ON THE MILD SOLUTIONS OF A CLASS OF SECOND-ORDER
   INTEGRO-DIFFERENTIAL INCLUSIONS

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Abstract. An existence result of the Filippov type for the mild solutions of a class of nonconvex second-order integro-
differential inclusions is obtained. By using a selection theorem due to Bressan and Colombo concerning the existence of
continuous selections of lower semicontinuous set-valued maps with decomposable values, we obtain the existence of mild
solutions continuously depending on a parameter for the problem considered. Based on this result, we deduce the existence of
a continuous selection of the solution set of the problem considered.

Keywords. Differential inclusion; Mild solution; Set-valued map.

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1. INTRODUCTION

In this paper we study second-order integro-differential inclusions of the form

\[ x''(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \ x'(0) = y_0, \]

where \( F : [0,T] \times X \to \mathcal{P}(X) \) is a set-valued map lipschitzian with respect to the second variable, \( X \) is a
separable Banach space, \( \{A(t)\}_{t \geq 0} \) is a family of linear closed operators from \( X \) into \( X \) that generates an
evolution system of operators \( \{S(t,s)\}_{t,s \in [0,T]}, \Delta = \{(t,s) \in [0,T] \times [0,T] ; t \geq s\}, K(\cdot, \cdot) : \Delta \to \mathbb{R} \) is con-
tinuous and \( x_0, y_0 \in X \). The framework of operators \( \{A(t)\}_{t \geq 0} \) that define problem (1.1) was introduced

Over the last years, one may see an increasing interest in the study of integro-differential equations and
inclusions; see, for instance, [1, 6, 13, 16] and the references therein. Even if there are first-order or
second-order integro-differential equations or inclusions, the results are, mainly, concern with existence,
uniqueness, controllability or other qualitative properties of the solutions. Most of the results on this
topic are based on fixed point techniques. The goal of this paper is to consider a second-order integro-
differential inclusion defined by a relatively new class of operators and to obtain several existence results
in the situation when the set-valued map that defines the problem is Lipschitz in the state variable.

In control theory, the infinitesimal properties of the reachable sets may be characterized only by tan-
gent cones in a generalized sense. It is essential to identify such kind of tangent cones existence results
like the ones in this paper. Our aim is to show that Filippov’s ideas [9] can be suitably adapted in order

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to prove the existence of mild solutions to problem (1.1). On one hand, using the classical measurable selection theorem of Kuratowski and Ryll-Nardzewski [15], we obtain a Filippov type existence result for problem (1.1). In the general theory of differential inclusions, a Filippov type theorem [9] means a result which proves the existence of a solution of a differential inclusion defined by a Lipschitz set-valued map starting from a given "almost" solution. At the same time, the result gives an estimate between the "almost" solution and the solution obtained. The proof of our result follows the general ideas in [9], where a similar result was proved for semilinear differential inclusions.

On the other hand, using a selection theorem due to Bressan and Colombo [5] concerning the existence of continuous selections of lower semicontinuous set-valued maps with decomposable values, we deduce the existence of mild solutions of problem (1.1) continuously depending on a parameter. The proof of continuous selections of lower semicontinuous set-valued maps with decomposable values, we deduce the existence of mild solutions of problem (1.1) continuously depending on a parameter. The proof of our result follows the general ideas in [9], where a similar result was proved for semilinear differential inclusions.

In [2, 3, 4, 7, 11, 12], existence results and qualitative properties of mild solutions were obtained for the following problem

\[ x''(t) \in A(t)x(t) + F(t,x(t)), \quad x(0) = x_0, \quad x'(0) = y_0, \quad (1.2) \]

with \( A(.) \) and \( F(.,.) \) as above.

On one hand, our results extend some results in [7] obtained for problem (1.2) to the integro-differential framework (1.1). On the other hand, our results extend similar results in [1, 6] obtained for some classes of first-order integro-differential inclusions to second-order integro-differential inclusions.

The paper is organized as follows. In Section 2, we present the notations and definitions which will be used in the sequel. Section 3 and Section 4 are devoted to our main results.

2. Preliminaries

Denote the interval \([0,T]\), where \( T > 0 \), by \( J \), and let \( X \) be a real Banach space with the norm \( |.| \). Denote by \( B \) the closed unit ball in \( X \), by \( \mathcal{L}(J) \) the \( \sigma \)-algebra of all Lebesgue measurable subsets of \( J \), by \( \mathcal{P}(X) \) the family of all nonempty subsets of \( X \) by \( \mathcal{B}(X) \) the family of all Borel subsets of \( X \). If \( A \subset J \), then \( \chi_A(.) : J \to \{0,1\} \) denotes the characteristic function of \( A \). For any subset \( A \subset X \), we denote by \( \text{cl}(A) \) the closure of \( A \). \( C(J,X) \) is the Banach space of all continuous functions \( z(.) : J \to X \) endowed with the norm \( |z(.)|_C = \sup_{t \in J} |z(t)| \). \( L^1(J,X) \) is the Banach space of all (Bochner) integrable functions \( z(.) : J \to X \) endowed with the norm \( |z(.)|_1 = \int_0^T |z(t)| \, dt \) and \( B(X) \) is the Banach space of linear bounded operators on \( X \).

