to \mathbb{R}^p$ is a C-convex function (see Definition 2.1), and F is the feasible set of (VP) defined by

$$F := \{x \in \mathbb{R}^n :\ g_j(x) \leqq 0,\ j = 1, \ldots, m\}, \tag{1.1}$$

where $g_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, \ldots, m$ are convex functions.

Below, we recall the concepts of some generalized ε-efficient solutions to problem (VP).

Definition 1.1. Let $\varepsilon \in C$ be given. $\bar{x} \in F$ is said to be

(i) [1, 10] a weakly C-efficient solution to problem (VP) if and only if

$$f(x) \nless_{\operatorname{int} C} f(\bar{x}),\ \ \forall x \in F,$$

where the inequality means $f(x) - f(\bar{x}) \notin -\operatorname{int} C$. At $C = \mathbb{R}^p_+$, problem (VP) is called a weak vector Pareto problem.

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(ii) [12] a weakly C-ε-efficient solution to problem (VP) if and only if

$$f(x) \nleq_{\operatorname{int} C} f(\bar{x}) - \varepsilon, \ \ \forall x \in F,$$

where the inequality means $f(x) - f(\bar{x}) + \varepsilon \notin -\operatorname{int} C$.

Remark 1.1. It is clear that if $\varepsilon = 0$, then a weakly C-ε-efficient solution to problem (VP) will be a weakly C-efficient solution; furthermore, if $C = \mathbb{R}^p_+$, then a weakly C-efficient solution to problem (VP) coincides with a classical weakly efficient solution [7]. Some examples for the mentioned solutions are referred to the paper [12].

In the past years, optimality conditions (and duality) for approximate (efficient) solutions in scalar/vector optimization problems have been studied by a great deal of researchers; see, e.g., [4, 6, 9, 11, 12, 13, 15, 16, 17, 18, 20, 21, 22, 23] and the references therein. It is worth noting that a kind of approximate solutions called quasi ε-solutions, which is motivated by the well-known Ekeland Variational Principle [8], for a scalar optimization problem was minutely studied in [11, 13, 15]; correspondingly, the quasi ε-efficient solutions for a vector optimization problem was investigated in [4, 6, 17, 18, 21, 22]. One of the main tools used in the mentioned papers is to employ the separation theorem of convex sets (see, e.g., [2, 19]) to formulate necessary conditions for approximate efficient solutions of a vector optimization problem, and to employ several kinds of generalized convexity of functions to provide sufficient conditions and duality relations for such approximate efficient solutions.

Besides, according to [5, Chapter 10], there are many approaches to establish approximate optimality conditions for a scalar (and also possible for vector case by scalarization methods) optimization problem; for example, ε-subdifferential approach, max-function approach, ε-saddle point approach, exact penalization approach, and duality-based approach to ε-optimality.

In this paper, we focus on the study of optimality conditions (both necessary and sufficient) for a weakly C-ε-efficient solution to problem (VP) by means of weakly C-ε-vector saddle point approach. The results presented in this paper generalize many previous ones in the literature. The rest of the paper is as follows. Section 2 provides some preliminaries and previous results. The main results of optimality conditions (both necessary and sufficient) for a weakly C-ε-efficient solution to problem (VP) is studied in Section 3. Finally, conclusions are given in Section 4.

2. Preliminaries and Previous Results

We denote by $\mathbb{R}^n$ the Euclidean space of dimension n. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. Let ϕ be a function from $\mathbb{R}^n$ to $\bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = [-\infty, +\infty]$. Here, ϕ is said to be proper if for all $x \in \mathbb{R}^n$, $\phi(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $\phi(x_0) \in \mathbb{R}$. We denote the domain of ϕ by $\operatorname{dom}\phi$, that is, $\operatorname{dom}\phi := \{x \in \mathbb{R}^n \colon \phi(x) < +\infty\}$. A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be proper convex if

$$\phi((1-\mu)x + \mu y) \leqq (1-\mu)\phi(x) + \mu\phi(y),$$

for all $\mu \in [0, 1]$, for all $x, y \in \mathbb{R}^n$.

Let $D = \mathbb{R}^p$ be a p-dimensional Euclidean vectorial space and let D^* be the dual space of D. The positive dual cone to C is denoted by

$$C^* := \{d^* \in D^* \colon \langle d^*, d\,\rangle \geqq 0, \ \forall d \in C\}.$$

Since $D = \mathbb{R}^p$, each element in D^* can be represented as a p-dimensional vector. As usual, we define the C-convex function as follows.

