J. Nonlinear Var. Anal. 3 (2019), No. 3, pp. 247-256
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.3.2019.3.02

ON THE MILD SOLUTIONS OF A CLASS OF SECOND-ORDER INTEGRO-DIFFERENTIAL INCLUSIONS

AURELIAN CERNEA[1,2]

[1] *Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania*

[2] *Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest, Romania*

Abstract. An existence result of the Filippov type for the mild solutions of a class of nonconvex second-order integro-differential inclusions is obtained. By using a selection theorem due to Bressan and Colombo concerning the existence of continuous selections of lower semicontinuous set-valued maps with decomposable values, we obtain the existence of mild solutions continuously depending on a parameter for the problem considered. Based on this result, we deduce the existence of a continuous selection of the solution set of the problem considered.

Keywords. Differential inclusion; Mild solution; Set-valued map.

2010 Mathematics Subject Classification. 34A60.

1. INTRODUCTION

In this paper we study second-order integro-differential inclusions of the form

$$x''(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0,\ x'(0) = y_0, \tag{1.1}$$

where $F : [0,T] \times X \to \mathscr{P}(X)$ is a set-valued map lipschitzian with respect to the second variable, X is a separable Banach space, $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{S(t,s)\}_{t,s\in[0,T]}$, $\Delta = \{(t,s) \in [0,T] \times [0,T]; t \geq s\}$, $K(.,.) : \Delta \to \mathbf{R}$ is continuous and $x_0, y_0 \in X$. The framework of operators $\{A(t)\}_{t\geq 0}$ that define problem (1.1) was introduced by Kozak [14] and, afterwards, improved by Henriquez [11].

Over the last years, one may see an increasing interest in the study of integro-differential equations and inclusions; see, for instance, [1, 6, 13, 16] and the references therein. Even if there are first-order or second-order integro-differential equations or inclusions, the results are, mainly, concern with existence, uniqueness, controllability or other qualitative properties of the solutions. Most of the results on this topic are based on fixed point techniques. The goal of this paper is to consider a second-order integro-differential inclusion defined by a relatively new class of operators and to obtain several existence results in the situation when the set-valued map that defines the problem is Lipschitz in the state variable.

In control theory, the infinitesimal properties of the reachable sets may be characterized only by tangent cones in a generalized sense. It is essential to identify such kind of tangent cones existence results like the ones in this paper. Our aim is to show that Filippov's ideas [9] can be suitably adapted in order

E-mail address: acernea@fmi.unibuc.ro.

Received November 26, 2018; Accepted April 20, 2019.

to prove the existence of mild solutions to problem (1.1). On one hand, using the classical measurable selection theorem of Kuratowsky and Ryll-Nardzewski [15], we obtain a Filippov type existence result for problem (1.1). In the general theory of differential inclusions, a Filippov type theorem [9] means a results which proves the existence of a solution of a differential inclusion defined by a Lipschitz set-valued map starting from a given "almost" solution. At the same time, the result gives an estimate between the "almost" solution and the solution obtained. The proof of our result follows the general ideas in [9], where a similar result was proved for semilinear differential inclusions.

On the other hand, using a selection theorem due to Bressan and Colombo [5] concerning the existence of continuous selections of lower semicontinuous set-valued maps with decomposable values, we deduce the existence of mild solutions of problem (1.1) continuously depending on a parameter. The proof of follows the general ideas presented in [6, 8]. This result is, in fact, a continuous (parametrized) version of Filippov theorem for the problem considered.

In [2, 3, 4, 7, 11, 12], existence results and qualitative properties of mild solutions were obtained for the following problem

$$x''(t) \in A(t)x(t) + F(t,x(t)), \quad x(0) = x_0,\ x'(0) = y_0, \tag{1.2}$$

with $A(.)$ and $F(.,.)$ are as above.

On one hand, our results extend some results in [7] obtained for problem (1.2) to the integro-differential framework (1.1). On the other hand, our results extend similar results in [1, 6] obtained for some classes of first-order integro-differential inclusions to second-order integro-differential inclusions.

The paper is organized as follows. In Section 2, we present the notations and definitions which will be used in the sequel. Section 3 and Section 4 are devoted to our main results.

2. Preliminaries

Denote the interval $[0,T]$, where $T > 0$, by J, and let X be a real Banach space with the norm $|.|$. Denote by B the closed unit ball in X, by $\mathscr{L}(J)$ the σ-algebra of all Lebesgue measurable subsets of J, by $\mathscr{P}(X)$ the family of all nonempty subsets of X and by $\mathscr{B}(X)$ the family of all Borel subsets of X. If $A \subset J$, then $\chi_A(.) : J \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$, we denote by $\mathrm{cl}(A)$ the closure of A. $C(J,X)$ is the Banach space of all continuous functions $z(.) : J \to X$ endowed with the norm $|z(.)|_C = \sup_{t\in J}|z(t)|$. $L^1(J,X)$ is the Banach space of all (Bochner) integrable functions $z(.) : J \to X$ endowed with the norm $|z(.)|_1 = \int_0^T |z(t)|dt$ and $B(X)$ is the Banach space of linear bounded operators on X.

