J. Nonlinear Var. Anal. 3 (2019), No. 1, pp. 79-85
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.3.2019.1.08

ASYMPTOTIC BEHAVIOR OF A DYNAMICAL SYSTEM ON A METRIC SPACE

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Abstract. An algorithm for minimizing an objective function on a set can often be viewed as a sequence of self-mappings of the set for which the objective function is a Lyapunov function. In this paper the set is a metric space, which is not necessarily bounded. We study the asymptotic behavior of trajectories of the dynamical system which is induced by the algorithm and generalize results which are known in the case where the metric space is bounded.

Keywords. Dynamical system; Lyapunov function; Metric space; Normal set.

2010 Mathematics Subject Classification. 37B25, 54E35.

1. Introduction and preliminaries

In this paper we study the asymptotic behavior of trajectories of a certain dynamical system which originates in a minimization problem. An algorithm for minimizing an objective function $f : K \to R^1$ on a set K can often be viewed as a sequence of self-mappings $A_t : K \to K$, $t = 1,2,\dots$, of the set K for which the objective function f is a Lyapunov function. More precisely,

$$f(A_t x) \le f(x)$$

for all $x \in K$ and all natural numbers t. In the present paper the set K is a metric space which is not necessarily bounded. We introduce the notion of a normal set of mappings and show that if the sequence $\{A_t\}_{t=1}^{\infty}$ has a subsequence which is a normal set, then the sequence of values of the Lyapunov function f tends to the infimum of f along any trajectory generated by $\{A_t\}_{t=1}^{\infty}$. From the point of view of the theory of dynamical systems, the sequence $\{A_t\}_{t=1}^{\infty}$ describes a nonstationary dynamical system with a Lyapunov function f. Also, some optimization procedures in Hilbert and Banach spaces can be represented in such a manner [8, 9, 13].

We generalize, in particular, the results which were obtained in [6] and presented in Chapter 4 of [15] in the case where K is a bounded, closed and convex set in a Banach space. In this case, our dynamical system was also studied in [13, 14]. In contrast with our previous results, here we no longer assume that K is bounded. In addition, K is, as a matter of fact, a general metric space. We also generalize some results of [16], which were obtained in the case where $A_t = A_1$ for all natural numbers t.

Assume that (K,d) is a metric space and that the function $f : K \to R^1$ is bounded from below. Set

$$\inf(f) := \inf\{f(x) : \ x \in K\}.$$

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Received August 22, 2018; Accepted January 30, 2019.

For each $x \in K$ and each $r > 0$, set

$$B(x,r) := \{y \in K : \ d(y,x) \leq r\}.$$

Denote by $\mathscr{A}$ the set of all self-mappings $A : K \to K$ such that

$$f(Ax) \leq f(x) \text{ for all } x \in K. \tag{1.1}$$

A nonempty set $E \subset \mathscr{A}$ is called normal if given $\varepsilon > 0$ and $M > \inf(f)$, there exists $\delta(\varepsilon, M) > 0$ such that for each $x \in K$ satisfying

$$M \geq f(x) \geq \inf(f) + \varepsilon$$

and each $A \in E$, the inequality

$$f(Ax) \leq f(x) - \delta(\varepsilon, M)$$

holds.

We are now ready to present our main results, Theorems 1.1–1.3. They are established in Sections 2-4, respectively.

Theorem 1.1. *Assume that $\{A_{t_k}\}_{k=1}^{\infty}$ is a subsequence of $\{A_t\}_{t=1}^{\infty} \subset \mathscr{A}$ such that the set $\{A_{t_k} : \ k = 1, 2, \dots\}$ is normal. Then, given any $\varepsilon > 0$ and $M > \inf(f)$, there exists a natural number N such that for each point $x \in K$ satisfying $f(x) \leq M$, we have*