Recall that a subset \( M \subset L^1(J,X) \) is said to be decomposable if, for any \( v(.) \), \( w(.) \in M \) and any subset \( B \in \mathcal{L}(J), v \chi_B + w \chi_A \in M \), where \( A = \cap B \). \( \mathcal{P}(J,X) \) denote the family of all decomposable closed subsets of \( L^1(J,X) \). Let \((\Lambda,d)\) be a separable metric space. We recall that a multifunction \( H(\cdot) : \Lambda \to \mathcal{P}(X) \) is said to be lower semicontinuous if, for any closed subset \( A \subset X \), \( \{ \lambda \in \Lambda ; H(\lambda) \subset A \} \) is closed. In what follows, \( \{ A(t) \}_{t \geq 0} \) is a family of linear closed operators from \( X \) into \( X \) that generates an evolution system of operators \( \{ S(t,s) \}_{t,s \in J} \). One knows from the hypothesis that the domains of \( A(t) \) and \( D(A(t)) \) are dense in \( X \) and are independent of \( t \).
Definition 2.1. [11, 14] A family of bounded linear operators \( S(t,s) : X \to X \), \( (t,s) \in \Delta := \{(t,s) \in J \times J; s \leq t\} \) is called an evolution operator of the equation

\[
x''(t) = A(t)x(t)
\]  
(2.1)

if the following conditions are satisfied:

i) if \( x \in X, (t,s) \to S(t,s)x \) is continuously differentiable and

a) \( S(t,t) = 0, t \in J \),

b) if \( t \in J \) and \( x \in X \), then \( \frac{\partial}{\partial t} S(t,s)x|_{t=s} = x \) and \( \frac{\partial}{\partial s} S(t,s)x|_{t=s} = -x; \)

ii) if \( (t,s) \in \Delta \), then \( \frac{\partial}{\partial t} S(t,s)x \in D(A(t)) \), the function \( (t,s) \to S(t,s)x \) is \( C^2 \) and

a) \( \frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x \),

b) \( \frac{\partial^2}{\partial t\partial s} S(t,s)x = S(t,s)A(t)x \),

c) \( \frac{\partial^2}{\partial s^2} S(t,s)x|_{t=s} = 0; \)

iii) if \( (t,s) \in \Delta \), then there exist \( \frac{\partial}{\partial t} S(t,s)x \), \( \frac{\partial}{\partial s} S(t,s)x \) and

a) \( \frac{\partial}{\partial t} S(t,s)x = A(t) \frac{\partial}{\partial s} S(t,s)x \) and the function \( (t,s) \to A(t) \frac{\partial}{\partial s} S(t,s)x \) is continuous,

b) \( \frac{\partial}{\partial s^2} S(t,s)x = \frac{\partial}{\partial s} S(t,s)A(t)x. \)

As an example of equation (2.1), one may consider the following problem ([11])

\[
\frac{\partial^2 u}{\partial t^2}(t,\tau) = \frac{\partial^2 u}{\partial \tau^2}(t,\tau) + \alpha(t) \frac{\partial u}{\partial t}(t,\tau), \quad t \in J, \tau \in [0,2\pi],
\]

\[
u(t,0) = u(t,\pi) = 0, \quad \frac{\partial u}{\partial \tau}(t,0) = \frac{\partial u}{\partial \tau}(t,2\pi), \quad t \in J,
\]

where \( \alpha(.) : J \to \mathbb{R} \) is continuous. The problem is modelled in the space \( X = L^2(\mathbb{R}, \mathbb{C}) \), \( 2\pi \)-periodic 2-integrable functions from \( \mathbb{R} \) to \( \mathbb{C} \) and \( A_1u = \frac{\partial u}{\partial \tau}(\tau) \) with domain \( H^2(\mathbb{R}, \mathbb{C}) \) and the Sobolev space of \( 2\pi \)-periodic functions whose derivatives belong to \( L^2(\mathbb{R}, \mathbb{C}) \). It is known that \( A_1 \) is the infinitesimal generator of strongly continuous cosine functions \( C(t) \) on \( X \). In addition, the spectrum of \( A_1 \) consists of eigenvalues \( -n^2, n \in \mathbb{Z} \) with associated eigenvectors \( u_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}, n \in \mathbb{N} \). The set \( u_n, n \in \mathbb{N} \) is an orthonormal basis of \( X \). In particular, \( A_1u = \sum_{n \in \mathbb{Z}} -n^2 < u, u_n > u_n, u \in D(A_1) \). The cosine function is given by

\[
C(t)u = \sum_{n \in \mathbb{Z}} \cos(nt) < u, u_n > u_n
\]

with the associated sine function

\[
S(t)z = t < u, u_0 > u_0 + \sum_{n \in \mathbb{Z}} \sin(nt) < u, u_n > u_n.
\]

It remains to define the operator \( A_2(t)u = \alpha(t) \frac{\partial u}{\partial \tau} \) with domain \( D(A_2(t)) = H^1(\mathbb{R}, \mathbb{C}) \) and put \( A(t) = A_1 + A_2(t) \). In [11], it was proved that this family generates an evolution operator as in Definition 2.1.

