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ATTRACTIVE POINTS OF FURTHER 2-GENERALIZED HYBRID MAPPINGS IN A COMPLETE CAT(0) SPACE

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Abstract. In this paper, a new nonlinear mapping, which is said to be further 2-generalized hybrid, is introduced. The new mapping includes further generalized hybrid and normally 2-generalized hybrid mappings as special cases. A Halpern-type iterative scheme is constructed for attractive points of further 2-generalized hybrid mappings in a complete CAT(0) space.

Keywords. Attractive point; Further generalized hybrid mapping; Further 2-generalized hybrid mapping; Hadamard space; Normally 2-generalized hybrid mapping.

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1. INTRODUCTION

Let C be a nonempty subset of a real Hilbert space H and let $T : C \to H$ be a nonlinear mapping. We denote the sets of attractive and fixed points of T by $A(T)$ and $F(T)$, respectively, i.e.,

$$A(T) = \{u \in H : \|Tv - u\| \leq \|v - u\|, \forall v \in C\}$$

and

$$F(T) = \{u \in C : Tu = u\}.$$

The concept of attractive points was first introduced in Hilbert space by Takahashi and Takeuchi [24]. The introduction was basically motivated to get rid of the closedness and convexity hypotheses imposed on nonempty subset $C \subset H$ in a celebrated Bailon's [2] nonlinear ergodic theorem. They established nonlinear ergodic theorem without convexity for generalized hybrid mappings in Hilbert spaces. Recall that a mapping $T : C \to H$ is said to be (α, β)-generalized hybrid [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \ \forall\, x, y \in C.$$

Observe that mapping T is reduced to a nonexpansive mapping, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C$$

if $\alpha = 1$ and $\beta = 0$. If $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, then it is said to be hybrid [15, 23], i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall\, x, y \in C.$$

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Also, it is said to be nonspreading [15, 16], i.e.,

$$2\|Tx-Ty\|^2 \le \|Tx-y\|^2+\|Ty-x\|^2, \quad \forall\, x,y \in C,$$

if $\alpha=2$ and $\beta=1$. Recall that a mapping T is said to be normally generalized hybrid [27] if there exist $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ such that
(a) $\alpha+\beta+\gamma+\delta \ge 0$, (b) $\alpha+\beta>0$ or $\alpha+\gamma>0$ and

$$\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \le 0, \quad \forall x,y \in C.$$

As a generalization of the class of normally generalized hybrid mapping, the classes of normally 2-generalized hybrid and further generalized hybrid mappings were introduced. Recall that a mapping T is said to be

(i) normally 2-generalized hybrid [17] if there exist $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3 \in \mathbb{R}$ such that (a) $\sum_{i=1}^{3}(\alpha_i+\beta_i) \ge 0$ and (b) $\sum_{i=1}^{3}\alpha_i>0$ and

$$\begin{aligned}&\alpha_1\|T^2x-Ty\|^2+\alpha_2\|Tx-Ty\|^2+\alpha_3\|x-Ty\|^2\\ +\quad &\beta_1\|T^2x-y\|^2+\beta_2\|Tx-y\|^2+\beta_3\|x-y\|^2 \le 0, \quad \forall\, x,y \in C.\end{aligned}$$

(ii) further generalized hybrid [22] if there exist $\alpha,\beta,\gamma,\delta,\varepsilon \in \mathbb{R}$ such that
(a) $\alpha+\beta+\gamma+\delta \ge 0$, $\varepsilon \ge 0$ (b) $\alpha+\beta>0$ or $\alpha+\gamma>0$ and

$$\begin{aligned}\alpha\|Tx-Ty\|^2 \quad &+ \quad \beta\|x-Ty\|^2+\gamma\|Tx-y\|^2\\ &+ \quad \delta\|x-y\|^2+\varepsilon\|x-Tx\|^2 \le 0, \quad \forall\, x,y \in C.\end{aligned}$$

Weak and strong convergence theorems for attractive points of the above mentioned generalized nonlinear mappings have been studied in Hilbert spaces by various authors recently; see, for example, [18, 22, 25, 27] and the references contained therein. In 2018, Khan [22] proposed an iterative scheme that converges weakly to a common attractive point of two further generalized hybrid mappings in Hilbert spaces. For the strong convergence, Kondo and Takahashi [18] constructed a Halpern's type iterative scheme that converges strongly to a attractive point of normally 2-generalized hybrid mappings in Hilbert spaces without the condition that the domain of the mapping to be closed. They proved the following theorem.