Definition 2.1. [3] Let Γ be a convex subset of $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be C-convex on the convex set Γ if for any $x, y \in \Gamma$ and $t \in [0,1]$,

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in C.$$

The following lemma plays a key role for obtaining our main results.

Lemma 2.1. [12, Corollary 3.1] *Let Γ be a convex subset of $\mathbb{R}^n$. Let $\varepsilon \in C$ and $f : \Gamma \to \mathbb{R}^p$ be a C-convex function. Then, exactly one of the following statements holds.*

(i) *There exists $x_0 \in \Gamma$ such that $f(x_0) - \varepsilon \in -\text{int}C$.*
(ii) *There exists $\lambda \in C^* \setminus \{0\}$ such that $\langle \lambda, f(x) - \varepsilon \rangle \geqq 0, \forall x \in \Gamma$.*

3. Main Results

In this section, we present our main results, i.e., establishing optimality conditions (both necessary and sufficient) for a weakly C-ε-efficient solution to problem (VP) by means of weakly C-ε-vector saddle point approach. Recall the weak vector problem (VP):

$$\min_{\text{int}C} f(x) \quad \text{subject to} \quad x \in F,$$

where $\min_{\text{int}C}$ means vector minimum with respect to cone $\text{int}C$, $f : \mathbb{R}^n \to \mathbb{R}^p$ is a C-convex function, and F is the feasible set of (VP) defined by (1.1).

For problem (VP), its associated vector Lagrangian function is

$$L(x,\mu) = f(x) + \sum_{j=1}^{m} \mu_j g_j(x) e,$$

where $(x,\mu) \in \mathbb{R}^n \times \mathbb{R}^m_+$, and $e := (1,\ldots,1) \in \mathbb{R}^p$.

Below, we give the definition of weakly C-ε-vector saddle points of the vector Lagrangian function L for (VP).

Definition 3.1. Let $\varepsilon \in C$ be given. We say that $(\bar{x}, \bar{\mu}) \in \mathbb{R}^n \times \mathbb{R}^m_+$ is a weakly C-ε-vector saddle point to problem (VP) if

$$L(x,\bar{\mu}) + \varepsilon \not<_{\text{int}C} L(\bar{x},\bar{\mu}) \not<_{\text{int}C} L(\bar{x},\mu) - \varepsilon,$$

for all $x \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m_+$.

Definition 3.2. We say that the Slater constraint qualification holds for problem (VP), if there exists $\hat{x} \in \mathbb{R}^n$ such that $g_j(\hat{x}) < 0$, $j = 1,\ldots,m$.

The following theorem shows the relation between a weakly C-ε-efficient solution and a weakly C-ε-efficient solution for problem (VP) under the fulfilment of the Slater constraint qualification (see Definition 3.2). Moreover, this result will play an important role in the Theorem 3.2.

Theorem 3.1. *Consider the problem (VP) with the feasible set given by (1.1). Let $\varepsilon \in C$ be given, and $\bar{x}$ a weakly C-ε-efficient solution to problem (VP). Assume that the Slater constraint qualification holds for* *problem* (VP). *Then there exist* $\bar{\lambda} \in C^* \setminus \{0\}$ *with* $\bar{\lambda}^T e = 1$, *and* $\bar{\mu}_j \geqq 0,\ j = 1, 2, \ldots, m$, *such that* $(\bar{x}, \bar{\mu})$ *is a weakly* C-ε*-vector saddle point of* (VP), *and* $\bar{\lambda}^T \varepsilon + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) \geqq 0$.

Proof. Since $\bar{x}$ is a weakly C-ε-efficient solution to problem (VP), the following system has no solution

$$f(x) - f(\bar{x}) + \varepsilon \in -\mathrm{int}C,$$
$$g_j(x) < 0,\ j = 1, \ldots, m,$$

where $x \in \mathbb{R}^n$. Then there exists $\bar{\lambda} \in C^* \setminus \{0\}$ with $\bar{\lambda}^T e = 1$ multiplying the above system such that

$$\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon < 0,$$
$$g_j(x) < 0,\ j = 1, \ldots, m,$$

also has no solution, where $x \in \mathbb{R}^n$. Define a set

$$A = \{(z_0, z) \in \mathbb{R} \times \mathbb{R}^m : \text{there exists } x \in \mathbb{R}^n \text{ such that}$$
$$\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon < z_0,\ g_j(x) < z_j, j = 1, 2, \ldots, m\}.$$