Definition 2.2. A continuous mapping \( z(.) \in C(\mathbb{J}, X) \) is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function \( g(.) \in L^1(\mathbb{J}, X) \) such that

\[
g(t) \in F(t,z(t)) \quad a.e. (\mathbb{J}),
\]  
(2.2)

\[
z(t) = -\frac{\partial}{\partial s} S(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s) \int_0^s K(s,\tau)g(\tau)d\tau ds, \quad t \in J.
\]  
(2.3)
In what follows, we say that \((z(\cdot), g(\cdot))\) is a trajectory-selection pair of (1.1) if \(g(\cdot)\) verifies (2.2) and \(z(\cdot)\) is defined by (2.3). The solution set of (1.1) will be denoted by

\[
\mathcal{S}(x_0, y_0) = \{(z(\cdot), g(\cdot)); (z(\cdot), g(\cdot))\text{ is a trajectory-selection pair of (1.1)}\}.
\]

Next we assume the following hypothesis.

**Hypothesis 2.1.** (i) There exists an evolution operator \(\{S(t, s)\}_{t, s \in \Delta}\) associated with the family \(\{A(t)\}_{t \geq 0}\).

(ii) There exist \(m, m_0 \geq 0\) such that \(|S(t, s)|_{B(X)} \leq m, |\frac{d}{ds}S(t, s)| \leq m_0\), for all \((t, s) \in \Delta\).

Obviously, condition (2.3) can be rewritten as

\[
z(t) = -\frac{d}{ds}S(t, 0)x_0 + S(t, 0)y_0 + \int_0^t \mathcal{W}(t, s)g(s)ds \quad \forall t \in J,
\]

where \(\mathcal{W}(t, s) = \int_s^t S(t, \tau)K(\tau, s)d\tau\). Denote \(k_0 := \sup_{(t, s) \in \Delta} |K(t, s)|\) and note that \(\mathcal{W}(t, s) \leq mk_0(t - s) \leq mk_0T\).

3. **A Filippov Type Theorem**

In what follows, \(v_0, w_0 \in X, f(\cdot) \in L^1(J, X)\) and \(x(\cdot) \in C(J, X)\) is a mild solution of the Cauchy problem

\[
x''(t) = A(t)x(t) + \int_0^t K(t, s)f(s)ds \quad x(0) = v_0, \quad x'(0) = w_0.
\]

**Hypothesis 3.1.** (i) Hypothesis 2.1 is satisfied.

(ii) \(F(\cdot, \cdot) : I \times X \to \mathcal{P}(X)\) has nonempty closed values and, for every \(x \in X\), \(F(\cdot, x)\) is measurable.

(iii) There exists \(l(\cdot) \in L^1(I, (0, \infty))\) such that

\[
d_H(F(t, y_1), F(t, y_2)) \leq l(t)|y_1 - y_2|, \quad \forall y_1, y_2 \in X,
\]

where \(d_H(B, C)\) is the Hausdorff distance between \(B, C \subset X\),

\[
d_H(B, C) = \max \{d^*(B, C), d^*(C, B)\}, \quad d^*(B, C) = \sup \{d(b, C); b \in B\}.
\]

(iv) The function \(t \to p(t) := d(f(t), F(t, x(t)))\) is integrable on \(I\).

(v) \(\Delta = \{(t, s) \in J \times I; J \geq s\}, K(\cdot, \cdot) : \Delta \to \mathbb{R}\) is continuous.

Set \(L(t) = e^{mk_0T \int_0^t l(u)du}, t \in J\). The next technical results summarized are well known in the theory of set-valued maps. For their proof, we refer the reader to [10].

**Lemma 3.1.** Let \(X\) be a separable Banach space. Let \(G : J \to \mathcal{P}(X)\) be a measurable set-valued map with nonempty closed values and \(f, h : I \to X, L : I \to (0, \infty)\) measurable functions. Then

(i) The function \(t \to d(h(t), G(t))\) is measurable.

(ii) If \(G(t) \cap (f(t) + l(t)B) \neq \emptyset \ a.e. (J)\) then \(t \to G(t) \cap (f(t) + l(t)B)\) has a measurable selection.