Recall that a subset $M \subset L^1(J,X)$ is said to be decomposable if, for any $v(\cdot), w(\cdot) \in M$ and any subset $B \in \mathscr{L}(J)$, $v\chi_B + w\chi_A \in M$, where $A = I\backslash B$. $\mathscr{D}(J,X)$ denote the family of all decomposable closed subsets of $L^1(J,X)$. Let (Λ, d) be a separable metric space. We recall that a multifunction $H(\cdot) : \Lambda \to \mathscr{P}(X)$ is said to be lower semicontinuous if, for any closed subset $A \subset X$, $\{\lambda \in \Lambda; H(\lambda) \subset A\}$ is closed. In what follows, $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{S(t,s)\}_{t,s\in J}$. One knows from the hypothesis that the domains of $A(t)$ and $D(A(t))$ are dense in X and are independent of t.

Definition 2.1. [11, 14] A family of bounded linear operators $S(t,s): X \to X$, $(t,s) \in \Delta := \{(t,s) \in J \times J; s \leq t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2.1}$$

if the following conditions are satisfied:
i) if $x \in X$, $(t,s) \to S(t,s)x$ is continuously differentiable and
  a) $S(t,t) = 0$, $t \in J$,
  b) if $t \in J$ and $x \in X$, then $\frac{\partial}{\partial t}S(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}S(t,s)x|_{t=s} = -x$;
ii) if $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}S(t,s)x \in D(A(t))$, the function $(t,s) \to S(t,s)x$ is C^2 and
  a) $\frac{\partial^2}{\partial t^2}S(t,s)x \equiv A(t)S(t,s)x$,
  b) $\frac{\partial^2}{\partial s^2}S(t,s)x \equiv S(t,s)A(t)x$,
  c) $\frac{\partial^2}{\partial s \partial t}S(t,s)x|_{t=s} = 0$;
iii) if $(t,s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x$ and
  a) $\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x \equiv A(t)\frac{\partial}{\partial s}S(t,s)x$ and the function $(t,s) \to A(t)\frac{\partial}{\partial s}S(t,s)x$ is continuous,
  b) $\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x \equiv \frac{\partial}{\partial t}S(t,s)A(s)x$.

As an example of equation (2.1), one may consider the following problem ([11])

$$\frac{\partial^2 u}{\partial t^2}(t,\tau) = \frac{\partial^2 u}{\partial \tau^2}(t,\tau) + \alpha(t)\frac{\partial u}{\partial t}(t,\tau), \quad t \in J, \tau \in [0,2\pi],$$

$$u(t,0) = u(t,\pi) = 0, \quad \frac{\partial u}{\partial \tau}(t,0) = \frac{\partial u}{\partial \tau}(t,2\pi),\ t \in J,$$

where $\alpha(.): J \to \mathbf{R}$ is continuous. The problem is modelled in the space $X = L^2(\mathbf{R},\mathbf{C})$, 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$ and $A_1 u = \frac{d^2 u(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R},\mathbf{C})$ and the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R},\mathbf{C})$. It is known that A_1 is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. In addition, the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $u_n(\tau) = \frac{1}{\sqrt{2\pi}}e^{in\tau}$, $n \in \mathbf{N}$. The set u_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 u = \sum_{n\in\mathbf{Z}} -n^2 < u, u_n > u_n$, $u \in D(A_1)$. The cosine function is given by

$$C(t)u = \sum_{n\in\mathbf{Z}} \cos(nt) < u, u_n > u_n$$

with the associated sine function

$$S(t)z = t < u, u_0 > u_0 + \sum_{n\in\mathbf{Z}^*} \frac{\sin(nt)}{n} < u, u_n > u_n.$$

It remains to define the operator $A_2(t)u = \alpha(t)\frac{du}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R},\mathbf{C})$ and put $A(t) = A_1 + A_2(t)$. In [11], it was proved that this family generates an evolution operator as in Definition 2.1.

Definition 2.2. A continuous mapping $z(.) \in C(J,X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $g(.) \in L^1(J,X)$ such that

$$g(t) \in F(t,z(t)) \quad a.e. (J), \tag{2.2}$$

$$z(t) = -\frac{\partial}{\partial s}S(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)\int_0^s K(s,\tau)g(\tau)d\tau ds,\ t \in J. \tag{2.3}$$

In what follows, we say that $(z(.),g(.))$ is a trajectory-selection pair of (1.1) if $g(.)$ verifies (2.2) and $z(.)$ is defined by (2.3). The solution set of (1.1) will be denoted by

$$\mathscr{S}(x_0,y_0)=\{(z(.),g(.));\ (z(.),g(.)) \text{ is a trajectory-selection pair of (1.1)}\}.$$

Next we assume the following hypothesis.