Now, we claim that A is convex. To this end, consider (z_0^1, z^1) and (z_0^2, z^2) in A with x_1 and x_2 the respectively associated elements from $\mathbb{R}^n$. For any $t \in [0, 1]$, $x = tx_1 + (1-t)x_2 \in \mathbb{R}^n$. By the C-convexity of f, we have that $\bar{\lambda}^T f$ is a convex function. This, along with the convexity of $g_j,\ j = 1, \ldots, m$ yields that

$$\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon \leqq t\Big(\bar{\lambda}^T f(x_1) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon\Big) + (1-t)\Big(\bar{\lambda}^T f(x_2) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon\Big)$$
$$\leqq tz_0^1 + (1-t)z_0^2,$$
$$g_j(x) \leqq tg_j(x_1) + (1-t)g_j(x_2) \leqq tz_j^1 + (1-t)z_j^2,\ j = 1 \ldots, m.$$

Thus, for every $t \in [0, 1]$, $t(z_0^1, z^1) + (1-t)(z_0^2, z^2) \in A$ with $x \in \mathbb{R}^n$ as associated element. Hence, the convexity of A follows.

Since A is convex, and $(0, 0) \notin A$. It follows from the Separation Theorem [5, Theorem 2.26], there exists $(\mu_0, \mu) \in \mathbb{R} \times \mathbb{R}^m$ with $(\mu_0, \mu) \neq (0, 0)$ such that

$$\mu_0 z_0 + \sum_{i=1}^{m} \mu_j z_j \geqq 0,\ \forall (z_0, z) \in A, \tag{3.1}$$

corresponding to $\bar{x} \in \mathbb{R}^n$, for $z_j > 0,\ j = 0, 1, \ldots, m$, $(z_0, z) \in A$. Also, for any $\alpha > 0$, $(z_0 + \alpha, z) \in A$. Therefore, from condition (3.1), one has

$$\mu_0(z_0 + \alpha) + \sum_{j=1}^{m} \mu_j z_j \geqq 0,\ \forall (z_0, z) \in A,$$

and

$$\mu_0 \geqq -\frac{1}{\alpha}\{\mu_0 z_0 + \sum_{j=1}^{m} \mu_j z_j\},$$

which leads to $\mu_0 \geqq 0$ as the limit $\alpha \to \infty$. Also, we use a similar method to prove $\mu \in \mathbb{R}^m_+$.

For any $x \in \mathbb{R}^n$, we consider a fixed $\beta_j > 0,\ j = 0, 1, \ldots, m$. Then, for any $\alpha_j > 0,\ j = 0, 1, \ldots, m$,

$$\Big(\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon + \alpha_0 \beta_0, g_1(x) + \alpha_1 \beta_1, \ldots, g_m(x) + \alpha_m \beta_m\Big) \in A.$$

Therefore, from (3.1), we have

$$\mu_0(\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon + \alpha_0\beta_0) + \sum_{j=1}^{m} \mu_j (g_j(x) + \alpha_j\beta_j) \geqq 0.$$

As $\alpha_j \to 0$, the above inequality yields

$$\mu_0(\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon) + \sum_{j=1}^{m} \mu_j g_j(x) \geqq 0, \ \forall x \in \mathbb{R}^n. \tag{3.2}$$

We claim that $\mu_0 \neq 0$. On the contrary, suppose that $\mu_0 = 0$. By the Slater constraint qualification, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_j(\hat{x}) < 0, \ j = 1,2,\dots,m$, which implies

$$\sum_{j=1}^{m} \mu_j g_j(\hat{x}) < 0$$

which contradicts (3.2). Therefore, $\mu_0 \neq 0$ and condition (3.2) can be expressed as

$$\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) + \bar{\lambda}^T \varepsilon + \sum_{j=1}^{m} \bar{\mu}_j g_j(x) \geqq 0, \ \forall x \in \mathbb{R}^n, \tag{3.3}$$

where $\bar{\mu}_j = \frac{\mu_j}{\mu_0}$ for $j = 1,2,\dots,m$. In particular, taking $x = \bar{x}$, the above inequality reduces to

$$\bar{\lambda}^T \varepsilon + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) \geqq 0. \tag{3.4}$$

As $g_j(\bar{x}) \leqq 0, i = 1,2,\dots,m$ we obtain from (3.3) that

$$\bar{\lambda}^T f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) \leqq \bar{\lambda}^T f(x) + \sum_{j=1}^{m} \bar{\mu}_j g_j(x) + \bar{\lambda}^T \varepsilon, \ \forall x \in \mathbb{R}^n,$$

which is equivalent to

$$0 \leqq \Big\langle \bar{\lambda}, \big(f(x) + \sum_{j=1}^{m} \bar{\mu}_j g_j(x) e + \varepsilon\big) - \big(f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) e\big) \Big\rangle, \ \forall x \in \mathbb{R}^n.$$