If Hypothesis 3.1 is satisfied and \(x(\cdot) \in C(J, X)\), then set-valued map \(t \to F(t, x(t))\) is measurable.

We are ready now to prove the main results of this section.

**Theorem 3.1.** Let \(\eta(t) = L(t)(\delta + mk_0T \int_0^t p(s)ds)\), where \(\delta \geq 0\). If Hypothesis 3.1 is satisfied, then, for any \(x_0, y_0 \in X\) with \(m_0|x_0 - v_0| + m|y_0 - w_0| \leq \delta\) and any \(\varepsilon > 0\), there exists \((z(\cdot), g(\cdot))\) a trajectory-selection pair of (1.1) such that

\[
|z(t) - x(t)| \leq \eta(t) + \varepsilon mk_0Tl(t) \quad \forall t \in J,
\]

\[
|g(t) - f(t)| \leq l(t)(\eta(t) + \varepsilon mk_0Tl(t)) + p(t) + \varepsilon \quad a.e. (J).
\]
Proof. Consider \( \varepsilon > 0 \) and set \( \gamma(t) = \delta + mk_0 T \int_0^t p(s) ds + \varepsilon mk_0 T t, z_0(t) \equiv x(t), g_0(t) \equiv f(t), t \in J \). One constructs the sequences \( z_n(\cdot) \in C(J,X), g_n(\cdot) \in L^1(J,X), n \geq 1 \) with the following properties

\[
\begin{align*}
z_n(t) &= -\frac{d}{ds} S(t,0)x_0 + S(t,0)\gamma_0 + \int_0^t S(t,s)g_n(s)ds, \quad \forall t \in J, \\
|z_1(t) - z_0(t)| &\leq \gamma(t) \quad \forall t \in J, \\
|g_1(t) - g_0(t)| &\leq p(t) + \varepsilon \quad a.e. (J), \\
g_n(t) &\in F(t,z_{n-1}(t)) \quad a.e. (J), n \geq 1, \\
|g_{n+1}(t) - g_n(t)| &\leq l(t)|z_n(t) - z_{n-1}(t)| \quad a.e. (J), n \geq 1.
\end{align*}
\]

From (3.1), (3.2) and (3.5), we have, for almost all \( t \in I \),

\[
|z_{n+1}(t) - z_n(t)| \leq \int_0^t |\mathcal{H}(t,s)| |g_{n+1}(t) - g_n(t)| dt_1
\]

\[
\leq mk_0 T \int_0^t l(t_1)|z_n(t_1) - z_{n-1}(t_1)| dt_1
\]

\[
\leq mk_0 T \int_0^t l(t_1) \int_0^{t_1} |\mathcal{H}(t_1,s)| ds dt_1,
\]

and

\[
|g_n(t_2) - g_{n-1}(t_2)| dt_2
\]

\[
\leq (mk_0 T)^2 \int_0^t l(t_1) \int_0^{t_1} l(t_2)|z_{n-1}(t_2) - z_{n-2}(t_2)| dt_2 dt_1
\]

\[
\leq (mk_0 T)^n \int_0^t l(t_1) \int_0^{t_1} l(t_2) \ldots \int_0^{t_{n-1}} l(t_n)|z_1(t_n) - x(t_n)| dt_n \ldots dt_1
\]

\[
\leq \gamma(t)(mk_0 T)^n \int_0^t l(t_1) \int_0^{t_1} l(t_2) \ldots \int_0^{t_{n-1}} l(t_n) dt_n \ldots dt_1
\]

\[
= \gamma(t) \left( \frac{(mk_0 T \int_0^t l(s) ds)^n}{n!} \right).
\]

Thus \( \{z_n(\cdot)\} \) is a Cauchy sequence in Banach space \( C(J,X) \). From (3.5) for almost all \( t \in J \), sequence \( \{g_n(t)\} \) is Cauchy in \( X \). Using (3.2) and the last inequality, we have

\[
|z_n(t) - x(t)| \leq |z_1(t) - x(t)| + \sum_{i=2}^{n-1} |z_{i+1}(t) - z_i(t)|
\]

\[
\leq \gamma(t)[1 + mk_0 T \int_0^t l(s) ds + \frac{(mk_0 T \int_0^t l(s) ds)^2}{2!} + \cdots]
\]

\[
\leq \gamma(t)e^{mk_0 T \int_0^t l(s) ds}
\]

\[
= \eta(t) + \varepsilon mk_0 T t \varepsilon.
\]

