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LEVITIN-POLYAK WELL-POSEDNESS FOR BILEVEL VECTOR VARIATIONAL INEQUALITIES

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Abstract. The purpose of this paper is to investigate the Levitin-Polyak well-posedness of a bilevel vector variational inequality. The (generalized) Levitin-Polyak well-posedness is extended to the bilevel vector variational inequality and some metric characterizations of these Levitin-Polyak well-posedness new concepts are discussed.

Keywords. Bilevel vector variational inequality; Levitin-Polyak well-posedness; Metric characterizations; Solution sequence.

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1. INTRODUCTION

The well-posedness is one of the most important and interesting subjects in the fields of optimization theory and applications. Tikhonov [24] introduced the notion of the well-posedness of minimization problems, which means the existence of a minimizer and the convergence of a subsequence of each approximation sequence to a solution. The Tikhonov well-posedness of a constrained minimization problem requires that every minimizing sequence should lie in the constraint set. In many practical situations, the minimizing sequence produced via a numerical optimization method usually fails to be feasible but gets closer and closer to the constraint set. Such a sequence is called a generalized minimizing sequence for constrained minimization problems. To take care of such a case, Levitin and Polyak [16] strengthened the concept of the Tikhonov well-posedness by requiring the existence and uniqueness of minimizers, and the convergence of every generalized minimizing sequence toward the unique minimizer, which has been known as the Levitin and Polyak (in short, LP) well-posedness. Up to now, the concept of the well-posedness has been generalized to several related problems: (vector) optimization problems [1, 13, 18, 23], (vector) variational inequalities [10, 11, 25], Nash equilibrium problems [19] and (vector) equilibrium problems [2, 6, 8, 14, 17], etc.

Some researchers have investigated the LP well-posedness for different problems. Revalski and Zhivkov [23] introduced the relations between different notions of the well-posedness of constrained optimization problems and showed the class of metric spaces in which Hadamard, strong and the LP well-posedness are equivalent. In the setting of Hilbert spaces, Fang, Huang and Cho [11] proved that

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under suitable conditions the well-posedness of a general mixed variational inequality is equivalent to the existence and the uniqueness of its solution. Recently, Ceng and Yao [4] derived some results for the well posedness of the generalized mixed variational inequality, the corresponding inclusion problem and the corresponding fixed point problem. Maingé [21] proposed an easily implementable algorithm for solving some classical monotone variational inequality problem over the set of solutions of mixed variational inequalities in Hilbert spaces. Xia [25] extended the notion of the LP well-posedness to a generalized variational inequality with generalized mixed variational inequality constraints in Banach spaces, and established some metric characterizations of its LP well-posedness.

Recently, many results were obtained on bilevel equilibrium problems and bilevel vector equilibrium problems. Moudafi [22] introduced a class of bilevel monotone equilibrium problems. He considered a bilevel problem involving two monotone equilibrium bifunctions and showed that the problem can be solved by a simple proximal method. Anh, Khanh and Van [1] considered the LP well-posedness of the bilevel equilibrium with equilibrium constraints and the bilevel optimization problems with equilibrium constraints. Chen, Cho and Wan [6] introduced and investigated some topological properties of solutions for the lower level mixed equilibrium problems and bilevel mixed equilibrium problems in reflexive Banach space. Khanh, Plubtieng and Sombut [14] introduced several types of the LP well-posedness for bilevel vector equilibrium and optimization problems with equilibrium constraints. Based on criteria and characterizations for these types of the LP well-posedness, they argued on diameters and Kuratowski's, Hausdorff's, or Istratescu's measures of the noncompactness of approximate solution sets under suitable conditions. Chen, Liou and Wen [7] proposed a dual model of bilevel vector equilibrium problem and established the primal-dual relationships under the cone-convexity and weak pseudomonotonicity assumptions. They further get the existence of solution of bilevel vector equilibrium problem. Chadli, Ansari and Al-Homidan [9] introduced a class of bilevel vector equilibrium problems (in short, BVEP) and established some existence results for the BVEP by using a vector Thikhonov-type regularization method. Anh and Hung [3] considered the strong BVEP and introduced the concepts of the LP well-posedness and the LP well-posedness in the generalized sense for such problems. However, to the best of our knowledge, there is no a result concerning the LP well-posedness of the bilevel vector variational inequality (in short, BVVI). The aim of this paper is to introduce and investigate the LP well-posedness for the BVVI.

This paper is organized as follows. In Section 2, we give a description of the problem and present some basic definitions and concepts. In Section 3, we give some metric characterizations of these concepts in the behavior of approximate solution sets under suitable conditions.

2. Preliminaries

Throughout this section, unless otherwise specified, X and Z are two real Banach spaces and K is a nonempty, closed and convex subset of X. Let $C,P \subset Z$ be two nonempty proper closed convex pointed cones with their topological interiors denoted by $\mathrm{int}C, \mathrm{int}P$, respectively. We denote by $L(X,Z)$ the space of all bounded linear operator from X into Z, and by $\langle q,x\rangle$ the value of $q \in L(X,Z)$ at $x \in X$. We denote by Z^* the dual space of Z, and by C^* the positive polar cone of C, i.e.,

$$C^* = \{\lambda \in Z^* : \langle \lambda, z\rangle \geq 0, \quad \forall z \in C\}.$$

Note that

$$\text{int}C^* = \{\lambda \in Z^* : \langle \lambda, z \rangle > 0, \quad \forall z \in \text{int}C\}.$$

For $s \in \text{int}C^*$ and $T : X \to L(X,Z)$, we define the operator $T_s : X \to X^*$ by

$$\langle T_s x, y \rangle = \langle s, \langle Tx, y \rangle \rangle.$$

Let $F, G : K \to L(X,Z)$ be two operators, and let $\varphi : K \to Z$ be a continuous vector-valued function.

We consider the following BVVI: find $x^* \in \Omega$ such that

$$\langle F(x^*), x^* - x \rangle \notin \text{int}C, \ \forall x \in \Omega,$$

where Ω is the solution set of the lower-level mixed vector variational inequality [in short, MVVI]: find $x \in K$ such that

$$\langle G(x), x - y \rangle + \varphi(x) - \varphi(y) \notin \text{int}P, \ \forall y \in K.$$

We denote by Γ the solution set of BVVI.