By Lemma 2.1, we have

$$f(x) + \sum_{j=1}^{m} \bar{\mu}_j g_j(x) e + \varepsilon - \big(f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) e\big) \notin -\mathrm{int} C, \ \forall x \in \mathbb{R}^n.$$

which implies

$$f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) e \not>_C f(x) + \sum_{j=1}^{m} \bar{\mu}_j g_j(x) e + \varepsilon, \ \forall x \in \mathbb{R}^n. \tag{3.5}$$

For any $\mu_j \geqq 0, \ j = 1,2,\dots,m$, the feasibility of $\bar{x}$ along with the nonnegativity of $\bar{\lambda}^T \varepsilon$ and (3.4) leads to

$$\bar{\lambda}^T f(\bar{x}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}) - \bar{\lambda}^T \varepsilon \leqq \bar{\lambda}^T f(\bar{x}) - \bar{\lambda}^T \varepsilon \leqq \bar{\lambda}^T f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}),$$

that is,

$$\bar{\lambda}^T f(\bar{x}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}) - \bar{\lambda}^T \varepsilon \leqq \bar{\lambda}^T f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}),$$

which implies

$$f(\bar{x})+\sum_{j=1}^{m}\mu_j g_j(\bar{x})e-\varepsilon \not>_C f(\bar{x})+\sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{x})e,\ \forall \mu\in\mathbb{R}^m_+.$$

The above inequality, along with (3.5), implies that $(\bar{x},\bar{\mu})$ is a weakly C-ε-vector saddle point of (VP), which satisfies (3.4). This completes the proof. □

Theorem 3.2 (Necessary Optimality Condition)**.** *Consider the problem (VP) with the feasible set given by (1.1). Let $\varepsilon\in C$ be given and $\bar{x}$ a weakly C-ε-efficient solution to problem (VP). Assume also that the Slater constraint qualification holds. Then there exist $\bar{\lambda}\in C^+\setminus\{0\}$ with $\bar{\lambda}^Te=1$, $\bar{\varepsilon}_j\in C\setminus\{0\}$, $j=0,\dots,m$, and $\bar{\mu}_j\geqq 0$, $j=1,\dots,m$, with $\bar{\alpha}_0+\sum_{j=1}^{m}\bar{\alpha}_j=\alpha$ such that*

$$0\in\partial_{\bar{\alpha}_0}(\bar{\lambda}^Tf)(\bar{x})+\sum_{j=1}^{m}\partial_{\bar{\alpha}_j}(\bar{\mu}_jg_j)(\bar{x}),$$

$$\alpha+\sum_{j=1}^{m}\bar{\mu}_jg_j(\bar{x})\geqq 0.$$

where $\alpha=\bar{\lambda}^T\varepsilon$, $\bar{\alpha}_j=\bar{\lambda}^T\bar{\varepsilon}_j$, $j=0,\dots,m$.

Proof. Note that $\bar{x}$ is a weakly C-ε-efficient solution of problem (VP). By using theorem 3.1, there exist $\bar{\mu}_j\geqq 0$, $j=1,\dots,m$, such that

$$\bar{\lambda}^Tf(\bar{x})+\sum_{j=1}^{m}\bar{\mu}_jg_j(\bar{x})\leqq\bar{\lambda}^Tf(x)+\sum_{j=1}^{m}\bar{\mu}_jg_j(x)+\bar{\lambda}^T\varepsilon,\ \forall x\in\mathbb{R}^n,$$

along with $\bar{\lambda}^T\varepsilon+\sum_{j=1}^{m}\bar{\mu}_jg_j(\bar{x})\geqq 0$. By the definition of ε-solutions (see [5, Chapter 10]), the above inequality implies that $\bar{x}$ is a $\bar{\lambda}^T\varepsilon$-solution of the following problem

$$\inf\ (\bar{\lambda}^Tf+\sum_{j=1}^{m}\bar{\mu}_jg_j)(x)\quad\text{subject to}\quad x\in\mathbb{R}^n.$$

From [12, Theorem 3.2] and [5, Theorem 10.9], we can obtain the desired results. □