Theorem 1.1. [18] Let C be a nonempty and convex subset of H, and let $T: C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{\lambda_n\},\{a_n\},\{b_n\}$ and $\{c_n\}$ be sequences of real numbers in $(0,1)$ satisfying some conditions. Let $x_1, z \in C$ and define a sequence $\{x_n\}$ in C as follows

$$x_{n+1}=\lambda_n z+(1-\lambda_n)(a_n x_n+b_n Tx_n+c_n T^2 x_n)$$

for all $n \in \mathbb{N}$. Then $x_n \to \bar{z}=P_{A(T)}z$.

Let (X,d) be a metric space and $x,y \in X$. An isometry $c:[0,d(x,y)] \to X$ satisfying $c(0)=x$ and $c(d(x,y))=y$ is called a geodesic path joining x to y. A geodesic segment between x and y is the image of a geodesic path joining x to y, which is denoted by $[x,y]$ when it is unique. A geodesic space is a metric space (X,d) in which every two points of X are joined by a geodesic segment. It is said to be a uniquely geodesic space if every two points of X are joined by only one geodesic segment. Let X be a uniquely geodesic space and $(1-t)x \oplus ty$ denote the unique point z of the geodesic segment joining x to y for each

$x,y \in X$ such that $d(z,x) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$. Set $[x,y] := \{(1-t)x \oplus ty : t \in [0,1]\}$. Then a subset $C \subset X$ is said to be convex if $[x,y] \subset C$ for all $x,y \in C$.

Lemma 1.1. [3] *Let X be a uniquely geodesic space. The following are equivalent:*

(i) *X is a CAT(0) space.*

(ii) *X satisfies the CAT(0)(CN) inequality i.e. If x,y,z are points in X and q is the midpoint of the segment $[y,z]$, then*

$$d^2(x,q) \leq \frac{1}{2}d^2(x,y) + \frac{1}{2}d^2(x,z) - \frac{1}{4}d^2(y,z).$$

A complete CAT(0) space is called a Hadamard space. There are many Hadamard spaces such as Hilbert spaces, the Hilbert ball [9], Euclidean space $\mathbb{R}^n$, $\mathbb{R}$-trees, hyperbolic space [21] and any complete simply connected Riemanian manifold having non-positive sectional curvature. Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X and for $x \in X$,

$$r(x,\{x_n\}) := \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}$$

and the asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

In a complete CAT(0) sapce, it is known that $A(\{x_n\})$ consists of exactly one point; see [7]. A sequence $\{x_n\}$ is said to be Δ-convergent to a point $x \in X$ if for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $A(\{x_{n_k}\}) = \{x\}$. In this case, x is called the Δ-limit of $\{x_n\}$ and it is written as $\Delta\text{-}\lim_{n\to\infty} x_n = x$.

The concept of the quasilinearisation in a complete CAT(0) space was introduced by Berg and Nicolev [1]. They denoted a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. The quasilinearisation is a map $\langle .,. \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d))$$

for every $a,b,c,d \in X$. Following the definition, it is easy to see that, for all $a,b,c,d,e \in X$,

$$\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b),$$

$$\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle,$$

and

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle.$$

The space X is said to satisfy the Cauchy Schwartz inequality if, for all $a,b,c,d \in X$,

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = d(a,b)d(c,d).$$

It is known (see [1]) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.

The concept of dual space in a complete CAT(0) space X was introduced by Kakavandi and Amini [12] is as follow: Let $C(X,\mathbb{R})$ be the space of all continuous real-valued functions on X. Consider a map $\Theta : \mathbb{R} \times X \times X \to C(X,\mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax}\rangle,\ (t \in \mathbb{R}, a,b,x \in X).$$

The Cauchy-Schwartz inequality implies $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm

$$L(\Theta(t,a,b)) = |t|d(a,b)(t \in \mathbb{R}, a,b \in X),$$

where the Lipschitz semi-norm $L(\phi)$ of any function $\phi : X \to \mathbb{R}$ is given by

$$L(\phi) = \sup\{\frac{\phi(x)-\phi(y)}{d(x,y)} : x,y \in X, x \neq y\}.$$

A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)),$$

for $t,s \in \mathbb{R}$ and $a,b,c,d \in X$. In a complete CAT(0) space, it was shown [12] that $D((t,a,b),(s,c,d)) = 0$ if and only if

$$t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle, \quad \forall x,y \in X.$$

Thus D induces an equivalence relation on $\mathbb{R} \times X \times X$ with equivalence class defined by

$$[t\overrightarrow{ab}] := \{s\overrightarrow{cd} : D((t,a,b),(s,c,d)) = 0\}.$$

The pair (X^*, D) is called the dual space of the metric space (X,d), where

$$X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$$

and the function D on X^* is a metric.