$$f(A_N \cdots A_1 x) \leq \inf(f) + \varepsilon.$$

Theorem 1.2. *Assume that for each $M > \inf(f)$, the function f is uniformly continuous on the set $\{x \in K : \ f(x) \leq M\}$. Assume further that $\{A_{t_k}\}_{k=1}^{\infty}$ is a subsequence of a sequence $\{A_t\}_{t=1}^{\infty} \subset \mathscr{A}$ such that the set $\{A_{t_k} : \ k = 1, 2, \dots\}$ is normal. Then, given any $\varepsilon > 0$ and $M_0 > \inf(f)$, there exist a natural number N and $\delta > 0$ such that for each point $x \in K$ satisfying $f(x) \leq M_0$ and each sequence $\{B_t\}_{t=1}^{N} \subset \mathscr{A}$ which satisfies*

$$d(B_t y, A_t y) \leq \delta, \ t = 1, \dots, N,$$

for each point $y \in K$ satisfying $f(y) \leq M_0$, the inequality

$$f(B_N \cdots B_1 x) \leq \inf(f) + \varepsilon$$

holds.

Theorem 1.3. *Assume that the function f is continuous. Assume further that $\{A_{t_k}\}_{k=1}^{\infty}$ is a subsequence of a sequence $\{A_t\}_{t=1}^{\infty} \subset \mathscr{A}$, where $A_t : K \to K$ is continuous for all natural numbers t, such that the set $\{A_{t_k} : \ k = 1, 2, \dots\}$ is normal. Then, given any $x \in K$ and $\varepsilon > 0$, there exist a natural number N and $\delta > 0$ such that for each sequence $\{x_t\}_{t=0}^{N} \subset K$ satisfying*

$$d(x_0, x) \leq \delta$$

and

$$d(A_t x_{t-1}, x_t) \leq \delta, \ t = 1, \dots, N,$$

the inequality

$$f(x_N) \leq \inf(f) + \varepsilon$$

holds.

Our main results generalize and extend the results of [6, 13, 14], which were obtained in the case where the whole sequence $\{A_t\}_{t=1}^{\infty}$ is normal and K is a bounded, closed and convex set in a Banach space. They also extend some results of [16], which were obtained in the case where $A_t = A_1$ for all natural numbers t.

If the minimization problem

$$f(x) \to \min$$

$$x \in K$$

is well posed [18], then our results imply the convergence of infinite products of the form $A_n \cdots A_1 x$ as $n \to \infty$ for all $x \in K$ to the unique point where the function f attains its minimum. Note that infinite products of operators find application in many areas of mathematics. See, for example, [1, 2, 3, 4, 5, 7, 10, 11, 12, 17] and the references therein.

2. Proof of Theorem 1.1

Let $\varepsilon > 0$ and $M > \inf(f)$ be given and let A_0 be the identity operator on K. We may assume without any loss of generality that $t_1 > 2$. There exists $\delta \in (0,1)$ such that the following property holds:

(a) for each point $x \in K$ satisfying $\inf(f) + \varepsilon \le f(x) \le M$ and each integer $k \ge 1$, we have

$$f(x) - f(A_{t_k} x) \ge \delta.$$

Choose a natural number $k_0 \ge 2$ such that

$$k_0 > \delta^{-1}(M - \inf(f)) \tag{2.1}$$

and set

$$N := t_{k_0}. \tag{2.2}$$

Let $x \in K$ satisfy

$$M \ge f(x). \tag{2.3}$$

We claim that

$$f(A_N \cdots A_1 x) = f(A_{t_{k_0}} \cdots A_1 x) \le \inf(f) + \varepsilon. \tag{2.4}$$

Suppose to the contrary that this does not hold. Then, in view of (1.1), we have

$$f(x) > \inf(f) + \varepsilon \tag{2.5}$$

and

$$f(A_t \cdots A_1 x) > \inf(f) + \varepsilon \tag{2.6}$$

for all $t = 1, \dots, N = t_{k_0}$. By (1.1) and (2.3), for all $t = 1, \dots, N = t_{k_0}$, we have

$$f(A_t \cdots A_1 x) \le M. \tag{2.7}$$

It follows from (2.3), (2.5), (2.6), (2.7) and property (a) that, for all $k = 1, \dots, k_0$,