From (3.3), (3.5) and (3.6), we obtain, for almost all \( t \in I \),

\[
|g_n(t) - f(t)| \leq \sum_{i=1}^{n-1} |g_{i+1}(t) - g_i(t)| + |g_1(t) - f(t)|
\]

\[
\leq l(t) \sum_{i=1}^{n-2} |z_i(t) - z_{i-1}(t)| + p(t) + \varepsilon
\]

\[
\leq l(t)(\eta(t) + \varepsilon (T T L(t)) + p(t) + \varepsilon.
\]
Denote by $z(.) \in C(J,X)$ the limit of the Cauchy sequence $z_n(.)$. From (3.7), the sequence $g_n(.)$ is integrably bounded. We have already proved that, for almost all $t \in J$, the sequence $\{g_n(t)\}$ is Cauchy in $X$. Consider $g(.) \in L^1(J,X)$ such that $g(t) = \lim_{n \to \infty} g_n(t)$. Taking the limit, we deduce that (2.2) for almost all $t \in J$. Taking the limit in (3.1) and using the Lebesque’s dominated convergence theorem, we get (2.4). From (3.6) and (3.7), we obtain the desired estimations.

Next, we construct the sequences $z_n(.)$, $g_n(.)$ with the properties in (3.1)-(3.5). This will be done by induction. Note that the set-valued map $t \to F(t,x(t))$ is measurable with closed values and

$$F(t,x(t)) \cap \{f(t) + (p(t) + \varepsilon)B\} \neq \emptyset \quad a.e. \ (J).$$

From Lemma 3.1, we find $g_1(.)$ a measurable selection of the set-valued map

$$G_1(t) := F(t,x(t)) \cap \{f(t) + (p(t) + \varepsilon)B\}.$$

Obviously, $g_1(.)$ satisfies (3.3). Define $z_1(.)$ as in (3.1) with $n = 1$. It follows that

$$|z_1(t) - x(t)| \leq | - \frac{\partial}{\partial s} S(t,0)(x_0 - v_0) + |S(t,0)(y_0 - w_0)| + \int_0^t \|U(t,s)(g_1(s) - f(s))ds|$$

$$\leq \delta + mk_0 \int_0^t (p(s) + \varepsilon)ds.$$

Assume that, for some $N \geq 1$, $z_n(.) \in C(J,X)$ and $g_n(.) \in L^1(J,X), n = 1, 2, ..., N$ satisfy (3.1)-(3.5). We define the set-valued map

$$G_{N+1}(t) := F(t,z_n(t)) \cap \{g_N(t) + l(t)|z_N(t) - z_{N-1}(t)|B\}, \quad t \in J.$$

From Lemma 3.1, the set-valued map $t \to F(t,z_N(t))$ is measurable. From the lipschitzianity of $F(t,.)$, we have that, for almost all $t \in I$, $G_{N+1}(t) \neq \emptyset$. With the aid of Lemma 3.1, we find a measurable selection $g_{N+1}(.)$ of $F(.z_N(.))$ such that

$$|g_{N+1}(t) - g_N(t)| \leq l(t)|z_N(t) - z_{N-1}(t)| \quad a.e. \ (J).$$

We define $x_{N+1}(.)$ as in (3.1) with $n = N + 1$ and the proof is complete.

4. A continuous version of the Filippov’s theorem

The following lemma is essential in the proof of our main results in this section.

**Lemma 4.1.** [5] Let $G(.,.) : J \times \Lambda \to \mathcal{P}(X)$ be $\mathcal{L}(J) \otimes \mathcal{B}(\Lambda)$-measurable with closed values such that $G(t,.)$ is lower semicontinuous for any $t \in J$. Then the set-valued map $H(.) : \Lambda \to \mathcal{P}(J,X)$ defined by

$$H(\lambda) = \{f \in L^1(J,X); \ f(t) \in G(t,\lambda) \quad a.e. \ (J)\}$$

has nonempty closed values and is lower semicontinuous if there exists a continuous mapping $q(.) : \Lambda \to L^1(J,X)$ such that

$$d(0,G(t,\lambda)) \leq q(\lambda)(t) \quad a.e. \ (J), \ \forall \lambda \in \Lambda.$$

**Lemma 4.2.** [5] Let $F(.) : \Lambda \to \mathcal{P}(J,X)$ be lower semicontinuous and let $\phi(.) : \Lambda \to L^1(J,X)$, $\psi(.) : \Lambda \to L^1(J,R)$ be continuous such that the set-valued map $G(.) : \Lambda \to \mathcal{P}(J,X)$ given by

$$G(s) = cl\{f(.) \in F(\lambda); \ |f(t) - \phi(\lambda)(t)| < \psi(\lambda)(t) \quad a.e. \ (J)\}$$

has nonempty values. Then $G(.)$ admits a continuous selection.
Hypothesis 4.1. (i) $\Lambda$ is a separable metric space and $b(\cdot), c(\cdot) : \Lambda \to X$, $a(\cdot) : \lambda \to (0, \infty)$ are continuous mappings.