Remark 2.1. (i) If $F \equiv 0$, then the problem BVVI is reduced to the MVVI, which was studied in [5, 12]. Furthermore, if $F, \varphi \equiv 0$, then the BVVI is reduced to the vector variational inequality considered in [26, 27].

(ii) It is easily seen that if $Z = R, C = P = R^1_+$, then X is a Hilbert space. Then the BVVI is reduced to the variational inequality with mixed variational inequality constraints: find $x^* \in \aleph$ such that

$$\langle F(x^*), x^* - x \rangle \leq 0, \quad \forall x \in \aleph,$$

where $\aleph$ is the solution set of the lower-level mixed variational inequality: find $x \in K$ such that

$$\langle G(x), x - y \rangle + \varphi(x) - \varphi(y) \leq 0, \quad \forall y \in K,$$

which was studied in [21].

As we will see in the sequel, our study for the BVVI is based on a vector regularization procedure. Letting $\eta \in \text{int}C, \xi \in \text{int}P, \alpha \geq 0$, we consider the following regularization BVVI: find $x^* \in \Omega_\alpha$ such that

$$\langle F(x^*), x^* - x \rangle - \frac{\alpha}{2} \eta \|x^* - x\|^2 \notin \text{int}C, \ \forall x \in \Omega_\alpha,$$

where Ω_α is the solution set of the lower-level regularization mixed vector variational inequality: find $x \in K$ such that

$$\langle G(x), x - y \rangle + \varphi(x) - \varphi(y) - \frac{\alpha}{2} \xi \|x - y\|^2 \notin \text{int}P, \ \forall y \in K.$$

We denote by Γ_α the solution set of the regularization BVVI.

Definition 2.1. Let $T : K \to L(X,Z)$ be an operator. T is said to be C-monotone if

$$\langle T(x) - T(y), x - y \rangle \in C, \quad \forall x, y \in K.$$

The following notions of the set-to-set distance andthe measure of non-compactness will be used in the sequel.

Definition 2.2. [15] Let A and B be nonempty subsets of X. The Hausdorff metric $H(\cdot,\cdot)$ between A and B is define by

$$H(A,B) = \max\{e(B,A), e(A,B)\},$$

where $e(A,B) = \sup\{d(a,B) : a \in A\}$ with $d(a,B) = \inf_{b\in B} \|a-b\|$.

Definition 2.3. [15] The Kuratowski measure of non-compactness of a set $A \subseteq X$ is defined by

$$\mu(A) = \inf\{\varepsilon > 0 : A \subset \cup_{i=1}^{n} A_i,\ diamA_i < \varepsilon, i = 1,2,...,n\},$$

where $diamA_i$ be the diameter of A_i defined by

$$diam(A_i) = \sup\{\|x_1 - x_2)\| : x_1, x_2 \in A_i\}.$$

Definition 2.4. A vector-valued function $\varphi : K \to Z$ is said to be C-convex if, for each $x,y \in K$ and $t \in (0,1)$,

$$t\varphi(x) + (1-t)\varphi(y) - \varphi(tx + (1-t)y) \in C.$$

The function φ is said to be C-concave if $-\varphi$ is C-convex.

It is easy to see that if φ is C-convex, then, for all $x_i \in K$, and $t_i \in (0,1)$ $i = 1,\ldots,n$, with $\Sigma_{i=1}^{n} t_i = 1$,

$$\Sigma_{i=1}^{n} t_i \varphi(x_i) - \varphi(\Sigma_{i=1}^{n} t_i x_i) \in C.$$

Definition 2.5. The operator $T : K \to L(X,Z)$ is said to be hemicontinuous if, for any given $x,y \in K$, the mapping $t \mapsto \langle T(ty + (1-t)x), y-x\rangle$ is continuous at 0^+.

Lemma 2.1. *For each $x \notin intC$, $y \in -C$ implies $x + y \notin intC$.*

Lemma 2.2. [12] *Let X and Z be real Banach spaces. Let K be a nonempty, bounded, closed and convex subset of X. Let $C \subset Z$ be a nonempty proper closed convex pointed cone with the topological interior $intC$. If $\Psi : K \to L(X,Z)$ is continuous, then the mapping $g_y : K \to Z$ defined by $g_y(x) = \langle \Psi x, y-x\rangle$ is continuous for any $y \in K$.*

Next, we assume that there exists $s \in \mathrm{int}P^*$ such that $\langle s,z\rangle \geq 0$ for all $z \notin -\mathrm{int}P$. This assumption originates from Theorem 4.1 in [9].

The following Lemma is important for proving Minty's lemma.

Lemma 2.3. *Let K be a nonempty, closed and convex subset of X. Let $G : K \to L(X,Z)$ be a hemicontinuous operator, and let $\varphi : K \to Z$ be a P-convex and continuous vector-valued function. Assume that there exists $s \in intP^*$ such that $\langle s,z\rangle \geq 0$ for all $z \notin -intP$. Let*

$$\Xi = \{x \in K : \langle G(y), y-x\rangle + \varphi(y) - \varphi(x) \notin -intP, \forall y \in K\}.$$

Then Ξ is convex.

Proof. If Ξ is a singleton, then it is trivial. Suppose $x_1, x_2 \in \Xi$, $t \in (0,1)$ and $x_t' = tx_1 + (1-t)x_2$. Thus, it suffices to show that

$$\langle G(y), y - x_t'\rangle + \varphi(y) - \varphi(x_t') \notin -\mathrm{int}P.$$

Since $x_1, x_2 \in \Xi$, we have

$$\langle G(y), y - x_1\rangle + \varphi(y) - \varphi(x_1) \notin -\mathrm{int}P, \quad \forall y \in K$$

and

$$\langle G(y), y-x_2\rangle + \varphi(y) - \varphi(x_2) \notin -\mathrm{int}P, \quad \forall y \in K.$$

Since P is a cone, we obtain that

$$\langle G(y), t(y-x_1)\rangle + t(\varphi(y) - \varphi(x_1)) \notin -\mathrm{int}P, \quad \forall y \in K$$

and

$$\langle G(y), (1-t)(y-x_2)\rangle + (1-t)(\varphi(y) - \varphi(x_2)) \notin -\mathrm{int}P, \quad \forall y \in K.$$