The concept of attractive points of a nonlinear map T was first studied in the setting of CAT(0) spaces by Kunwai, Kaewkhao and Inthakon [19]. They defined and denoted the set of attractive points $A(T)$ by

$$A(T) = \{z \in X : d(z,Ty) \leq d(z,y), \forall x \in C\}$$

and established some of its properties. In 2015, Kaewkhao, Inthakon and Kunwai [10] proved the Δ-convergence of a Mann-type scheme to a point in the set of attractive points of normally generalized hybrid mappings. Recently, Cuntavepanit and Phuengrattana [5] studied the class of further generalized hybrid mappings [22] in Hadamard spaces. They established the demiclosed principle and proved the Δ-convergence for attractive points.

It is our purpose in this paper to introduce a further 2-generalized hybrid mapping, which includes normally 2-generalized hybrid and further generalized hybrid mappings as special cases, in a complete CAT(0) space. We then construct a Halpern's type iterative scheme to find an attractive point of such mappings. Our results improved and generalize many results in the literature.

2. PRELIMINARIES

Throughout this paper, the symbols " $\to$ " and " $\rightharpoonup$ " represent the strong and the Δ-convergence, respectively. The following results play vital roles in establishing our main result.

Lemma 2.1. [7] *Let X be a CAT(0) space and $x,y \in X$, $t \in [0,1]$. Then*

(i) $d(z,tx \oplus (1-t)y) \leq td(z,x) + (1-t)d(z,y)$;
(ii) $d^2(z,tx \oplus (1-t)y) \leq td^2(z,x) + (1-t)d^2(z,y) - t(1-t)d^2(x,y)$.

The following notation was introduced by Dompongsa, Kaewkhao and Panyanak [8] in CAT(0) spaces. Let $x_1, x_2, ..., x_n$ be points in CAT(0) spaces. For $\lambda_1, \lambda_2, ..., \lambda_n \in (0,1)$ with $\sum_{i=1}^n \lambda_i = 1$, we write

$$\oplus_{i=1}^n \lambda_i x_i = (1-\lambda_n)(\frac{\lambda_1}{1-\lambda_n}x_1 \oplus \frac{\lambda_2}{1-\lambda_n}x_2 \oplus ... \oplus \frac{\lambda_{n-1}}{1-\lambda_n}x_{n-1}) \oplus \lambda_n x_n,$$

where the definition of $\oplus$ is an ordered one in the sense that it depends on the order of points $x_1, ..., x_n$.

The following Lemma can be deduced from [4, Lemma 12]. For completeness, we give the shorter proof here.

Lemma 2.2. *Let X be a complete CAT(0) space and $x,y \in X$, $t_i \in (0,1)$. Then*

$$d^2(z, \oplus_{i=0}^n t_i x_i) \leq \sum_{i=0}^n t_i d^2(z,x_i) - t_j t_k d^2(x_j,x_k),$$

where $j,k \in \{0,1,...,n\}$ and $\sum_{i=0}^n t_i = 1$

Proof. Without loss of generality, we assume $j=0, k=1$ and apply induction on n. For $n=1$, the result follows from (ii) of Lemma 2.1. Suppose the result is true for $n=m$, for some $m \geq 1$, i.e.,

$$d^2(z, \oplus_{i=0}^m t_i x_i) \leq \sum_{i=0}^m t_i d^2(z,x_i) - t_0 t_1 d^2(x_0,x_1).$$

We now prove for $n = m+1$. By convexity of distance function in a complete CAT(0) space, we get

$$\begin{aligned} d^2(z, \oplus_{i=0}^{m+1} t_i x_i) &= d^2(z, (1-\alpha_{m+1})(\frac{t_1 x_1}{1-t_{m+1}} \oplus ... \oplus \frac{t_m x_m}{1-t_{m+1}}) \oplus t_{m+1}x_{m+1}) \\ &= d^2(z, (1-t_{m+1}) \oplus_{i=0}^m \frac{t_i x_i}{1-t_{m+1}} \oplus t_{m+1}x_{m+1}) \\ &\leq \sum_{i=0}^m t_i d^2(z,x_i) - t_0 t_1 d^2(x_0,x_1) + t_{m+1} d^2(z,x_{m+1}) \\ &= \sum_{i=0}^{m+1} t_i d^2(z,x_i) - t_0 t_1 d^2(x_0,x_1). \end{aligned}$$

This completes the proof. □

Lemma 2.3. [6] *Let X be a CAT(0) space and $x,y,z \in X$. Then, for all $t \in [0,1]$,*

$$d^2(z,tx \oplus (1-t)y) \leq t^2 d^2(z,x) + (1-t)^2 d^2(z,y) - 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle.$$

Lemma 2.4. [13] *Let X be a complete CAT(0) space. Then every bounded sequence in X has a Δ-convergence subsequence.*

Kakavandi [11] introduced the notion of the $\mathbb{S}$ property for complete CAT(0) spaces and used it to characterize the concept of the Δ-convergence in the space. A complete CAT(0) X is said to satisfy the $\mathbb{S}$ property if, for any $(x,y) \in X \times X$, there exists a point $y_x \in X$ such that $[\overrightarrow{xy}] = [\overrightarrow{y_x x}]$.