Hypothesis 2.1. (i) There exists an evolution operator $\{S(t,s)\}_{t,s\in I}$ associated with the family $\{A(t)\}_{t\geq 0}$.
(ii) There exist $m,m_0\geq 0$ such that $|S(t,s)|_{B(X)}\leq m$, $|\frac{\partial}{\partial s}S(t,s)|\leq m_0$, for all $(t,s)\in\Delta$.

Obviously, condition (2.3) can be rewritten as

$$z(t)=-\frac{\partial}{\partial s}S(t,0)x_0+S(t,0)y_0+\int_0^t \mathscr{U}(t,s)g(s)ds \quad \forall t\in J, \tag{2.4}$$

where $\mathscr{U}(t,s)=\int_s^t S(t,\tau)K(\tau,s)d\tau$. Denote $k_0:=\sup_{(t,s)\in\Delta}|K(t,s)|$ and note that $|\mathscr{U}(t,s)|\leq mk_0(t-s)\leq mk_0T$.

3. A Filippov type theorem

In what follows, $v_0,w_0\in X$, $f(.)\in L^1(J,X)$ and $x(.)\in C(J,X)$ is a mild solution of the Cauchy problem

$$x''(t)=A(t)x(t)+\int_0^t K(t,s)f(s)ds \quad x(0)=v_0,\ x'(0)=w_0.$$

Hypothesis 3.1. (i) Hypothesis 2.1 is satisfied.
(ii) $F(.,.):I\times X\to\mathscr{P}(X)$ has nonempty closed values and, for every $x\in X$, $F(.,x)$ is measurable.
(iii) There exists $l(.)\in L^1(I,(0,\infty))$ such that

$$d_H(F(t,y_1),F(t,y_2))\leq l(t)|y_1-y_2|,\quad \forall\, y_1,y_2\in X,$$

where $d_H(B,C)$ is the Hausdorff distance between $B,C\subset X$,

$$d_H(B,C)=\max\{d^*(B,C),d^*(C,B)\},\quad d^*(B,C)=\sup\{d(b,C);b\in B\}.$$

(iv) The function $t\to p(t):=d(f(t),F(t,x(t)))$ is integrable on I.
(v) $\Delta=\{(t,s)\in J\times I;J\geq s\}$, $K(.,.):\Delta\to\mathbf{R}$ is continuous.

Set $L(t)=e^{mk_0T\int_0^t l(u)du}$, $t\in J$. The next technical results summarized are well known in the theory of set-valued maps. For their proof, we refer the reader to [10].

Lemma 3.1. *Let X be a separable Banach space. Let $G:J\to\mathscr{P}(X)$ be a measurable set-valued map with nonempty closed values and $f,h:I\to X,L:I\to(0,\infty)$ measurable functions. Then*
(i) The function $t\to d(h(t),G(t))$ is measurable,
(ii) If $G(t)\cap(f(t)+l(t)B)\neq\emptyset$ a.e. (J) then $t\to G(t)\cap(f(t)+l(t)B)$ has a measurable selection.
If Hypothesis 3.1 is satisfied and $x(.)\in C(J,X)$, then set-valued map $t\to F(t,x(t))$ is measurable.

We are ready now to prove the main results of this section.

Theorem 3.1. *Let $\eta(t)=L(t)(\delta+mk_0T\int_0^t p(s)ds)$, where $\delta\geq 0$. If Hypothesis 3.1 is satisfied, then, for any $x_0,y_0\in X$ with $m_0|x_0-v_0|+m|y_0-w_0|\leq\delta$ and any $\varepsilon>0$, there exists $(z(.),g(.))$ a trajectory-selection pair of (1.1) such that*

$$|z(t)-x(t)|\leq\eta(t)+\varepsilon mk_0TtL(t)\quad \forall t\in J,$$

$$|g(t)-f(t)|\leq l(t)(\eta(t)+\varepsilon mk_0TtL(t))+p(t)+\varepsilon\quad a.e.\ (J).$$

Proof. Consider $\varepsilon > 0$ and set $\gamma(t) = \delta + mk_0T\int_0^t p(s)ds + \varepsilon mk_0Tt$, $z_0(t) \equiv x(t)$, $g_0(t) \equiv f(t)$, $t \in J$. One constructs the sequences $z_n(.) \in C(J,X)$, $g_n(.) \in L^1(J,X)$, $n \geq 1$ with the following properties

$$z_n(t) = -\frac{\partial}{\partial s}S(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)g_n(s)ds, \quad \forall t \in J, \tag{3.1}$$

$$|z_1(t) - z_0(t)| \leq \gamma(t) \quad \forall t \in J, \tag{3.2}$$

$$|g_1(t) - g_0(t)| \leq p(t) + \varepsilon \quad a.e.\ (J), \tag{3.3}$$

$$g_n(t) \in F(t, z_{n-1}(t)) \quad a.e.\ (J),\ n \geq 1, \tag{3.4}$$

$$|g_{n+1}(t) - g_n(t)| \leq l(t)|z_n(t) - z_{n-1}(t)| \quad a.e.\ (J), n \geq 1. \tag{3.5}$$