It follows that

$$\langle s, \langle G(y), t(y-x_1)\rangle + t(\varphi(y) - \varphi(x_1))\rangle \geq 0 \tag{2.1}$$

and

$$\langle s, \langle G(y), (1-t)(y-x_2)\rangle + (1-t)(\varphi(y) - \varphi(x_2))\rangle \geq 0. \tag{2.2}$$

Adding (2.1) and (2.2), we get

$$\langle s, \langle G(y), y-x_t'\rangle + \varphi(y) - t\varphi(x_1) - (1-t)\varphi(x_2)\rangle \geq 0. \tag{2.3}$$

Since φ is P-convex, we have $t\varphi(x_1) + (1-t)\varphi(x_2) - \varphi(x_t') \in P$. Hence

$$\langle s, t\varphi(x_1) + (1-t)\varphi(x_2) - \varphi(x_t')\rangle \geq 0. \tag{2.4}$$

Adding (2.3) and (2.4), we have

$$\langle s, \langle G(y), y-x_t'\rangle + \varphi(y) - \varphi(x_t')\rangle \geq 0.$$

Therefore,

$$\langle G(y), y-x_t'\rangle + \varphi(y) - \varphi(x_t') \notin -\mathrm{int}P,$$

i.e., $x_t' \in \Xi$. This shows that Ξ is convex. This completes the proof. □

Now, we provide the following example to support Lemma 2.3.

Example 2.1. Let $X = R$, $Z = R^2$, $K = [-5,5]$, $P = R^2_+ = \{(x_1,x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\}$ be a fixed closed convex cone in Z. Define $G(x) = (x,x), \langle G(x), y\rangle = y(x,x)$ and $\varphi : x \mapsto (x+10, 6x+10)$. It is obvious that K is a nonempty, closed and convex subset of X, $G(x) \in L(X,Z)$, G is hemicontinuous and φ is P-convex and we can see that, for each $z \notin -\mathrm{int}P$, there exists $s \in \mathrm{int}P^*$ such that $\langle s, z\rangle \geq 0$. Hence, all conditions of Lemma 2.3 hold. It follows from a direction computation that, for any $x \in [-5,-1]$, we have

$$\langle G(y), y-x\rangle + \varphi(y) - \varphi(x) = (y-x)(y+1, y+6) \notin -\mathrm{int}P, \quad \forall y \in K.$$

Hence, $\Xi = [-5,-1]$ is convex.

Now, we are ready to the Minty's lemma for the BVVI.

Proposition 2.1. *(Minty's lemma) Let K be a nonempty, closed and convex subset of X. Let $F, G : K \to L(X,Z)$ be two hemicontinuous operators. Let F be C-monotone, G be P-monotone and $\varphi : K \to Z$ be a P-convex and continuous vector-valued function. Assume that there exists $s \in intP^*$ such that $\langle s, z\rangle \geq 0$ for all $z \notin -intP$. Then the following statements are equivalent:*

(i) find $x^* \in \Omega$ such that $\langle F(x^*), x^* - x\rangle \notin intC$, $\forall x \in \Omega$, where

$$\Omega = \{x \in K | \langle G(x), x-y\rangle + \varphi(x) - \varphi(y) \notin intP, \forall y \in K\};$$

(ii) find $x^* \in \Omega$ such that $\langle F(x), x^* - x\rangle \notin intC$, $\forall x \in \Omega$, where

$$\Omega = \{x \in K | \langle G(y), x-y\rangle + \varphi(x) - \varphi(y) \notin intP, \forall y \in K\}.$$

Proof. (i)⇒(ii). Since G is P-monotone, we have

$$\langle G(x) - G(y), x-y\rangle \in P, \quad \forall x, y \in K.$$

That is,

$$\langle G(y), x-y\rangle - \langle G(x), x-y\rangle \in -P, \quad \forall x, y \in K. \tag{2.5}$$

Combining (2.5) with

$$\langle G(x), x-y\rangle + \varphi(x) - \varphi(y) \notin \text{int}P$$

and using Lemma 2.1, we obtain that

$$\langle G(y), x-y\rangle + \varphi(x) - \varphi(y) \notin \text{int}P, \quad \forall y \in K.$$

We also can find

$$\langle F(x), x^* - x\rangle \notin \text{int}C, \quad \forall x \in \Omega.$$

(ii)⇒(i). Fixing $y \in K$, we know that $y_t = x + t(y-x) \in K$, $\forall t \in (0,1)$, since K is convex. Replacing y by y_t in $\langle G(y), x-y\rangle + \varphi(x) - \varphi(y) \notin \text{int}P$, we have

$$\langle G(y_t), t(x-y)\rangle + \varphi(x) - \varphi(y_t) \notin \text{int}P, \quad \forall y \in K. \tag{2.6}$$

It follows from the P-convexity of φ that

$$-t\varphi(y) - (1-t)\varphi(x) + \varphi(y_t) \in -P. \tag{2.7}$$

Combining (2.6) with (2.7) and using Lemma 2.1, we get

$$\langle G(y_t), x-y\rangle + \varphi(x) - \varphi(y) \notin \text{int}P, \quad \forall y \in K.$$

Note that G is hemicontinuous and $Z \setminus \text{int}P$ is closed. Letting $t \to 0^+$, we deduce that

$$\langle G(x), x-y\rangle + \varphi(x) - \varphi(y) \notin \text{int}P, \quad \forall y \in K, \tag{2.8}$$

which implies $x \in \Omega$. Since $\langle G(y), x-y\rangle + \varphi(x) - \varphi(y) \notin \text{int}P$, we have

$$\langle G(y), y-x\rangle + \varphi(y) - \varphi(x) \notin -\text{int}P.$$

From Lemma 2.3, we obtain that Ω is a convex set.

For any given $x \in \Omega$, $x_t = x^* + t(x-x^*) \in \Omega$, $\forall t \in (0,1)$ since Ω is convex. It follows that

$$\langle F(x_t), x^* - x\rangle \notin \text{int}C, \quad \forall x \in \Omega. \tag{2.9}$$

Using the facts that F is hemicontinuous and $Z \setminus \text{int}C$ is closed and letting $t \to 0^+$, we have

$$\langle F(x^*), x^* - x\rangle \notin \text{int}C, \quad \forall x \in \Omega.$$

The proof is complete. □

Remark 2.2. Proposition 2.1 generalize the corresponding results in [12].

Now, we provide an example to support Proposition 2.1.