Lemma 2.5. [11] *Let X be a complete CAT(0) space that satisfies the $\mathbb{S}$ property. Let $\{x_n\}$ be a bounded sequence in X and $x \in X$. Then $\Delta - \lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$ for all $y \in X$.*

Lemma 2.6. [11] *Let X be a complete CAT(0) space, $\{x_n\}$ a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{x_n x}, \overrightarrow{yx}\rangle \leq 0$, $\forall y \in X$.*

Kaewkhao, Inthakon and Kunwai [10] proved that the set of attractive points of map T is closed and convex in a complete CAT(0) space that satisfy $(\bar{Q}_4)$ condition. A complete CAT(0) space is said to satisfy $(\bar{Q}_4)$ condition [11] if, for any $x,y,p,q \in X$, $d(p,x) < d(x,q)$ and $d(p,y) < d(y,q)$ imply $d(p,m) \leq d(m,q)$, $\forall m \in [x,y]$.

Lemma 2.7. [10] *Let (X,d) be a complete CAT(0) space satisfying the $(\bar{Q}_4)$ condition. Let C be a nonempty subset of X and $T : C \to X$ be any map. Then, $A(T)$ is closed and convex.*

Let l^∞ be the Banach space of bounded sequences with supremum norm and let $\mu : l^\infty \to \mathbb{R}$ be a bounded and linear functional on l^∞. Let $\mu(f)$ (or $\mu_n(x_n)$) denote the value of μ at $f = (x_1,x_2,x_3,.....) \in l^\infty$. A mean μ_n is a linear functional defined on l^∞ satisfying $\mu(e) = \|\mu\| = 1$, where $e = (1,1,1,...)$. And a Banach limit on l^∞ is a mean μ_n such that $\mu_n(x_{n+1}) = \mu_n(x_n)$. The existence a Banach limit on l^∞ is well known. In fact, if μ is a Banach limit on l^∞, then, for $f = (x_1,x_2,x_3,.....) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1,x_2,x_3,.....) \in l^\infty$ and $x_n \to l \in \mathbb{R}$, then $\mu(f) = \mu_n(x_n) = l$. For the proof of the existence of a Banach limit and its elementary properties; see [26].

Lemma 2.8. [10, 19] *Let C be a nonempty subset of a complete CAT(0) space X and $T : C \to C$ be a mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then*

(i) *the sequences $\{d(x_n,y)\}$ and $\{d(Tx_n,y)\}$ are bounded for all $y \in C$;*
(ii) *$\mu_n d(x_n,y) = \mu d(Tx_n,y)$ for any Banach limit μ_n on l^∞.*

Lemma 2.9. [28] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n,\ n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\}$ is a real sequence satisfying the following conditions $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. [20] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} \leq a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1}$$

and

$$a_k \leq a_{m_k+1}$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main Results

In this section, X is considered to be a complete CAT(0) space. We start by introducing a new generalized nonlinear mapping called further 2-generalized hybrid mapping in Hadamard spaces. We then construct a Halpern type iterative scheme that converges strongly to a attractive point of a further 2-generalized hybrid mapping.

Definition 3.1. Let X be a complete CAT(0) space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be further 2-generalized hybrid if there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$ such that (i) $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0, \varepsilon_1, \varepsilon_2 \geq 0$, (ii) $\sum_{i=1}^{3} \alpha_i > 0$ and

$$\begin{aligned}\text{(iii) } & \alpha_1 \mathrm{d}^2(\mathrm{T}^2\mathrm{x}, \mathrm{Ty}) + \alpha_2 \mathrm{d}^2(\mathrm{Tx}, \mathrm{Ty}) + \alpha_3 \mathrm{d}^2(\mathrm{x}, \mathrm{Ty}) + \beta_1 \mathrm{d}^2(\mathrm{T}^2\mathrm{x}, \mathrm{y}) \\ + \quad & \beta_2 d^2(Tx, y) + \beta_3 d^2(x, y) + \varepsilon_1 d^2(x, T^2x) + \varepsilon_2 d^2(x, Tx) \leq 0.\end{aligned}$$

for all $x, y \in C$.