From (3.1), (3.2) and (3.5), we have, for almost all $t \in I$,

$$\begin{aligned} |z_{n+1}(t) - z_n(t)| &\leq \int_0^t |\mathscr{U}(t,t_1)|.|g_{n+1}(t_1) - g_n(t_1)|dt_1 \\ &\leq mk_0T\int_0^t l(t_1)|z_n(t_1) - z_{n-1}(t_1)|dt_1 \\ &\leq mk_0T\int_0^t l(t_1)\int_0^{t_1} |\mathscr{U}(t_1,t_2)|, \end{aligned}$$

and

$$\begin{aligned} &|g_n(t_2) - g_{n-1}(t_2)|dt_2 \\ &\leq (mk_0T)^2\int_0^t l(t_1)\int_0^{t_1} l(t_2)|z_{n-1}(t_2) - z_{n-2}(t_2)|dt_2dt_1 \\ &\leq (mk_0T)^n\int_0^t l(t_1)\int_0^{t_1} l(t_2)\ldots\int_0^{t_{n-1}} l(t_n)|z_1(t_n) - x(t_n)|dt_n\ldots dt_1 \\ &\leq \gamma(t)(mk_0T)^n\int_0^t l(t_1)\int_0^{t_1} l(t_2)\ldots\int_0^{t_{n-1}} l(t_n)dt_n\ldots dt_1 \\ &= \gamma(t)\frac{(mk_0T\int_0^t l(s)ds)^n}{n!}. \end{aligned}$$

Thus $\{z_n(.)\}$ is a Cauchy sequence in Banach space $C(J,X)$. From (3.5) for almost all $t \in J$, sequence $\{g_n(t)\}$ is Cauchy in X. Using(3.2) and the last inequality, we have

$$\begin{aligned} |z_n(t) - x(t)| &\leq |z_1(t) - x(t)| + \sum_{i=2}^{n-1}|z_{i+1}(t) - z_i(t)| \\ &\leq \gamma(t)[1 + mk_0T\int_0^t l(s)ds + \frac{(mk_0T\int_0^t l(s)ds)^2}{2!} + \cdots] \\ &\leq \gamma(t)e^{mk_0T\int_0^t l(s)ds} \\ &= \eta(t) + \varepsilon mk_0TtL(t). \end{aligned} \tag{3.6}$$

From (3.3), (3.5) and (3.6), we obtain, for almost all $t \in I$,

$$\begin{aligned} |g_n(t) - f(t)| &\leq \sum_{i=1}^{n-1}|g_{i+1}(t) - g_i(t)| + |g_1(t) - f(t)| \\ &\leq l(t)\sum_{i=1}^{n-2}|z_i(t) - z_{i-1}(t)| + p(t) + \varepsilon \\ &\leq l(t)(\eta(t) + \varepsilon tL(t)mk_0T) + p(t) + \varepsilon. \end{aligned} \tag{3.7}$$

Denote by $z(.) \in C(J,X)$ the limit of the Cauchy sequence $z_n(.)$. From (3.7), the sequence $g_n(.)$ is integrably bounded. We have already proved that, for almost all $t \in J$, the sequence $\{g_n(t)\}$ is Cauchy in X. Consider $g(.) \in L^1(J,X)$ such that $g(t) = \lim_{n\to\infty} g_n(t)$. Taking the limit, we deduce that (2.2) for almost all $t \in J$. Taking the limit in (3.1) and using theLebesque's dominated convergence theorem, we get (2.4). From (3.6) and (3.7), we obtain the desired estimations.

Next, we construct the sequences $z_n(.), g_n(.)$ with the properties in (3.1)-(3.5). This will be done by induction. Note that the set-valued map $t \to F(t,x(t))$ is measurable with closed values and

$$F(t,x(t)) \cap \{f(t) + (p(t)+\varepsilon)B\} \neq \emptyset \quad a.e.\ (J).$$

From Lemma 3.1, we find $g_1(.)$ a measurable selection of the set-valued map

$$G_1(t) := F(t,x(t)) \cap \{f(t) + (p(t)+\varepsilon)B\}.$$

Obviously, $g_1(.)$ satisfies (3.3). Define $z_1(.)$ as in (3.1) with $n=1$. It follows that

$$\begin{aligned}|z_1(t)-x(t)| &\leq |-\frac{\partial}{\partial s}S(t,0)(x_0-v_0)| + |S(t,0)(y_0-w_0)| + |\int_0^t \mathscr{U}(t,s)(g_1(s)-f(s))ds| \\ &\leq \delta + mk_0\int_0^t (p(s)+\varepsilon)ds.\end{aligned}$$