Example 2.2. Let $X = R$, $Z = R^2$ and $K = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Let

$$P = R^2_+ = \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\},$$

and

$$C = R^2_- = \{(x_1, x_2) \in R^2 : x_1 \leq 0, x_2 \leq 0\}$$

be two fixed closed convex cones in Z. Define $G(x) = (x, x+10)$, $F(x) = (-x-3, -x-5)$,

$$\langle G(x), y\rangle = y(x, x+10),\ \langle F(x), y\rangle = y(-x-3, -x-5)$$

and $\varphi : x \mapsto (10x+20, -10x)$. It is obvious that $G(x), F(x) \in L(X, Z)$ and φ is P-convex. Since

$$\langle G(x) - G(y), x-y\rangle = (x-y)(x-y, x-y) \in P,$$

and

$$\langle F(x) - F(y), x-y\rangle = \langle(-x+y, y-x), x-y\rangle = (x-y)(y-x, y-x) \in C.$$

we find that F is C-monotone and G is P-monotone. It is easy to see that F and G are continuous, and we can see that, for each $z \notin -\text{int}P$, there exists $s \in \text{int}P^*$ such that $\langle s, z\rangle \geq 0$. Hence, all conditions of Proposition 2.1 hold. It follows from a direction computation that, for $x^* = -\frac{\pi}{2}$,

$$\langle F(x^*), x^* - x\rangle = (x^* - x)(-x^* - 3, -x^* - 5) \notin \text{int}C,\ \forall x \in \Omega = [-\frac{\pi}{2}, 0],$$

where Ω is the solution set of the lower-level mixed vector variational inequality

$$\langle G(x), x-y\rangle + \varphi(x) - \varphi(y) = (x-y)(x+10, x) \notin \text{int}P,\ \forall y \in K.$$

For $x^* = -\frac{\pi}{2}$, we deduce that

$$\langle F(x), x^* - x\rangle = (x^* - x)(-x-3, -x-5) \notin \text{int}C,\ \forall x \in \Omega = [-\frac{\pi}{2}, 0],$$

where Ω is the solution set of the lower-level mixed vector variational inequality

$$\langle G(y), x-y\rangle + \varphi(x) - \varphi(y) = (x-y)(y+10, y) \notin \text{int}P, \quad \forall y \in K.$$

Hence, for a given $x^* \in K$, (i) and (ii) are equivalent.

Now, we are ready to the Minty's lemma for the regularization MVVI.

Lemma 2.4. *Let K be a nonempty, closed and convex subset of X. Let $G : K \to L(X, Z)$ be a hemicontinuous operator and let G be P-monotone. Let $\varphi : K \to Z$ be a P-convex and continuous vector-valued function, $\alpha \geq 0, \xi \in \text{int}P$. Then the following statements are equivalent:*

(i) find $x \in K$ such that $\langle G(x), x-y\rangle + \varphi(x) - \varphi(y) - \frac{\alpha}{2}\xi\|x-y\|^2 \notin \text{int}P, \forall y \in K$;
(ii) find $x \in K$ such that $\langle G(y), x-y\rangle + \varphi(x) - \varphi(y) - \frac{\alpha}{2}\xi\|x-y\|^2 \notin \text{int}P, \forall y \in K$.

Proof. $(i) \Rightarrow (ii)$. Since G is P-monotone, we have

$$\langle G(x) - G(y), x-y\rangle \in P, \quad \forall x, y \in K.$$

It implies

$$\langle G(y), x-y\rangle - \langle G(x), x-y\rangle \in -P. \tag{2.10}$$

Note that

$$\langle G(x), x-y\rangle + \varphi(x) - \varphi(y) - \frac{\alpha}{2}\xi\|x-y\|^2 \notin \text{int}P.$$

From (2.10) and Lemma 2.1, we obtain

$$\langle G(y), x-y\rangle + \varphi(x) - \varphi(y) - \frac{\alpha}{2}\xi\|x-y\|^2 \notin \text{int}P, \quad \forall y \in K.$$

(ii)⇒(i). For any given $y \in K$, we know that $y_t = x + t(y-x) \in K, \forall t \in (0,1)$, since K is convex. Replacing y by y_t in $\langle G(y), x-y\rangle + \varphi(x) - \varphi(y) - \frac{\alpha}{2}\xi\|x-y\|^2 \notin \text{int}P$, we have

$$\langle G(y_t), t(x-y)\rangle + \varphi(x) - \varphi(y_t) - \frac{\alpha}{2}\xi t^2\|x-y\|^2 \notin \text{int}P, \quad \forall y \in K. \tag{2.11}$$

It follows from the P-convexity of φ that

$$-t\varphi(y) - (1-t)\varphi(x) + \varphi(y_t) \in -P. \tag{2.12}$$

Combining (2.11) with (2.12) and using Lemma 2.1, we get

$$\langle G(y_t), x-y\rangle + \varphi(x) - \varphi(y) - \frac{\alpha}{2}\xi t\|x-y\|^2 \notin \text{int}P, \quad \forall y \in K.$$

Since G is hemicontinuous and $Z \setminus \text{int}P$ is closed, we deduce that

$$\langle G(x), x-y\rangle + \varphi(x) - \varphi(y) \notin \text{int}P, \quad \forall y \in K. \tag{2.13}$$

Since $-\frac{\alpha}{2}\xi\|x-y\|^2 \in -P$, by (2.13)and Lemma 2.1, we get

$$\langle G(x), x-y\rangle + \varphi(x) - \varphi(y) - \frac{\alpha}{2}\xi\|x-y\|^2 \notin \text{int}P, \ \forall y \in K. \tag{2.14}$$

The proof is complete. □

The following lemma is easy to obtain.

Lemma 2.5. *Let K be a nonempty, closed and convex subset of X and let $F: K \to L(X,Z)$ be a hemicontinuous operator. Let F be C-monotone, $\alpha \geq 0, \eta \in intC$. Then the following statements are equivalent:*

(i) find $x \in K$ such that $\langle F(x), x-y\rangle - \frac{\alpha}{2}\eta\|x-y\|^2 \notin intC, \forall y \in K$;
(ii) find $x \in K$ such that $\langle F(y), x-y\rangle - \frac{\alpha}{2}\eta\|x-y\|^2 \notin intC, \forall y \in K$.