Remark 3.1. If $\alpha_1 = \beta_1 = \varepsilon_2 = 0$, then the mapping is reduced to a further generalized hybrid mapping in the sense of [5]. Also, the mapping is reduced to a normally 2-generalized hybrid mapping if $\varepsilon_1 = \varepsilon_2 = 0$, i.e., there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that (i) $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0$, (ii) $\sum_{i=1}^{3} \alpha_i > 0$ and

$$\begin{aligned}\text{(iii) } \alpha_1 \mathrm{d}^2(\mathrm{T}^2\mathrm{x}, \mathrm{Ty}) \quad + \quad & \alpha_2 d^2(Tx, Ty) + \alpha_3 d^2(x, Ty) + \beta_1 d^2(T^2x, y) \\ + \quad & \beta_2 d^2(Tx, y) + \beta_3 d^2(x, y) \leq 0.\end{aligned}$$

for all $x, y \in C$.

Lemma 3.1. *Let C be a nonempty subset of a complete CAT(0) space X which satisfy the $(\mathbb{S})$ property and let $T : C \to C$ be a further 2-generalized hybrid mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta - \lim_{n\to\infty} x_n = z$ and $d(x_n, Tx_n) \to 0$, $d(x_n, T^2x_n) \to 0$ as $n \to \infty$. Then $z \in A(T)$.*

Proof. Since $\{x_n\}$ is bounded, one sees from Lemma 2.7 that $\{d(x_n, y)\}$, $\{d(Tx_n, y)\}$ and $\{d(T^2x_n, y)\}$ are bounded. Now replacing x with x_n in (iii) of Definition 3.1, we get

$$\begin{aligned}& \alpha_1 d^2(T^2x_n, Ty) + \alpha_2 d^2(Tx_n, Ty) + \alpha_3 d^2(x_n, Ty) + \beta_1 d^2(T^2x_n, y) \\ + \quad & \beta_2 d^2(Tx_n, y) + \beta_3 d^2(x_n, y) + \varepsilon_1 d^2(x_n, T^2x_n) + \varepsilon_2 d^2(x_n, Tx_n) \leq 0.\end{aligned}$$

for all $y \in C$. By applying Banach limit on both sides of the above inequality and the boundedness of $\{x_n\}$, we get

$$\begin{aligned}& \alpha_1 \mu_n d^2(T^2x_n, Ty) + \alpha_2 \mu_n d^2(Tx_n, Ty) + \alpha_3 \mu_n d^2(x_n, Ty) + \beta_1 \mu_n d^2(T^2x_n, y) \\ + \quad & \beta_2 \mu_n d^2(Tx_n, y) + \beta_3 \mu_n d^2(x_n, y) + \varepsilon_1 \mu_n d^2(x_n, T^2x_n) + \varepsilon_2 \mu_n d^2(x_n, Tx_n) \leq 0.\end{aligned}$$

for all $y \in C$. This implies

$$(\alpha_1 + \alpha_2 + \alpha_3)\mu_n d^2(x_n, Ty) + (\beta_1 + \beta_2 + \beta_3)\mu_n d^2(x_n, y) \leq 0,$$

for all $y \in C$. Since $\sum_{i=1}^{3} \alpha_i > 0$, we obtain from definition 3.1 that

$$\mu_n d^2(x_n, Ty) \leq -\frac{(\beta_1 + \beta_2 + \beta_3)}{(\alpha_1 + \alpha_2 + \alpha_3)} \mu_n d^2(x_n, y),$$

for all $y \in C$. Using the fact that $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0$, we get

$$\mu_n d^2(x_n, Ty) \leq \mu_n d^2(x_n, y), \tag{3.1}$$

for all $y \in C$. Since X satisfy $\mathbb{S}$ property and $\Delta - \lim_{n\to\infty} x_n = z$, we find from Lemma 2.5 that $\lim_{n\to\infty}$ $\langle \overrightarrow{zx_n}, \overrightarrow{zy} \rangle = 0$ for all $y \in C$. By definition of quasilinearization, we get

$$\lim_{n\to\infty} (d^2(x_n,z) - d^2(x_n,y) + d^2(z,y)) = 0,$$

for all $y \in C$. Thus

$$\mu(d^2(x_n,z) - d^2(x_n,y) + d^2(z,y)) = 0.$$

From (3.1), we get

$$-\mu_n d^2(x_n,y) \leq -\mu_n d^2(x_n,Ty).$$

This implies

$$\begin{aligned} -\mu_n d^2(x_n,y) &+ \mu(d^2(x_n,z) + d^2(z,y) + d^2(z,Ty)) \\ &\leq -\mu_n d^2(x_n,Ty) + \mu(d^2(x_n,z) + d^2(z,y) + d^2(z,Ty)). \end{aligned}$$