Assume that, for some $N \geq 1$ $z_n(.) \in C(J,X)$ and $g_n(.) \in L^1(J,X), n = 1,2,...N$ satisfy (3.1)-(3.5). We define the set-valued map

$$G_{N+1}(t) := F(t,z_N(t)) \cap \{g_N(t) + l(t)|z_N(t)-z_{N-1}(t)|B\}, \quad t \in J.$$

From Lemma 3.1, the set-valued map $t \to F(t,z_N(t))$ is measurable. From the lipschitzianity of $F(t,.)$, we have that, for almost all $t \in I$, $G_{N+1}(t) \neq \emptyset$. With the aid of Lemma 3.1, we find a measurable selection $g_{N+1}(.)$ of $F(.,z_N(.))$ such that

$$|g_{N+1}(t) - g_N(t)| \leq l(t)|z_N(t)-z_{N-1}(t)| \quad a.e.\ (J).$$

We define $x_{N+1}(.)$ as in (3.1) with $n = N+1$ and the proof is complete. □

4. A CONTINUOUS VERSION OF THE FILIPPOV'S THEOREM

The the following lemma is essential in the proof of our main results in this section.

Lemma 4.1. [5] *Let $G(.,.): J\times\Lambda \to \mathscr{P}(X)$ be $\mathscr{L}(J)\otimes\mathscr{B}(\Lambda)$-measurable with closed values such that $G(t,.)$ is lower semicontinuous for any $t \in J$. Then the set-valued map $H(.): \Lambda \to \mathscr{D}(J,X)$ defined by*

$$H(\lambda) = \{f \in L^1(J,X);\quad f(t) \in G(t,\lambda)\quad a.e.\ (J)\}$$

has nonempty closed values and is lower semicontinuous iff there exists a continuous mapping $q(.): \Lambda \to L^1(J,X)$ such that

$$d(0,G(t,\lambda)) \leq q(\lambda)(t) \quad a.e.\ (J),\ \forall \lambda \in \Lambda.$$

Lemma 4.2. [5] *Let $F(.): \Lambda \to \mathscr{D}(J,X)$ be lower semicontinuous and let $\phi(.): \Lambda \to L^1(J,X)$, $\psi(.): \Lambda \to L^1(J,\mathbf{R})$ be continuous such that the set-valued map $G(.): \Lambda \to \mathscr{D}(J,X)$ given by*

$$G(s) = cl\{f(.) \in F(\lambda);\quad |f(t)-\phi(\lambda)(t)| < \psi(\lambda)(t)\quad a.e.\ (J)\}$$

has nonempty values. Then $G(.)$ admits a continuous selection.

Hypothesis 4.1. (i) Λ is a separable metric space and $b(.), c(.) : \Lambda \to X$, $a(.) : \lambda \to (0,\infty)$ are continuous mappings.

(ii) There exists the continuous mappings $f(.), p(.) : \Lambda \to L^1(J,X)$, $x(.) : \Lambda \to C(J,X)$ such that

$$(x(\lambda))''(t) = A(t)x(\lambda)(t) + \int_0^t K(t,u)f(\lambda)(u)du \quad \forall \lambda \in \Lambda, t \in J$$

and

$$d(f(\lambda)(t), F(t, x(\lambda)(t)) \le p(\lambda)(t) \quad a.e.\ (J),\ \forall\ \lambda \in \Lambda.$$

Next, We use the following notations $L(t) = \int_0^t l(u)du$ and

$$\begin{aligned}\xi(\lambda)(t) &:= k_0 e^{mk_0 L(t)}(tTa(\lambda) + m_0|b(\lambda) - x(\lambda)(0)| + m|c(\lambda) - (x(\lambda))'(0)|) \\ &\quad + mT\int_0^t p(\lambda)(u)e^{mk_0T(L(t)-L(u))}du.\end{aligned} \tag{4.1}$$

Theorem 4.1. *Assume that Hypotheses 3.1 and 4.1 are satisfied. Then there exist the continuous mappings $z(.) : \Lambda \to C(J,X)$, $g(.) : \Lambda \to L^1(J,X)$ such that, for any $\lambda \in \Lambda$, $(z(\lambda)(.), g(\lambda)(.))$ is a trajectory-selection of*

$$z''(t) \in A(t)z(t) + \int_0^t K(t,u)F(u,z(u))du, \quad z(0) = b(\lambda),\ z'(0) = c(\lambda)$$

and

$$|z(\lambda)(t) - x(\lambda)(t)| \le \xi(\lambda)(t) \quad \forall (t,s) \in J \times \Lambda, \tag{4.2}$$

$$|g(\lambda)(t) - f(\lambda)(t)| \le l(t)\xi(\lambda)(t) + p(\lambda)(t) + a(\lambda) \quad a.e.\,(J),\ \forall \lambda \in \Lambda. \tag{4.3}$$