Proposition 2.2. *Let K be a nonempty, closed and convex subset of X and let $G: K \to L(X,Z)$ be a hemicontinuous operator. Let $\varphi: K \to Z$ be a P-convex vector-valued function, $\alpha \geq 0, \xi \in intP$. If*

$$\aleph = \{x \in K : \langle G(y), y-x\rangle + \varphi(y) - \varphi(x) \notin -intP, \forall y \in K\}$$

and

$$\Delta = \{x \in K : \langle G(y), y-x\rangle + \varphi(y) - \varphi(x) + \frac{\alpha}{2}\xi\|x-y\|^2 \notin -intP, \forall y \in K\},$$

then $\Delta = \aleph$. Furthermore, if $\aleph$ is convex, then Δ is convex.

Proof. If $x \in \aleph$, then

$$\langle G(y), y-x\rangle + \varphi(y) - \varphi(x) \notin -\text{int}P, \quad \forall y \in K.$$

Due to $\frac{\alpha}{2}\xi\|x-y\|^2 \in P$, we obtain from Lemma 2.1 that

$$\langle G(y), y-x\rangle + \varphi(y) - \varphi(x) + \frac{\alpha}{2}\xi\|x-y\|^2 \notin -\text{int}P, \quad \forall y \in K, \tag{2.15}$$

which implies $x \in \Delta$. Therefore, $\aleph \subseteq \Delta$. Letting $x \in \Delta$, we obtain that

$$\langle G(y), y-x\rangle + \varphi(y) - \varphi(x) + \frac{\alpha}{2}\xi\|x-y\|^2 \notin -\text{int}P, \quad \forall y \in K. \tag{2.16}$$

For any given $y \in K$, we know that $y_t = x + t(y-x) \in K, \forall t \in (0,1)$, since K is convex. It follows that

$$\langle G(y_t), t(y-x)\rangle + \varphi(y_t) - \varphi(x) + \frac{\alpha}{2}\xi t^2\|x-y\|^2 \notin -\text{int}P, \quad \forall y \in K. \tag{2.17}$$

It follows from the P-convexity of φ that

$$t\varphi(y) + (1-t)\varphi(x) - \varphi(y_t) \in P. \tag{2.18}$$

Combining (2.17) with (2.18) and using Lemma 2.1, we obtain

$$\langle G(y_t), y-x\rangle + \varphi(y) - \varphi(x) + \frac{\alpha}{2}\xi t\|x-y\|^2 \notin -\text{int}P, \quad \forall y \in K.$$

Since G is hemicontinuous and $Z \setminus intP$ is closed, we deduce that

$$\langle G(x), y-x\rangle + \varphi(y) - \varphi(x) \notin -\text{int}P, \quad \forall y \in K, \tag{2.19}$$

which implies $x \in \aleph$. Therefore, $\Delta \subseteq \aleph$. i.e., $\aleph = \Delta$. So, if $\aleph$ is convex, then Δ is convex. □

Next, we prove a result which is one of the highlights of this paper.

Proposition 2.3. *Let K be a nonempty, closed and convex subset of X and let $F, G : K \to L(X,Z)$ be two hemicontinuous operators. Let F be C-monotone and let G be P-monotone. Let $\varphi : K \to Z$ be a P-convex vector-valued function, $\eta \in intC, \xi \in intP$ and $\alpha \geq 0$. Assume that there exists $s \in intP^*$ such that $\langle s,z\rangle \geq 0$ for all $z \notin -intP$. Then, $x^* \in \Gamma$ if and only if $x^* \in \Omega_\alpha$ with*

$$\langle F(x^*), x^*-y\rangle - \frac{\alpha}{2}\eta\|x^*-y\|^2 \notin intC, \quad \forall y \in \Omega_\alpha, \tag{2.20}$$

where Ω_α is the solution set of the lower-level regularization mixed vector variational inequality

$$\langle G(x), x-z\rangle + \varphi(x) - \varphi(z) - \frac{\alpha}{2}\xi\|x-z\|^2 \notin intP, \quad \forall z \in K. \tag{2.21}$$

Proof. If $x^* \in \Gamma$, then

$$\langle F(x^*), x^*-y\rangle \notin \text{int}C, \quad \forall y \in \Omega, \tag{2.22}$$

where Ω is the solution set of the lower-level mixed vector variational inequality

$$\langle G(x), x-z\rangle + \varphi(x) - \varphi(z) \notin \text{int}P, \quad \forall z \in K. \tag{2.23}$$

Since $\xi \in intP$, we have

$$-\frac{\alpha}{2}\xi\|x-z\|^2 \in -P. \tag{2.24}$$

By (2.23), (2.24) and Lemma 2.1, we obtain that

$$\langle G(x), x-z\rangle + \varphi(x) - \varphi(z) - \frac{\alpha}{2}\xi\|x-z\|^2 \notin \text{int}P, \quad \forall z \in K.$$

This means that $x \in \Omega_\alpha$. By using the similar argument proving above, we have

$$\langle F(x^*), x^*-y\rangle - \frac{\alpha}{2}\eta\|x^*-y\|^2 \notin \text{int}C, \quad \forall y \in \Omega_\alpha.$$

Conversely, by Lemma 2.4, we find that (2.21) is equivalent to

$$\langle G(z), x-z\rangle + \varphi(x) - \varphi(z) - \frac{\alpha}{2}\xi\|x-z\|^2 \notin \text{int}P, \quad \forall z \in K. \tag{2.25}$$

Since K is convex, we have $z_t = tz+(1-t)x \in K$, $\forall t \in (0,1)$. Replacing z by z_t in (2.25), one has

$$t\langle G(z_t), x-z\rangle + \varphi(x) - \varphi(z_t) - \frac{\alpha t^2}{2}\xi\|x-z\|^2 \notin \text{int}P, \quad \forall z \in K.$$

Since φ is P-convex, we get

$$-t\varphi(z) - (1-t)\varphi(x) + \varphi(z_t) \in -P.$$

Hence,

$$\langle G(z_t), x-z\rangle + \varphi(x) - \varphi(z) - \frac{\alpha t}{2}\xi\|x-z\|^2 \notin \text{int}P, \quad \forall z \in K. \tag{2.26}$$

Since $Z \setminus \text{int}P$ is closed and G is hemicontinuous, we deduce that

$$\langle G(x), x-z\rangle + \varphi(x) - \varphi(z) \in Z \setminus \text{int}P, \quad \forall z \in K,$$

which implies that

$$\langle G(x), x-z\rangle + \varphi(x) - \varphi(z) \notin \text{int}P, \quad \forall z \in K. \tag{2.27}$$