(ii) There exists the continuous mappings $f(\cdot), p(\cdot) : \Lambda \to L^1(J, X)$, $x(\cdot) : \Lambda \to C(J, X)$ such that

\[
(x(\lambda))''(t) = A(t)x(\lambda)(t) + \int_0^t K(t,u)f(\lambda)(u)du \quad \forall \lambda \in \Lambda, t \in J
\]

and

\[
d(f(\lambda)(t), F(t,x(\lambda))(t)) \leq p(\lambda)(t) \quad \text{a.e. (J), } \forall \lambda \in \Lambda.
\]

Next, We use the following notations $L(t) = \int_0^t l(u)du$ and

\[
\xi(\lambda)(t) := k_0 e^{m_0 L(t)}(t \Lambda(\lambda) + m_0 b(\lambda) - x(\lambda)(0)) + m|c(\lambda) - (x(\lambda))' (0)|
\]

\[
+ mT \int_0^t p(\lambda)(u)e^{m_0 T(L(t)-L(u))}du.
\]

Theorem 4.1. Assume that Hypotheses 3.1 and 4.1 are satisfied. Then there exist the continuous mappings $z(\cdot) : \Lambda \to C(J, X)$, $g(\cdot) : \Lambda \to L^1(J, X)$ such that, for any $\lambda \in \Lambda$, $(z(\lambda)(\cdot), g(\lambda)(\cdot))$ is a trajectory-selection of

\[
z''(t) \in A(t)z(t) + \int_0^t K(t,u)F(u,z(u))du, \quad z(0) = b(\lambda), \quad z'(0) = c(\lambda)
\]

and

\[
|z(\lambda)(t) - x(\lambda)(t)| \leq \xi(\lambda)(t) \quad \forall (t,s) \in J \times \Lambda,
\]

\[
|g(\lambda)(t) - f(\lambda)(t)| \leq \xi(\lambda)(t) + p(\lambda)(t) + a(\lambda) \quad \text{a.e. (J), } \forall \lambda \in \Lambda.
\]

Proof. We make the following notations $\varepsilon_n(\lambda) = a(\lambda) \frac{n+1}{n+2}$, $n = 0, 1, \ldots,$

\[
d(\lambda) = m_0 b(\lambda) - x(\lambda)(0) + m|c(\lambda) - (x(\lambda))'(0)|
\]

and

\[
p_n(\lambda)(t) = (m_0 T)^n \int_0^t p(\lambda)(u) \frac{(L(t)-L(u))^{n-1}}{(n-1)!} du
\]

\[
+ (m_0 T)^{n-1} \frac{(L(t))^{n-1}}{(n-1)!} (m_0 T \xi_n(\lambda) + d(\lambda)) \quad n \geq 1.
\]

Setting $z_0(\lambda)(t) = x(\lambda)(t)$, $\forall \lambda \in \Lambda$, we define the set-valued maps $H_0(\cdot), G_0(\cdot)$ by

\[
H_0(\lambda) = \{v \in L^1(J, X) ; \quad v(t) \in F(t,x(\lambda)(t)) \quad \text{a.e. (J)}, \}
\]

\[
G_0(\lambda) = cl\{v \in H_0(\lambda) ; \quad |v(t) - f(\lambda)(t)| < p(\lambda)(t) + \varepsilon_0(\lambda)\}.
\]

One has

\[
d(f(\lambda)(t), F(t,x(\lambda)(t))) \leq p(\lambda)(t) < p(\lambda)(t) + \varepsilon_0(\lambda).
\]

From Lemma 4.2, one finds that set $G_0(\lambda)$ is not empty. Putting $F_0(\lambda)(t) = F(t,x(\lambda)(t))$, one has

\[
d(0, F_0(\lambda)(t)) \leq \|f(\lambda)(t)| + p(\lambda)(t) = p_*(\lambda)(t)
\]

where $p_*(\cdot) : \Lambda \to L^1(J, X)$ is continuous. From Lemmas 4.1 and 4.2, we find a continuous selection $g_0$ of $G_0$ such that

\[
g_0(\lambda)(t) \in F(t,x(\lambda)(t)) \quad \text{a.e. (J), } \forall \lambda \in \Lambda,
\]

and

\[
|g_0(\lambda)(t) - f(\lambda)(t)| \leq p_0(\lambda)(t) = p(\lambda)(t) + \varepsilon_0(\lambda) \quad \forall \lambda \in \Lambda, t \in J.
\]
Define
\[ z_1(\lambda)(t) = -\frac{\partial}{\partial s} S(t, 0) b(\lambda) + S(t, 0) c(\lambda) + \int_0^t \mathcal{H}(t, r) g_0(\lambda)(r) dr. \]

It follows that
\[
|z_1(\lambda)(t) - z_0(\lambda)(t)| \\
\leq m_0 |b(\lambda) - x(\lambda)(0)| + m|c(\lambda) - (x(\lambda))'(0)| + mk_0 T \int_0^t |g_0(\lambda)(u) - f(\lambda)(u)| du \\
\leq d(\lambda) + mk_0 T \int_0^t p_0(\lambda)(u) du + mk_0 T \epsilon_0(\lambda) \\
\leq p_1(\lambda)(t).
\]