Proof. We make the following notations $\varepsilon_n(\lambda) = a(\lambda)\frac{n+1}{n+2}$, $n = 0,1,...,$

$$d(\lambda) = m_0|b(\lambda) - x(\lambda)(0)| + m|c(\lambda) - (x(\lambda))'(0)|$$

and

$$\begin{aligned}p_n(\lambda)(t) &= (mk_0T)^n \int_0^t p(\lambda)(u)\frac{(L(t)-L(u))^{n-1}}{(n-1)!}du \\ &\quad + (mk_0T)^{(n-1)}\frac{(L(t))^{n-1}}{(n-1)!}(mk_0Tt\varepsilon_n(\lambda)t + d(\lambda)) \quad n \ge 1.\end{aligned}$$

Setting $z_0(\lambda)(t) = x(\lambda)(t)$, $\forall \lambda \in \Lambda$, we define the set-valued maps $H_0(.), G_0(.)$ by

$$H_0(\lambda) = \{v \in L^1(J,X); \quad v(t) \in F(t, x(\lambda)(t)) \quad a.e.\,(J)\},$$

$$G_0(\lambda) = \mathrm{cl}\{v \in H_0(\lambda); \quad |v(t) - f(\lambda)(t)| < p(\lambda)(t) + \varepsilon_0(\lambda)\}.$$

One has

$$d(f(\lambda)(t), F(t, x(\lambda)(t)) \le p(\lambda)(t) < p(\lambda)(t) + \varepsilon_0(\lambda).$$

From Lemma 4.2, one finds that set $G_0(\lambda)$ is not empty. Putting $F_0^*(t,\lambda) = F(t, x(\lambda)(t))$, one has

$$d(0, F_0^*(t,\lambda)) \le |f(\lambda)(t)| + p(\lambda)(t) = p_*(\lambda)(t)$$

where $p_*(.) : \Lambda \to L^1(J,X)$ is continuous. From Lemmas 4.1 and 4.2, we find a continuous selection g_0 of G_0 such that

$$g_0(\lambda)(t) \in F(t, x(\lambda)(t)) \quad a.e.\,(J),\ \forall \lambda \in \Lambda,$$

and

$$|g_0(\lambda)(t) - f(\lambda)(t)| \le p_0(\lambda)(t) = p(\lambda)(t) + \varepsilon_0(\lambda) \quad \forall \lambda \in \Lambda,\ t \in J.$$

Define
$$z_1(\lambda)(t) = -\frac{\partial}{\partial s}S(t,0)b(\lambda) + S(t,0)c(\lambda) + \int_0^t \mathscr{U}(t,r)g_0(\lambda)(r)dr.$$
It follows that
$$\begin{aligned}
&|z_1(\lambda)(t) - z_0(\lambda)(t)|\\
&\le m_0|b(\lambda) - x(\lambda)(0)| + m|c(\lambda) - (x(\lambda))'(0)| + mk_0T\int_0^t |g_0(\lambda)(u) - f(\lambda)(u)|du\\
&\le d(\lambda) + mk_0T\int_0^t p_0(\lambda)(u)(u)du + mk_0Tt\varepsilon_0(\lambda)\\
&\le p_1(\lambda)(t).
\end{aligned}$$

As in [6, 8], we construct the sequences $g_n(.) : \Lambda \to L^1(J,X)$, $z_n(.) : \Lambda \to C(J,X)$ such that
a) $g_n(.) : \Lambda \to L^1(J,X)$, $z_n(.) : \Lambda \to C(J,X)$ are continuous.
b) $g_n(\lambda)(t) \in F(t, z_n(\lambda)(t))$, a.e. (J), $\lambda \in \Lambda$.
c) $|g_n(\lambda)(t) - g_{n-1}(\lambda)(t)| \le l(t)p_n(\lambda)(t)$, a.e. (J), $\lambda \in \Lambda$.
d) $z_{n+1}(\lambda)(t) = -\frac{\partial}{\partial s}S(t,0)b(\lambda) + S(t,0)c(\lambda) + \int_0^t \mathscr{U}(t,r)g_n(\lambda)(r)dr$, $\forall t \in J, \lambda \in \Lambda$.
If $g_i(.), z_i(.)$ with a)-c) are constructed, we define $z_{n+1}(.)$ as in d). From c) and d), we have
$$\begin{aligned}
&|z_{n+1}(\lambda)(t) - z_n(\lambda)(t)|\\
&\le mk_0T\int_0^t |g_n(\lambda)(u) - g_{n-1}(\lambda)(u)|du\\
&\le mk_0T\int_0^t l(u)p_n(\lambda)(u)du \qquad (4.4)\\
&= (mk_0T)^{n+1}\int_0^t p(\lambda)(u)\frac{(L(t)-L(u))^n}{n)!}du + (mk_0T)^n\frac{(L(t))^n}{n!}(mk_0Tt\varepsilon_n(\lambda)t + d(\lambda))\\
&< p_{n+1}(\lambda)(t).
\end{aligned}$$
We also have
$$d(g_n(\lambda)(t), F(t, z_{n+1}(\lambda)(t)) \le l(t)|z_{n+1}(\lambda)(t) - z_n(\lambda)(t)| < l(t)p_{n+1}(\lambda)(t). \qquad (4.5)$$
For any $\lambda \in \Lambda$, define the set-valued maps
$$H_{n+1}(\Lambda) = \{v \in L^1(J,X); \quad v(t) \in F(t, z_{n+1}(\lambda)(t)) \quad a.e.(J)\},$$
$$G_{n+1}(\lambda) = \mathrm{cl}\{v \in H_{n+1}(\lambda); \quad |v(t) - g_n(\lambda)(t)| < L(t)p_{n+1}(\lambda)(t)\ a.e.(J)\}.$$
In order to show that $G_{n+1}(\lambda)$ is nonempty, we notice that
$$t \to w_n(\lambda)(t) = a(\lambda)\cdot\frac{(mk_0T)^{n+1}tl(t)(L(t))^n}{(n+2)(n+3)n!}$$
is strictly positive and measurable for any λ. Taking into account (4.5), we deduce
$$\begin{aligned}
d(g_n(\lambda)(t), F(t, z_{n+1}(\lambda)(t)) &\le l(t)|z_{n+1}(\lambda)(t) - z_n(\lambda)(t)| - w_n(\lambda)(t)\\
&\le l(t)p_{n+1}(\lambda)(t).
\end{aligned}$$
From Lemma 4.2, one sees that there exists $v(.) \in L^1(J,X)$ such that $v(t) \in F(t, z_n(\lambda)(t))$ a.e. (J) and
$$|v(t) - g_n(\lambda)(t)| < d(g_n(\lambda)(t), F(t, z_n(\lambda)(t)) + w_n(\lambda)(t).$$