Therefore, $x \in \Omega$. It follows from Proposition 2.2 and Lemma 2.3 that Ω_α is convex. For any given $y \in \Omega_\alpha$, $\forall t \in (0,1)$, let $y_t = ty+(1-t)x^*$. Then $y_t \in \Omega_\alpha$. By Lemma 2.5, we have that (2.20) is equivalent to

$$\langle F(y), x^*-y\rangle - \frac{\alpha}{2}\xi\|x^*-y\|^2 \notin \text{int}C. \tag{2.28}$$

Replacing y by y_t in (2.28), one has

$$\langle F(y_t), x^*-y\rangle - \frac{\alpha t}{2}\xi\|x^*-y\|^2 \notin \text{int}C.$$

Since $Z \setminus \text{int}C$ is closed and F is hemicontinuous, we have

$$\langle F(x^*), x^*-y\rangle \notin \text{int}C, \quad \forall y \in \Omega. \tag{2.29}$$

Therefore, $x^* \in \Gamma$. This completes the proof. □

3. LP WELL-POSEDNESS FOR BVVI

In this section, we introduce the concept of the LP approximating sequences for the BVVI. In terms of the LP approximating sequences, we generalize the concept of the LP well-posedness to the BVVI. Some metric characterizations of these LP well-posedness concepts in the behavior of approximate solution sets are also discussed.

In practice, it is not easy to get the precise solutions of the BVVI. So, it is necessary to consider the approximation solutions of the BVVI. Let $\varepsilon \in [0,+\infty[$ be the tolerance error, $\eta \in \text{int}C$, $\xi \in \text{int}P$, and $\alpha \geq 0$ a fixed number. We consider the following regularization ε-bilevel vector variational inequality [in short, ε-BVVI]: find $x^* \in \Omega_\alpha(\varepsilon)$ such that

$$\langle F(x^*), x^*-y\rangle - \frac{\alpha}{2}\eta\|x^*-y\|^2 - \varepsilon\eta \notin \text{int}C, \quad \forall y \in \Omega_\alpha(\varepsilon),$$

where $\Omega_\alpha(\varepsilon)$ is the solution set of the lower-level regularization ε-mixed vector variational inequality [in short, ε-MVVI]: find $x \in K$ such that

$$\langle G(x), x-z\rangle + \varphi(x) - \varphi(z) - \frac{\alpha}{2}\xi\|x-z\|^2 - \varepsilon\xi \notin \text{int}P, \quad \forall z \in K.$$

We denote by $\Gamma_\alpha(\varepsilon)$ the solution set of the regularization ε-BVVI.

Definition 3.1. Let $(\varepsilon_n)_{n\in N}$ be a positive sequence with $0<\varepsilon_n\to 0$ as $n\to+\infty$. A sequence $(x_n)_{n\in N}\subseteq K$ is said to be an LP approximating solution sequence for the BVVI if, for each $n\in N$,

$$d(x_n,K)\le\varepsilon_n,$$

$$\langle F(x_n),x_n-y\rangle-\frac{\alpha}{2}\eta\|x_n-y\|^2-\varepsilon_n\eta\notin \operatorname{int}C,\ \forall y\in\Omega_\alpha(\varepsilon_n),$$

where $\Omega_\alpha(\varepsilon_n)$ is the solution set of the lower-level regularization ε-mixed vector variational inequality

$$\langle G(x_n),x_n-z\rangle+\varphi(x_n)-\varphi(z)-\frac{\alpha}{2}\xi\|x_n-z\|^2-\varepsilon_n\xi\notin \operatorname{int}P,\quad \forall z\in K.$$

Definition 3.2. The BVVI is said to be LP well-posed if the following conditions hold:

(i) Γ is a singleton set, i.e.,$\Gamma=\{x^*\}$;
(ii) every LP approximating solution sequence $(x_n)_{n\in N}$ for BVVI converges to x^*.

Definition 3.3. BVVI is said to be generalized LP well-posed if $\Gamma\neq\emptyset$ and every LP approximating sequence $(x_n)_{n\in N}$ for the BVVI has a subsequence which converges to some point of Γ.

Remark 3.1. By Definition 3.2 and Definition 3.3, we know that if the BVVI is LP well-posed or generalized LP well-posed, then Γ is compact.

Proposition 3.1. *Let K be a nonempty, closed and convex subset of X and let $F,G:K\to L(X,Z)$ be two continuous operators. Let F be C-monotone and let G be P-monotone. Let $\varphi:K\to Z$ be a P-convex, continuous vector-valued function, $\eta\in intC,\xi\in intP$ and $\alpha\ge 0$. Assume that there exists $s\in intP^*$ such that $\langle s,z\rangle\ge 0$ for all $z\notin -intP$. Then*

$$\Gamma=\cap_{\varepsilon>0}\Gamma_\alpha(\varepsilon_1).$$

Proof. We claim that $\Omega_\alpha(\varepsilon_1)\subset\Omega_\alpha(\varepsilon_2)$, $0\le\varepsilon_1<\varepsilon_2$. To prove that, we suppose that $x\in\Omega_\alpha(\varepsilon_1)$. Then

$$\langle G(x),x-z\rangle+\varphi(x)-\varphi(z)-\frac{\alpha}{2}\xi\|x-z\|^2-\varepsilon_1\xi\notin \operatorname{int}P,\quad \forall z\in K.$$

From $(\varepsilon_2-\varepsilon_1)\xi\in -P$ and Lemma 2.1, we obtain that

$$\langle G(x),x-z\rangle+\varphi(x)-\varphi(z)-\frac{\alpha}{2}\xi\|x-z\|^2-\varepsilon_2\xi\notin \operatorname{int}P,\quad \forall z\in K.$$

This implies that $x\in\Omega_\alpha(\varepsilon_2)$.