By rearranging, we get

$$\begin{aligned} d^2(z,Ty) &+ \mu(d^2(x_n,z) - d^2(x_n,y) + d^2(z,y)) \\ &\leq d^2(z,y) + \mu(d^2(x_n,z) - d^2(x_n,Ty) + d^2(z,Ty)). \end{aligned}$$

Thus, $d(z,Ty) \leq d(z,y)$. Hence $z \in A(T)$. This completes the proof. □

Theorem 3.1. *Let C be a nonempty, convex subset of a complete CAT(0) space X which satisfy property $(\mathbb{S})$ and condition $(\bar{Q}_4)$. Let $T : C \to C$ be a further 2-generalized hybrid mapping such that $A(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$*

$$\begin{cases} y_n = \alpha_n x_n \oplus \beta_n T x_n \oplus \gamma_n T^2 x_n \\ x_{n+1} = \delta_n u \oplus (1-\delta_n) y_n, \end{cases} . \tag{3.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are real sequences satisfying $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a,b] \subset (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $(C_1) : \lim_{n\to\infty} \delta_n = 0$, $(C_2) : \sum_{n=1}^{\infty} \delta_n = +\infty$. Then $\{x_n\}$ converges strongly to $z = P_{A(T)}(u)$.

Proof. From Lemma 2.7, we see that $A(T)$ is closed and convex. So, $z = \mathrm{P}_{A(T)}(u)$ is well defined. Let $z = \mathrm{P}_{A(T)}(u) \in A(T)$. From equations (3.2), Lemma 2.1 and definition of $A(T)$, we have

$$\begin{aligned} d(z,y_n) &= d(z, \alpha_n x_n \oplus \beta_n T x_n \oplus \gamma_n T^2 x_n) \\ &\leq \alpha_n d(z,x_n) + \beta_n d(z,Tx_n) + \gamma_n d(z,T^2 x_n) \\ &\leq \alpha_n d(z,x_n) + \beta_n d(z,x_n) + \gamma_n d(z,x_n). \\ &= d(z,x_n). \end{aligned}$$

Similarly,

$$\begin{aligned} d(z,x_{n+1}) &= d(z, \delta_n u \oplus (1-\delta_n) y_n) \\ &= \delta_n d(z,u) + (1-\delta_n) d(z,y_n) \\ &\leq \delta_n d(z,u) + (1-\delta_n) d(z,x_n). \end{aligned} \tag{3.3}$$

By induction, we obtain

$$d(z,x_{n+1}) \leq \max\{d(z,u), d(z,x_n)\}, \quad \forall\, n \geq 1,$$

which implies that $\{d(z,x_n)\}$, $\{x_n\}$, $\{Tx_n\}$ and $\{T^2x_n\}$ are bounded. By (3.2) and Lemma 2.2, we have

$$\begin{aligned} d^2(z,y_n) &= d^2(z,\alpha_n x_n \oplus \beta_n Tx_n \oplus \gamma_n T^2x_n) \\ &\leq \alpha_n d^2(z,x_n) + \beta_n d^2(z,Tx_n) + \gamma_n d^2(z,T^2x_n) - \alpha_n\beta_n d^2(x_n,Tx_n) \\ &\leq \alpha_n d^2(z,x_n) + \beta_n d^2(z,x_n) + \gamma_n d^2(z,x_n) - \alpha_n\beta_n d^2(x_n,Tx_n) \\ &= d^2(z,x_n) - \alpha_n\beta_n d^2(x_n,Tx_n). \end{aligned}$$

Thus,

$$d^2(z,y_n) \leq d^2(z,x_n) - \alpha_n\beta_n d^2(x_n,Tx_n). \tag{3.4}$$

Similarly,

$$d^2(z,y_n) \leq d^2(z,x_n) - \alpha_n\gamma_n d^2(x_n,T^2x_n). \tag{3.5}$$

It follows from (3.2), Lemma 2.1, (3.4) and (3.5) that

$$\begin{aligned} d^2(z,x_{n+1}) &= d^2(z,\delta_n u \oplus (1-\delta_n)y_n) \\ &\leq \delta_n d^2(z,u) + (1-\delta_n)d^2(z,y_n) \\ &\leq \delta_n d^2(z,u) + (1-\delta_n)d^2(z,x_n) \\ &- (1-\delta_n)\alpha_n\beta_n d^2(x_n,Tx_n) \\ &= \delta_n[d^2(z,u) - d^2(z,x_n) + \alpha_n\beta_n d^2(x_n,Tx_n)] \\ &+ d^2(z,x_n) - \alpha_n\beta_n d^2(x_n,Tx_n). \end{aligned}$$