Therefore, $G_{n+1}(\lambda)$ is not empty. Define $F^*_{n+1}(t,\lambda) = F(t, z_{n+1}(\lambda)(t))$. We may write

$$\begin{aligned} d(0, F^*_{n+1}(t,\lambda)) &\le l(t)|z_{n+1}(\lambda)(t) - z_n(\lambda)(t)| \\ &\le |g_n(\lambda)(t)| + l(t)p_{n+1}(\lambda)(t) \\ &= p^*_{n+1}(\lambda)(t) \quad a.e.\,(J) \end{aligned}$$

where $p^*_{n+1}(.) : \Lambda \to L^1(J,X)$ is continuous. Using Lemmas 4.1 and 4.2, we find a continuous function $g_{n+1}(.) : \Lambda \to L^1(J,X)$ such that

$$g_{n+1}(\Lambda)(t) \in F(t, z_{n+1}(\lambda)(t)) \quad a.e.\,(J), \quad \forall \lambda \in \Lambda$$

and

$$|g_{n+1}(\lambda)(t) - g_n(\lambda)(t)| \le l(t)p_{n+1}(\lambda)(t) \quad a.e.\,(J), \quad \forall \lambda \in \Lambda.$$

From (4.4) and d), we deduce

$$\begin{aligned} |z_{n+1}(\lambda)(.) - z_n(\lambda)(.)|_C &\le mk_0T|g_{n+1}(\lambda)(.) - g_n(\lambda)(.)|_1 \\ &\le \frac{(mk_0TL(T))^n}{n!}(mk_0T|p(\lambda)(.)|_1 + mk_0T^2a(\lambda) + d(\lambda)). \end{aligned} \tag{4.6}$$

Thus $g_n(\lambda)(.)$, $z_n(\lambda)(.)$ are Cauchy sequences in $L^1(J,X)$ and $C(J,X)$, respectively. Since the mapping

$$\lambda \to MK_0T|p(\lambda)(.)|_1 + MK_0T^a(\lambda) + d(\lambda)$$

is continuous, we get that $\lambda \to g(\lambda)(.)$ is continuous from Λ into $L^1(J,X)$. As above, (4.6) implies that $z_n(\lambda)(.)$ is Cauchy in $C(J,X)$ locally uniformly with respect to λ. Therefore, $\lambda \to z(\lambda)(.)$ is continuous from Λ into $C(J,X)$. Note that $z_n(\lambda)(.)$ converges uniformly to $z(\lambda)(.)$ and

$$d(g_n(\lambda)(t), F(t, z(\lambda)(t)) \le l(t)|z_n(\lambda)(t) - z(\lambda)(t)| \quad a.e.\ (J),\ \forall \lambda \in \Lambda.$$

Passing to the limit along a subsequence of $g_n(.)$ which converges pointwise to $g(.)$, we find that

$$g(\lambda)(t) \in F(t, z(\lambda)(t)) \quad a.e.\ (J),\ \forall \lambda \in \Lambda.$$

Passing to the limit in d), we obtain that

$$z(\lambda)(t) = -\frac{\partial}{\partial s}S(t,0)b(\lambda) + S(t,0)c(\lambda) + \int_0^t \mathscr{U}(t,r)g(\lambda)(r)dr.$$