Next, we show that $\Gamma_\alpha(\varepsilon_1)\subset\Gamma_\alpha(\varepsilon_2)$, $0\le\varepsilon_1\le\varepsilon_2$. If $\bar{x}\in\Gamma_\alpha(\varepsilon_1)$, then

$$\langle F(\bar{x}),\bar{x}-y\rangle-\frac{\alpha}{2}\eta\|\bar{x}-y\|^2-\varepsilon_1\eta\notin \operatorname{int}C,\quad \forall y\in\Omega_\alpha(\varepsilon_1). \tag{3.1}$$

From $(\varepsilon_2-\varepsilon_1)\eta\in \operatorname{int}C$ and Lemma 2.1, we obtain that $\bar{x}\in\Omega_\alpha(\varepsilon_1)\subset\Omega_\alpha(\varepsilon_2)$ with

$$\langle F(\bar{x}),\bar{x}-y\rangle-\frac{\alpha}{2}\eta\|\bar{x}-y\|^2-\varepsilon_2\eta\notin \operatorname{int}C,\quad \forall y\in\Omega_\alpha(\varepsilon_1)\subset\Omega_\alpha(\varepsilon_2).$$

Suppose by contradiction that there is $y\in\Omega_\alpha(\varepsilon_2)$ and $y\notin\Omega_\alpha(\varepsilon_1)$ such that

$$\langle F(\bar{x}),\bar{x}-y\rangle-\frac{\alpha}{2}\eta\|\bar{x}-y\|^2-\varepsilon_2\eta\in \operatorname{int}C.$$

Then,

$$\exists z'\in K,\ \langle G(y),y-z'\rangle+\varphi(y)-\varphi(z')-\frac{\alpha}{2}\xi\|y-z'\|^2-\varepsilon_1\xi\in \operatorname{int}P, \tag{3.2}$$

and

$$\langle G(y), y-z\rangle + \varphi(y) - \varphi(z) - \frac{\alpha}{2}\xi\|y-z\|^2 - \varepsilon_2\xi \notin \mathrm{int}P, \quad \forall z \in K.$$

It follows that

$$\langle G(y), y-z'\rangle + \varphi(y) - \varphi(z') - \frac{\alpha}{2}\xi\|y-z'\|^2 - \varepsilon_2\xi \notin \mathrm{int}P. \tag{3.3}$$

Taking the limit $\varepsilon_1, \varepsilon_2 \to 0$ in (3.2) and (3.3), respectively, this reaches a contradiction. Hence,

$$\Gamma_\alpha(\varepsilon_1) \subset \Gamma_\alpha(\varepsilon_2).$$

It follows from Proposition 2.3 that $\Gamma \subset \cap_{\varepsilon>0}\Gamma_\alpha(\varepsilon)$.

On the other hand, since $\Gamma_\alpha(\varepsilon_1) \subset \Gamma_\alpha(\varepsilon_2)$, we deduce that

$$\bigcap_{\varepsilon>0}\Gamma_\alpha(\varepsilon) = \lim_{\varepsilon\to 0}\Gamma_\alpha(\varepsilon).$$

That is,

$$\bigcap_{\varepsilon>0}\Gamma_\alpha(\varepsilon) = \{x \in K : \langle F(x), x-y\rangle - \frac{\alpha}{2}\eta\|x-y\|^2 \notin \mathrm{int}C,\ \forall y \in \Omega_\alpha\},$$

where

$$\Omega_\alpha = \{x \in K : \langle G(x), x-z\rangle + \varphi(x) - \varphi(z) - \frac{\alpha}{2}\xi\|x-z\|^2 \notin \mathrm{int}P,\ \forall z \in K\}.$$

From Proposition 2.3, it follows that $\Gamma \supset \cap_{\varepsilon>0}\Gamma_\alpha(\varepsilon)$. Therefore,

$$\Gamma = \bigcap_{\varepsilon>0}\Gamma_\alpha(\varepsilon).$$

This completes the proof. □

The following result derives a Furi-Vignoli-type characterization for the LP well-posedness of the BVVI. It shows that the LP well-posedness of the BVVI can be characterized by the behavior of the diameter of the approximating solution set.

Theorem 3.1. *Let K be a nonempty, closed and convex subset of X and let $F, G : K \to L(X,Z)$ be two continuous operators. Let F be C-monotone and let G be P-monotone. Let $\varphi : K \to Z$ be a continuous P-convex vector-valued function. Assume that there exists $s \in intP^*$ such that $\langle s, z\rangle \geq 0$ for all $z \notin -intP$. Then, BVVI is LP well-posed if and only if*

$$\Gamma_\alpha(\varepsilon) \neq \emptyset, \forall \varepsilon > 0 \text{ and } diam(\Gamma_\alpha(\varepsilon)) \to 0 \text{ as } \varepsilon \to 0. \tag{3.4}$$

Proof. Suppose that the BVVI is LP well-posed and $x^* \in \Gamma$ is its unique solution. By Proposition 3.1, we have $\Gamma_\alpha(\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. Assume by contradiction that $diam(\Gamma_\alpha(\varepsilon)) \nrightarrow 0$ as $\varepsilon \to 0$. Then there exists $\rho > 0$ and $0 < \varepsilon_n \to 0$ and $(u_n)_{n\in N}, (v_n)_{n\in N}$ with $u_n, v_n \in \Gamma_\alpha(\varepsilon_n)$ such that

$$\|u_n - v_n\| > \rho, \quad \forall n \in N. \tag{3.5}$$

Since $u_n, v_n \in \Gamma_\alpha(\varepsilon_n)$, $\forall n \in N$, we obtain that

$$d(u_n, K) \leq \varepsilon_n,$$

and

$$\langle F(u_n), u_n - y\rangle - \frac{\alpha}{2}\eta\|u_n - y\|^2 - \varepsilon_n\eta \notin \mathrm{int}C, \quad \forall y \in \Omega_\alpha(\varepsilon_n),$$

where $\Omega_\alpha(\varepsilon_n)$ is the solution set of the lower-level regularization ε_n-mixed vector variational inequality

$$\langle G(u_n), u_n - z\rangle + \varphi(u_n) - \varphi(z) - \frac{\alpha}{2}\xi\|u_n - z\|^2 - \varepsilon_n\xi \notin \text{int}P, \quad \forall z \in K,$$

$$d(v_n, K) \leq \varepsilon_n,$$

and

$$\langle F(v_n), v_n - y\rangle - \frac{\alpha}{2}\eta\|v_n - y\|^2 - \varepsilon_n\eta \notin \text{int}C, \quad \forall y \in \Omega_\alpha(\varepsilon_n),$$

where $\Omega_\alpha(\varepsilon_n)$ is the solution set of the lower-level regularization ε_n-mixed vector variational inequality

$$\langle G(v_n), u_n - z\rangle + \varphi(v_n) - \varphi(z) - \frac{\alpha}{2}\xi\|v_n - z\|^2 - \varepsilon_n\xi \notin \text{int}P, \quad \forall z \in K.$$