Putting

$$k_1 = \sup\{|d^2(z,u) - d^2(z,x_n)| + \alpha_n\beta_n d^2(x_n,Tx_n)\}$$

and

$$k_2 = \sup\{|d^2(z,u) - d^2(z,x_n)| + \alpha_n\gamma_n d^2(x_n,T^2x_n)\},$$

we get

$$d^2(z,x_{n+1}) \leq d^2(z,x_n) - \alpha_n\beta_n d^2(x_n,Tx_n) + \delta_n k_1,$$

and

$$d^2(z,x_{n+1}) \leq d^2(z,x_n) - \alpha_n\gamma_n d^2(x_n,T^2x_n) + \delta_n k_2.$$

These imply

$$\alpha_n\beta_n d^2(x_n,Tx_n) \leq d^2(z,x_n) - d^2(z,x_{n+1}) + \delta_n k_1, \tag{3.6}$$

and

$$\alpha_n\gamma_n d^2(x_n,T^2x_n) \leq d^2(z,x_n) - d^2(z,x_{n+1}) + \delta_n k_2. \tag{3.7}$$

We next consider the following cases.

Case 1. Assume sequence $\{d(z,x_n)\}$ is non-increasing. Since it is bounded, it is convergent. Thus, we have that

$$d^2(z,x_n) - d^2(z,x_{n+1}) \to 0 \; as \; n \to \infty. \tag{3.8}$$

Using C_1, equations (3.6), (3.7) and (3.8) we have $\alpha_n\beta_n d^2(x_n,Tx_n) \to 0$ and $\alpha_n\gamma_n d^2(x_n,T^2x_n) \to 0$ as $n \to \infty$. Using the fact that $\alpha_n,\beta_n,\gamma_n \in [a,b] \subset (0,1)$, we obtain

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0, \quad \lim_{n\to\infty} d(x_n,T^2x_n) = 0. \tag{3.9}$$

From (3.2) and (3.9) we get

$$\begin{aligned} d(x_n,y_n) &\leq \alpha_n d(x_n,x_n)+\beta_n d(x_n,Tx_n) \\ &+ \gamma_n d^2(x_n,T^2x_n)\to 0 \ as\ n\to\infty. \end{aligned} \tag{3.10}$$

Using (3.2) and $C1$, we get

$$\begin{aligned} d(y_n,x_{n+1}) &= d(y_n,\delta_n u\oplus(1-\delta_n)y_n) \\ &\leq \delta_n d(y_n,u)+(1-\delta_n)d(y_n,y_n) \\ &= \delta_n d(y_n,u)\to 0 \ as\ n\to\infty. \end{aligned} \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{3.12}$$

Since X is a complete CAT(0) space and $\{x_n\}$ is bounded, one obtains from Lemma 2.4 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that Δ-$\lim_{k\to\infty} x_{n_k}=z$. From $\lim_{k\to\infty} d(x_{n_k},Tx_{n_k})=0$ and $\lim_{k\to\infty} d(x_{n_k},T^2x_{n_k})=0$, we obtain from Lemma 3.1 that $z\in A(T)$. Using Lemma 2.6, we have

$$\limsup_{n\to\infty}\langle\overrightarrow{uz},\overrightarrow{x_nz}\rangle=\limsup_{k\to\infty}\langle\overrightarrow{uz},\overrightarrow{x_{n_k}z}\rangle\leq 0, \tag{3.13}$$

for all $u\in X$. Using the quasilinearization property, the Cauchy-Swartz inequality, (3.10) and (3.13), we have that

$$\begin{aligned} \limsup_{n\to\infty}\langle\overrightarrow{uz},\overrightarrow{y_nz}\rangle &= \limsup_{n\to\infty}(\langle\overrightarrow{uz},\overrightarrow{y_nx_n}\rangle+\langle\overrightarrow{uz},\overrightarrow{x_nz}\rangle) \\ &\leq \limsup_{n\to\infty}(d(u,z)d(y_n,x_n)+\langle\overrightarrow{uz},\overrightarrow{x_nz}\rangle)\leq 0. \end{aligned} \tag{3.14}$$

Using inequality (3.14) and condition $C1$, we get

$$\limsup_{n\to\infty}(\delta_n d(z,u)+2(1-\delta_n)\langle\overrightarrow{uz},\overrightarrow{y_nz}\rangle)\leq 0. \tag{3.15}$$

From (3.2) and Lemma 2.3, we get

$$d^2(z,x_{n+1})\leq\delta_n^2 d^2(z,u)+(1-\delta_n)^2d^2(z,y_n)+2\delta_n(1-\delta_n)\langle\overrightarrow{uz},\overrightarrow{y_nz}\rangle.$$

Thus,

$$d^2(z,x_{n+1})\leq(1-\delta_n)d^2(z,x_n)+\delta_n(\delta_n d^2(z,u)+2(1-\delta_n)\langle\overrightarrow{uz},\overrightarrow{y_nz}\rangle). \tag{3.16}$$

By Lemma 2.9, inequalities (3.15) and (3.16), we obtain that $x_n\to z$ as $n\to\infty$.