On the other hand, adding inequalities c) for all n and using the fact that $\sum_{i\ge 1} p_i(\lambda)(t) \le \xi(\lambda)(t)$, we get

$$\begin{aligned} |g_{n+1}(\lambda)(t) - f(\lambda)(t)| &\le \sum_{l=0}^{n} |g_{l+1}(\lambda)(u) - g_l(\lambda)(u)| + |g_0(\lambda)(t) - f(\lambda)(t)| \\ &\le \sum_{l=0}^{n} l(t)p_{l+1}(\lambda)(t) + p(\lambda)(t) + \varepsilon_0(\lambda) \\ &\le l(t)\xi(\lambda)(t) + p(\lambda)(t) + a(\lambda). \end{aligned} \tag{4.7}$$

In a similar way, we obtain from (4.4) that

$$|z_{n+1}(\lambda)(t) - x(\lambda)(t)| \le \sum_{l=0}^{n} p_l(\lambda)(t) \le \xi(\lambda)(t). \tag{4.8}$$

It remains to pass to the limit in (4.7) and (4.8) in order to obtain (4.2) and (4.3), respectively. □

Theorem 4.1 allows us to obtain a continuous selections of the solution set of problem (1.1).

Hypothesis 4.2. Hypothesis 3.1 is satisfied and there exists $p_0(.) \in L^1(J,(0,\infty))$ with $d(0,F(t,0)) \leq p_0(t)$ a.e. (J).

Corollary 4.1. *Assume that Hypothesis 4.2 is satisfied. Then there exists a function* $z(.,.) : J \times X \to X^2$ *such that*

a) $z(.,(\xi,\eta)) \in \mathscr{S}(\xi,\eta))$, $\forall (\xi,\eta) \in X^2$.

b) $(\xi,\eta) \to z(.,(\xi,\eta))$ *is continuous from* X^2 *into* $C(I,X)$.

Proof. Taking $S = X^2$, $b(\xi,\eta) = \xi$, $c(\xi,\eta) = \eta$, $\forall (\xi,\eta) \in X^2$, $a(.) : X^2 \to (0,\infty)$ an arbitrary continuous function, $f(.) = 0$, $x(.) = 0$, $p(\xi,\eta)(t) = p_0(t)$, $\forall (\xi,\eta) \in X^2$, $t \in J$ in Theorem 4.1, we obtain the desired conclusion immediately. □

REFERENCES

[1] A. Anguraj, C. Murugesan, Continuous selections of set of mild solutions of evolution inclusions, Electron. J. Differential Equations 2005 (2005), 21.

[2] A. Baliki, M. Benchohra, J.R. Graef, Global existence and stability of second order functional evolution equations with infinite delay, Electron. J. Qual. Theory Differ. Equ. 2016 (2016), 23.

[3] A. Baliki, M. Benchohra, J.J. Nieto, Qualitative analysis of second-order functional evolution equations, Dynamic Syst. Appl. 24 (2015), 559-572.

[4] M. Benchohra, I. Medjadj, Global existence results for second order neutral functional differential equations with state-dependent delay, Comment. Math. Univ. Carolin. 57 (2016), 169-183.

[5] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, Studia Math. 90 (1988), 69-86.

[6] A. Cernea, On the mild solutions of a class of evolution inclusions, Int. J. Evol. Equ. 3 (2009), 447-457.

[7] A. Cernea, Some remarks on the solutions of a second-order evolution inclusion, Dyn. Sys. Appl. 27 (2018), 319-330.

[8] R.M. Colombo, A. Fryszkowski, T. Rzezuchowski and V. Staicu, Continuous selections of solution sets of Lipschitzean differential inclusions, Funkcial. Ekvac. 34 (1991), 321-330.

[9] A.F. Filippov, Classical solutions of differential equations with multivalued right-hand side, SIAM J. Control Optim. 5 (1967), 609-621.

[10] H. Frankowska, A priori estimates for operational differential inclusions, J. Differential Equations, 84 (1990), 100-128.

[11] H.R. Henriquez, Existence of solutions of nonautonomous second order functional differential equations with infinite delay, Nonlinear Anal. 74 (2011), 3333-3352.

[12] H.R. Henriquez, V. Poblete, J.C. Pozo, Mild solutions of non-autonomous second order problems with nonlocal initial conditions, J. Math. Anal. Appl. 412 (2014), 1064-1083.

[13] K. Karthikeyan, A. Anguraj, K. Malar, J.J. Trujillo, Existence of mild and classical solutions for nonlocal impulsive integrodifferential equations in Banach spaces with measure of noncompactness, Int. J. Differ. Equ. 2014 (2014), 319250.

[14] M. Kozak, A fundamental solution of a second-order differential equation in a Banach space, Univ. Iagel. Acta. Math. 32 (1995), 275-289.

[15] K. Kuratowski, C. Ryll-Nardzewski, A general theorem on selectors, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 13 (1965), 397-403.

[16] B. Liu, Controllability of neutral functional differential and integrodifferential inclusions with infinite delay, J. Optim. Theory Appl. 123 (2004), 573-593.