As in [6, 8], we construct the sequences \( g_n(\cdot) : \Lambda \rightarrow L^1(J, X) \), \( z_n(\cdot) : \Lambda \rightarrow C(J, X) \) such that
a) \( g_n(\cdot) : \Lambda \rightarrow L^1(J, X) \), \( z_n(\cdot) : \Lambda \rightarrow C(J, X) \) are continuous.
b) \( g_n(\lambda)(t) \in F(t, z_n(\lambda)(t)) \), a.e. \((J), \lambda \in \Lambda.\)
c) \|g_n(\lambda)(t) - g_{n-1}(\lambda)(t)\| \leq l(t)p_n(\lambda)(t), \text{ a.e. } (J), \lambda \in \Lambda.
d) \( z_{n+1}(\lambda)(t) = -\frac{d}{dt} S(t, 0) b(\lambda) + S(t, 0) c(\lambda) + \int_0^t \mathcal{H}(t, r) g_n(\lambda)(r) dr, \forall t \in J, \lambda \in \Lambda.\)

If \( g(.), z(.) \) with a-c) are constructed, we define \( z_{n+1}(.) \) as in d). From c) and d), we have
\[
|z_{n+1}(\lambda)(t) - z_n(\lambda)(t)| \\
\leq mk_0 T \int_0^t |g_n(\lambda)(u) - g_{n-1}(\lambda)(u)| du \\
\leq mk_0 T \int_0^t l(t)p_n(\lambda)(u) du \\
= (mk_0 T)^{n+1} \int_0^t p(\lambda)(u) \frac{(L(t) - L(u))^n}{n!} du + (mk_0 T)^n \frac{(L(t))^n}{n!} (mk_0 T \epsilon_n(\lambda) t + d(\lambda)) \\
< p_{n+1}(\lambda)(t).
\]

We also have
\[
d(g_n(\lambda)(t), F(t, z_{n+1}(\lambda)(t)) \leq l(t)|z_{n+1}(\lambda)(t) - z_n(\lambda)(t)| < l(t)p_{n+1}(\lambda)(t). \tag{4.5}
\]

For any \( \lambda \in \Lambda \), define the set-valued maps
\[
H_{n+1}(\lambda) = \{v \in L^1(J, X); \quad v(t) \in F(t, z_{n+1}(\lambda)(t)) \text{ a.e. } (J)\},
\]
\[
G_{n+1}(\lambda) = \text{cl}\{v \in H_{n+1}(\lambda); \quad |v(t) - g_n(\lambda)(t)| < L(t)p_{n+1}(\lambda)(t) \text{ a.e. } (J)\}.
\]

In order to show that \( G_{n+1}(\lambda) \) is nonempty, we notice that
\[
t \rightarrow w_n(\lambda)(t) = a(\lambda) \cdot \frac{(mk_0 T)^{n+1} l(t)(L(t))^n}{(n+2)(n+3)n!}
\]
is strictly positive and measurable for any \( \lambda \). Taking into account (4.5), we deduce
\[
d(g_n(\lambda)(t), F(t, z_{n+1}(\lambda)(t)) \leq l(t)|z_{n+1}(\lambda)(t) - z_n(\lambda)(t) - w_n(\lambda)(t) \\
\leq l(t)p_{n+1}(\lambda)(t).
\]