Case 2. Assume sequence $\{d(z,x_n)\}$ is not non-increasing. Putting $\{a_n\}=\{d^2(z,x_n)\}$, we find that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that, for all $i\in\mathbb{N}$, $a_{n_i}\leq a_{n_i+1}$. For some sufficiently large N and for all $n\geq N$, we define a map $\tau:\mathbb{N}\to\mathbb{N}$ by

$$\tau(n)=max\{k\leq n:a_k\leq a_{k+1}\}.$$

Then, it follows from Lemma 2.10 that $\tau(n)$ is non-decreasing with $\tau(n)\to\infty$ as $n\to\infty$ and $a_{\tau(n)}\leq a_{\tau(n)+1}$, $a_n\leq a_{\tau(n)+1}$. Using the fact that $\delta_{\tau(n)}\to 0$ as $\tau(n)\to\infty$, equation (3.6) and (3.7), we obtain $d(x_{\tau(n)},Tx_{\tau(n)})\to 0$, $d(x_{\tau(n)},T^2x_{\tau(n)})\to 0$ as $\tau(n)\to 0$. Following similar argument as in Case 1, we see from (3.15) that

$$\limsup_{n\to\infty}(\delta_{\tau(n)}d(z,u)+2(1-\delta_{\tau(n)})\langle\overrightarrow{uz},\overrightarrow{y_{\tau(n)}z}\rangle)\leq 0. \tag{3.17}$$

It follows from equation (3.16) that

$$a_{\tau(n)+1} \leq a_{\tau(n)} + \delta_{\tau(n)}[\delta_{\tau(n)} d^2(z,u) + 2(1-\delta_{\tau(n)})\langle \overrightarrow{uz}, \overrightarrow{y_n z}\rangle - a_{\tau(n)}].$$

From the fact that $a_{\tau(n)} \leq a_{\tau(n)+1}$ and $\delta_{\tau(n)} > 0$, the above inequalities give

$$a_{\tau(n)} \leq \delta_{\tau(n)} d^2(z,u) + 2(1-\delta)\langle \overrightarrow{uz}, \overrightarrow{y_n z}\rangle.$$

Thus, using (3.17) we get

$$\lim_{\tau(n)\to\infty} a_{\tau(n)} = \lim_{\tau(n)\to\infty} a_{\tau(n)+1} = 0.$$

Since $0 \leq a_n \leq a_{\tau(n)+1}$, this implies that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(z,x_n) = 0$. Therefore, $x_n \to z$ as $n \to \infty$. Hence in view of the above two cases, we see that $\{x_n\}$ converges strongly to $z = \mathrm{P}_{A(T)}(u) \in A(T)$. This completes the proof. □

In view of Remark 3.1, the following results can be obtained by applying Theorem 3.1.

Corollary 3.1. *Let C be a nonempty, convex subset of a complete CAT(0) space X which satisfy property $(\mathbb{S})$ and condition $(\bar{Q}_4)$. Let $T : C \to C$ be a further generalized hybrid mapping such that $A(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.2). Then $\{x_n\}$ converges strongly to $z = P_{A(T)}(u)$.*

Proof. Since a further 2-generalized hybrid is reduced to a further generalized hybrid mapping if $\alpha_1 = \beta_1 = \varepsilon_2 = 0$. It follows from Theorem 3.1 that sequence $\{x_n\}$ converges strongly to $z = \mathrm{P}_{A(T)}(u)$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty, convex subset of a complete CAT(0) space X which satisfy property $(\mathbb{S})$ and condition $(\bar{Q}_4)$. Let $T : C \to C$ be a normally 2-generalized hybrid mapping such that $A(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.2). Then $\{x_n\}$ converges strongly to $z = P_{A(T)}(u)$.*

Proof. If $\varepsilon_1 = \varepsilon_2 = 0$, then a further 2-generalized hybrid is reduced to a further generalized hybrid mappings. From Theorem 3.1 we see that $\{x_n\}$ converges strongly to $z = \mathrm{P}_{A(T)}(u)$. This completes the proof. □

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