Theorem 3.3 (Sufficient Optimality Condition)**.** *Consider the problem (VP) with the feasible set given by (1.1), and $\varepsilon\in C$. Suppose that f is a C-convex function and g_j, $j=1,\dots,m$ are convex functions. Further, assume that $0\in\partial_{\alpha_0}(\bar{\lambda}^Tf)(\bar{x})+\sum_{j=1}^{m}\partial_{\alpha_j}(\bar{\mu}_jg_j)(\bar{x})$, and $\alpha+\sum_{j=1}^{m}\bar{\mu}_jg_j(\bar{x})\geqq 0$ hold for $(\bar{x},\bar{\mu})\in\mathbb{R}^n\times\mathbb{R}^m_+$, $\alpha_j\geqq 0$, $j=0,\dots,m$, satisfying $\alpha_0+\sum_{j=1}^{m}\alpha_j=\alpha$, where $\alpha=\bar{\lambda}^T\varepsilon$, $\alpha_j=\bar{\lambda}^T\varepsilon_j$, $j=1,\dots,m$. Then $\bar{x}$ is a weakly C-2ε-efficient solution of (VP).*

Proof. Since $0\in\partial_{\alpha_0}(\bar{\lambda}^Tf)(\bar{x})+\sum_{j=1}^{m}\partial_{\alpha_j}(\bar{\mu}_jg_j)(\bar{x})$ hold for $(\bar{x},\bar{\mu})\in\mathbb{R}^n\times\mathbb{R}^m_+$, $\alpha_j\geqq 0$, $j=0,\dots,m$, satisfying $\alpha_0+\sum_{j=1}^{m}\alpha_j=\alpha$, where $\alpha=\bar{\lambda}^T\varepsilon$, $\alpha_j=\bar{\lambda}^T\varepsilon_j$, $j=1,\dots,m$. Then, there exist $\xi_0\in\partial_{\alpha_0}(\bar{\lambda}^Tf)(\bar{x})$, and $\xi_j\in\partial_{\alpha_j}(\bar{\mu}_jg_j)(\bar{x})$, $j=1,\dots,m$ such that

$$\xi_0+\sum_{j=1}^{m}\xi_j=0\in\partial_{\alpha_0}(\bar{\lambda}^Tf)(\bar{x})+\sum_{j=1}^{m}\partial_{\alpha_j}(\bar{\mu}_jg_j)(\bar{x}).\tag{3.6}$$

As well, since f is a C-convex function, and for $\bar{\lambda} \in C^* \setminus \{0\}$, $\bar{\lambda}^T f$ is a convex function. $\bar{\mu}_j g_j$, $j = 1, \dots, m$ are also convex functions. By [5, Definition 2.109] of the ε-subdifferential, one has

$$\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) \geqq \langle \xi_0, x - \bar{x} \rangle - \alpha_0,\ \xi_0 \in \partial_{\alpha_0}(\bar{\lambda}^T f)(\bar{x}),$$

and

$$\bar{\mu}_j g_j(x) - \bar{\mu}_j g_j(\bar{x}) \geqq \langle \xi_j, x - \bar{x} \rangle - \alpha_j,\ \xi_j \in \partial_{\alpha_j}(\bar{\mu}_j g_j)(\bar{x}),\ j = 1, \dots, m.$$

Summing the above inequalities along with condition (3.6) leads to

$$\bar{\lambda}^T f(x) + \sum_{j=1}^{m} \bar{\mu}_j g_j(x) \geqq \bar{\lambda}^T f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) - (\alpha_0 + \sum_{j=1}^{m} \alpha_j).$$

For any x, which is feasible for (VP), $g_j(x) \leqq 0$, $j = 1, \dots, m$, which along with the condition $\alpha + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) \geqq 0$ and the fact that $\alpha_0 + \sum_{j=1}^{m} \alpha_j = \alpha$, we imply

$$\bar{\lambda}^T f(x) \geqq \bar{\lambda}^T f(\bar{x}) - 2\alpha,\ \forall x \in F,$$

which is equivalent to

$$\langle \bar{\lambda}, f(x) - f(\bar{x}) + 2\varepsilon \rangle \geqq 0,\ \forall x \in F.$$

By using Lemma 2.1, there exists no $x \in F$ such that $f(x) - f(\bar{x}) + 2\varepsilon \in -\mathrm{int}C$, which means that $\bar{x}$ is a weakly C-2ε-solution of (VP). □

4. Conclusions

In this paper, we studied optimality conditions (both necessary and sufficient) for a weakly C-ε-efficient solution to problem (VP) via a weakly C-ε-vector saddle point approach. The idea is motivated by [5, Chapter 10]. It will be interesting to investigate the optimality conditions for a (weakly) C-ε-efficient solution to problem (VP) with other approaches.

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