From Lemma 4.2, one sees that there exists \( v(.) \in L^1(J, X) \) such that \( v(t) \in F(t, z_n(\lambda)(t)) \text{ a.e. } (J) \) and
\[
|v(t) - g_n(\lambda)(t)| < d(g_n(\lambda)(t), F(t, z_n(\lambda)(t)) + w_n(\lambda)(t).
\]
Therefore, \( G_{n+1}(\lambda) \) is not empty. Define \( F^*_{n+1}(t, \lambda) = F(t, z_{n+1}(\lambda)(t)) \). We may write
\[
d(0,F^*_{n+1}(t, \lambda)) \leq l(t)|z_{n+1}(\lambda)(t) - z_n(\lambda)(t)|
\leq |g_n(\lambda)(t)| + l(t)p_{n+1}(\lambda)(t)
= p^*_{n+1}(\lambda)(t) \text{ a.e. } (J)
\]
where \( p^*_{n+1}(\cdot): \Lambda \rightarrow L^1(J, X) \) is continuous. Using Lemmas 4.1 and 4.2, we find a continuous function \( g_{n+1}(\cdot): \Lambda \rightarrow L^1(J, X) \) such that
\[
g_{n+1}(\Lambda)(t) \in F(t, z_{n+1}(\lambda)(t)) \text{ a.e. } (J), \quad \forall \lambda \in \Lambda
\]
and
\[
|g_{n+1}(\lambda)(t) - g_n(\lambda)(t)| \leq l(t)p_{n+1}(\lambda)(t) \text{ a.e. } (J), \quad \forall \lambda \in \Lambda.
\]
From (4.4) and d), we deduce
\[
|z_{n+1}(\lambda)(\cdot) - z_n(\lambda)(\cdot)|_C \leq mk_0T|g_{n+1}(\lambda)(\cdot) - g_n(\lambda)(\cdot)|_1
\leq \left(\frac{mk_0T(T)}{n!} + mk_0T^2a(\lambda) + d(\lambda)\right).
\]
Thus \( g_n(\lambda)(\cdot), z_n(\lambda)(\cdot) \) are Cauchy sequences in \( L^1(J, X) \) and \( C(J, X) \), respectively. Since the mapping
\[
\lambda \rightarrow MK_0T|p(\lambda)(\cdot)|_1 + MK_0T^a(\lambda) + d(\lambda)
\]
is continuous, we get that \( \lambda \rightarrow g(\lambda)(\cdot) \) is continuous from \( \Lambda \) into \( L^1(J, X) \). As above, (4.6) implies that \( z_n(\lambda)(\cdot) \) is Cauchy in \( C(J, X) \) locally uniformly with respect to \( \lambda \). Therefore, \( \lambda \rightarrow z(\lambda)(\cdot) \) is continuous from \( \Lambda \) into \( C(J, X) \). Note that \( z_n(\lambda)(\cdot) \) converges uniformly to \( z(\lambda)(\cdot) \) and
\[
d(g_n(\lambda)(t), F(t, z(\lambda)(t)) \leq l(t)|z_n(\lambda)(t) - z(\lambda)(t)| \text{ a.e. } (J), \quad \forall \lambda \in \Lambda.
\]
Passing to the limit along a subsequence of \( g_n(\cdot) \) which converges pointwise to \( g(\cdot) \), we find that
\[
g(\lambda)(t) \in F(t, z(\lambda)(t)) \text{ a.e. } (J), \quad \forall \lambda \in \Lambda.
\]
Passing to the limit in d), we obtain that
\[
z(\lambda)(t) = -\frac{2}{d_s}S(t, 0)b(\lambda) + S(t, 0)c(\lambda) + \int_0^t \mathcal{U}(t, \tau)g(\lambda)(\tau)d\tau.
\]
On the other hand, adding inequalities c) for all \( n \) and using the fact that \( \sum_{i \geq 1} p_i(\lambda)(t) \leq \xi(\lambda)(t) \), we get
\[
|g_{n+1}(\lambda)(t) - f(\lambda)(t)| \leq \sum_{i=0}^n |g_{i+1}(\lambda)(u) - g_i(\lambda)(u)| + |g_0(\lambda)(t) - f(\lambda)(t)|
\leq \sum_{i=0}^n l(t)p_{i+1}(\lambda)(t) + p(\lambda)(t) + \varepsilon_0(\lambda)
\leq l(t)\xi(\lambda)(t) + p(\lambda)(t) + a(\lambda).
\]
In a similar way, we obtain from (4.4) that
\[
|z_{n+1}(\lambda)(t) - x(\lambda)(t)| \leq \sum_{i=0}^n p_i(\lambda)(t) \leq \xi(\lambda)(t).
\]
It remains to pass to the limit in (4.7) and (4.8) in order to obtain (4.2) and (4.3), respectively.

\[ \square \]

Theorem 4.1 allows us to obtain a continuous selections of the solution set of problem (1.1).
Hypothesis 4.2. Hypothesis 3.1 is satisfied and there exists \( p_0(\cdot) \in L^1(J,(0,\infty)) \) with \( d(0,F(t,0)) \leq p_0(t) \) a.e. \((J)\).

Corollary 4.1. Assume that Hypothesis 4.2 is satisfied. Then there exists a function \( z(\cdot,\cdot): J \times X \to X^2 \) such that

a) \( z(\cdot,(\xi,\eta)) \in \mathcal{S}(\xi,\eta), \forall (\xi,\eta) \in X^2. \)

b) \( (\xi,\eta) \to z(\cdot,(\xi,\eta)) \) is continuous from \( X^2 \) into \( C(I,X) \).

Proof. Taking \( S = X^2, b(\xi,\eta) = \xi, c(\xi,\eta) = \eta, \forall (\xi,\eta) \in X^2, a(\cdot): X^2 \to (0,\infty) \) an arbitrary continuous function, \( f(\cdot) = 0, x(\cdot) = 0, p(\xi,\eta)(t) = p_0(t), \forall (\xi,\eta) \in X^2, t \in J \) in Theorem 4.1, we obtain the desired conclusion immediately. \( \square \)

References