$$f(A_{t_k} \cdots A_1 x) \le f(A_{t_k - 1} \cdots A_1 x) - \delta. \tag{2.8}$$

By (1.1), (2.2), (2.3) and (2.8), we have

$$\begin{aligned} M - \inf(f) &\geq f(x) - f(A_{t_{k_0}} \cdots A_1 x) \\ &= \sum_{t=1}^{N} (f(A_{t-1} \cdots A_0 x) - f(A_t \cdots A_0 x)) \\ &\geq \sum_{k=1}^{k_0} (f(A_{t_k - 1} \cdots A_0 x) - f(A_{t_k} \cdots A_0 x)) \\ &\geq k_0 \delta \end{aligned}$$

and

$$k_0 \leq \delta^{-1}(M - \inf(f)).$$

This, however, contradicts (2.1). The contradiction we have reached proves (2.4) and Theorem 1.1 itself.

3. Proof of Theorem 1.2

Let $\varepsilon > 0$ and $M_0 > \inf(f)$ be given. We may assume without any loss of generality that $t_1 > 0$. There exists $\delta_0 \in (0,1)$ such that the following property holds:

(a) for each point $x \in K$ satisfying $\inf(f) + \varepsilon \leq f(x) \leq M_0$ and each integer $k \geq 1$, we have

$$f(x) - f(A_{t_k} x) \geq 2\delta_0.$$

Choose a natural number $k_0 \geq 2$ such that

$$k_0 > \delta^{-1}(M_0 - \inf(f)) \tag{3.1}$$

and set

$$N := t_{k_0}. \tag{3.2}$$

There exists $\delta \in (0, \delta_0)$ such that the following property holds:

(b) for each $y_1, y_2 \in K$ satisfying

$$d(y_1, y_2) \leq \delta \text{ and } f(y_1), f(y_2) \leq M_0,$$

we have

$$|f(y_1) - f(y_2)| \leq \delta_0.$$

Let $x \in K$ satisfy

$$M_0 \geq f(x) \tag{3.3}$$

and let $\{B_t\}_{t=1}^N \subset \mathscr{A}$ satisfy

$$d(B_t y, A_t y) \leq \delta,\ t = 1, \ldots, N, \tag{3.4}$$

for each point $y \in K$ satisfying $f(y) \leq M_0$.

We claim that

$$f(B_N \cdots B_1 x) \leq \inf(f) + \varepsilon. \tag{3.5}$$

Suppose to the contrary that this does not hold. Then, in view of (1.1), we have

$$f(x) > \inf(f) + \varepsilon \tag{3.6}$$

and

$$f(B_t \cdots B_1 x) > \inf(f) + \varepsilon \tag{3.7}$$

for all $t = 1, \ldots, N = t_{k_0}$.

Let B_0 be the identity operator on K. By (1.1) and (3.3), for all $t = 0, \dots, N = t_{k_0}$, we have

$$f(B_t \cdots B_0 x) \le M_0. \tag{3.8}$$

Let

$$k \in \{1, \dots, k_0\}. \tag{3.9}$$

In view of (3.6)–(3.9), we also have

$$\inf(f) + \varepsilon \le f(B_{t_k - 1} \cdots B_0 x) \le M_0. \tag{3.10}$$

Property (a) and (3.10) imply that

$$f(A_{t_k} B_{t_k - 1} \cdots B_0 x) \le f(B_{t_k - 1} \cdots B_0 x) - 2\delta_0. \tag{3.11}$$

It follows from (3.2), (2.4), (3.9) and (3.10) that

$$d(B_{t_k} B_{t_k - 1} \cdots B_0 x, A_{t_k} B_{t_k - 1} \cdots B_0 x) \le \delta. \tag{3.12}$$

Property (b), (3.1), (3.10) and (3.12) imply that

$$|f(B_{t_k} B_{t_k - 1} \cdots B_0 x) - f(A_{t_k} B_{t_k - 1} \cdots B_0 x)| \le \delta_0. \tag{3.13}$$

In view of (3.9), (3.11) and (3.13),

$$f(B_{t_k} B_{t_k - 1} \cdots B_0 x) \le f(A_{t_k} B_{t_k - 1} \cdots B_0 x) + \delta_0 \quad \le f(B_{t_k - 1} \cdots B_0 x) - 2\delta_0 + \delta_0,$$

$$f(B_{t_k} B_{t_k - 1} \cdots B_0 x) \le f(B_{t_k - 1} \cdots B_0 x) - \delta_0, \ k = 1, \dots, N. \tag{3.14}$$

It now follows from (3.1), (3.3) and (3.14) that

$$\begin{aligned} M_0 - \inf(f) &\ge f(x) - f(B_N \cdots B_1 x) \\ &= \sum_{t=1}^{N} (f(B_{t-1} \cdots B_0 x) - f(B_t \cdots B_0 x)) \\ &\ge \sum_{k=1}^{k_0} (f(B_{t_k - 1} \cdots B_0 x) - f(B_{t_k} \cdots B_0 x)) \\ &\ge k_0 \delta_0 \end{aligned}$$

and

$$k_0 \le \delta_0^{-1}(M_0 - \inf(f)).$$

This, however, contradicts (3.1). The contradiction we have reached proves (3.5) and Theorem 1.2 itself.

4. Proof of Theorem 1.3

Let $x \in K$ and $\varepsilon > 0$ be given and let $A_0 : K \to K$ denote the identity operator on K. By Theorem 1.1, there exists a natural number N such that

$$f(A_N \cdots A_1 x) \le \inf(f) + \varepsilon/4. \tag{4.1}$$

We may assume that $N > 2$. There exists

$$\delta_N \in (0, \varepsilon/4)$$

such that for each

$$y \in B(A_N \cdots A_1 x, \delta_N),$$

we have

$$|f(y) - f(A_N \cdots A_1 x)| \leq \varepsilon/4. \tag{4.2}$$

There exists a sequence of positive numbers $\{\delta_i\}_{i=0}^N$ such that for each $i \in \{1, \ldots, N\}$,

$$\delta_{i-1} < \delta_i/4$$

and the following property holds:

(a) for each point $y \in B(A_{i-1} \cdots A_0 x, \delta_{i-1})$, we have

$$d(A_i y, A_i \cdots A_0 x) \leq \delta_i/4.$$

Set

$$\delta := \delta_0. \tag{4.3}$$

Assume that $\{x_t\}_{t=0}^N \subset K$ satisfies

$$d(x_0, x) \leq \delta \tag{4.4}$$

and

$$d(A_t x_{t-1}, x_t) \leq \delta, \ \ t = 1, \ldots, N. \tag{4.5}$$

We show by induction that, for all $t = 0, \ldots, N$,

$$d(x_t, A_t \cdots A_0 x) \leq \delta_t. \tag{4.6}$$

In view of (4.3) and (4.4), inequality (4.6) does hold for $t = 0$.

Assume now that $t \in \{0, \ldots, N-1\}$ and that (4.6) holds. Property (a) (with $i = t+1$) and (4.6) imply that

$$d(A_{t+1} x_t, A_{t+1} \cdots A_0 x) \leq \delta_{t+1}/4. \tag{4.7}$$

It follows from (4.3), (4.5) and (4.7) that

$$\begin{aligned} d(x_{t+1}, A_{t+1} \cdots A_0 x) &\leq d(A_{t+1} x_t, x_{t+1}) + d(A_{t+1} x_t, A_{t+1} \cdots A_1 x) \\ &\leq \delta + \delta_{t+1}/4 \\ &\leq \delta_{t+1}. \end{aligned}$$

Thus inequaity (4.6) indeed holds for all $t = 0, \ldots, N$. In particular,

$$d(x_N, A_N \cdots A_1 x) \leq \delta_N. \tag{4.8}$$

By (4.8) and the choice of δ_N (see (4.2)),

$$|f(x_N) - f(A_N \cdots A_1 x)| \leq \varepsilon/4. \tag{4.9}$$

In view of (4.1) and (4.9), we obtain

$$f(x_N) \leq f(A_N \cdots A_1 x) + \varepsilon/4 \leq \inf(f) + \varepsilon/2.$$

This completes the proof of Theorem 1.3.

Acknowledgments

The first author was partially supported by the Israel Science Foundation (Grant No. 820/17), by the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund.

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