Thus $(u_n)_{n\in N}, (v_n)_{n\in N}$ are LP approximating sequences for the BVVI. By the LP well-posedness, they converge to the unique solution of the BVVI which contradicts (3.5). Conversely, suppose that condition (3.4) holds. Then $x^* \in \Gamma$ is the unique solution of the BVVI. Indeed, if $\bar{x}(\neq x^*)$ is another solution of BVVI, then

$$0 < \|x^* - \bar{x}\| \leq diam(\Gamma_\alpha(\varepsilon)) \to 0 \; as \; \varepsilon \to 0, \tag{3.6}$$

which is a contradiction. Let $(x_n)_{n\in N}$ be a LP approximating sequence for the BVVI. Then there exists $0 < \varepsilon_n \to 0$ such that

$$d(x_n, K) \leq \varepsilon_n,$$

$$\langle F(x_n), x_n - y\rangle - \frac{\alpha}{2}\eta\|x_n - y\|^2 - \varepsilon_n\eta \notin \text{int}C, \; \forall y \in \Omega_\alpha(\varepsilon_n),$$

where $\Omega_\alpha(\varepsilon_n)$ is the solution set of the lower-level regularization ε_n-mixed vector variational inequality

$$\langle G(x_n), x_n - z\rangle + \varphi(x_n) - \varphi(z) - \frac{\alpha}{2}\xi\|x_n - z\|^2 - \varepsilon_n\xi \notin \text{int}P, \quad \forall z \in K.$$

Hence, $x_n \in \Gamma_\alpha(\varepsilon_n)$. Since $x^* \in \Gamma_\alpha(\varepsilon)$, we obtain that

$$0 < \|x_n - x^*\| \leq diam(\Gamma_\alpha(\varepsilon_n)) \to 0 \; as \; n \to +\infty. \tag{3.7}$$

This implies that $x_n \to x^*$. So, the BVVI is LP well-posed. This completes the proof. □

As an application of Theorem 3.1, we immediately obtain the following result, which shows the LP well-posedness for the MVVI.

Corollary 3.1. *Let X and Z be real Banach spaces and let K be a nonempty, closed and convex subset of X. Let $P \subset Z$ be a nonempty proper closed convex pointed cone with the topological interior, intP. Let $G : K \to L(X,Z)$ be a hemicontinuous and P-monotone operator and let $\varphi : K \to Z$ be a continuous P-convex vector-valued function. Assume that there exists $s \in intP^*$ such that $\langle s, z\rangle \geq 0$ for all $z \notin -intP$. Then, MVVI is LP well-posed if and only if*

$$\Omega_\alpha(\varepsilon) \neq \emptyset, \forall \varepsilon > 0 \; and \; diam(\Omega_\alpha(\varepsilon)) \to 0 \; as \; \varepsilon \to 0.$$

Under the suitable conditions, we can establish a metric characterization of the generalized well-posedness for BVVI via the Kuratowski measure of noncomapctness of the approximating solution sets.

Theorem 3.2. *Let K be a nonempty, closed and convex subset of X and let $F,G : K \to L(X,Z)$ be two continuous operators. Let F be C-monotone and let G be P-monotone. Let $\varphi : K \to Z$ be a P-convex vector-valued function. Assume that there exists $s \in intP^*$ such that $\langle s,z\rangle \geq 0$ for all $z \notin -intP$. Then, BVVI is LP well-posed in the generalized sense if and only if*

$$\Gamma_\alpha(\varepsilon) \neq \emptyset, \forall \varepsilon > 0 \text{ and } \mu(\Gamma_\alpha(\varepsilon)) \to 0 \text{ as } \varepsilon \to 0. \tag{3.8}$$

Proof. Suppose that BVVI is LP well-posed in the generalized sense. By Remark 3.1, we know that Γ is nonempty and compact. Then $\Gamma_\alpha(\varepsilon) \neq \emptyset$ since $\Gamma \subset \Gamma_\alpha(\varepsilon)$ for all $\varepsilon > 0$. Now we show that

$$\mu(\Gamma_\alpha(\varepsilon)) \to 0, \text{ as } \varepsilon \to 0.$$

Observe that, for any $\varepsilon > 0$,

$$H(\Gamma_\alpha(\varepsilon),\Gamma) = \max\{e(\Gamma_\alpha(\varepsilon),\Gamma), e(\Gamma,\Gamma_\alpha(\varepsilon))\} = e(\Gamma_\alpha(\varepsilon),\Gamma). \tag{3.9}$$

With the compactness of Γ, it follows that

$$\mu(\Gamma_\alpha(\varepsilon)) \leq 2H(\Gamma_\alpha(\varepsilon),\Gamma) + \mu(\Gamma) = 2e(\Gamma_\alpha(\varepsilon),\Gamma).$$

To prove (3.8), it is sufficient to show that

$$e(\Gamma_\alpha(\varepsilon),\Gamma) \to 0, \text{ as } \varepsilon \to 0.$$

Indeed, if $e(\Gamma_\alpha(\varepsilon),\Gamma) \nrightarrow 0, \text{ as } \varepsilon \to 0$, then there exist $\sigma > 0$ and $(\varepsilon_n)_{n\in N} \subset R^+$ with $\varepsilon_n \to 0$, and $x_n \in \Gamma_\alpha(\varepsilon)$ such that

$$x_n \notin \Gamma + B(0,\sigma), \ \forall n \in N, \tag{3.10}$$

where $B(0,\sigma)$ is the closed ball centered at 0 with radius σ. Since $x_n \in \Gamma_\alpha(\varepsilon_n)$, we have that $(x_n)_{n\in N}$ is an LP approximating sequence for BVVI. Since BVVI is LP well-posed in the generalized sense, there exists a subsequence $(x_{n_k})_{k\in N}$ of $(x_n)_{n\in N}$ which converges to some point of Γ. This contradicts (3.10). Hence

$$e(\Gamma_\alpha(\varepsilon),\Gamma) \to 0, \text{ as } \varepsilon \to 0. \tag{3.11}$$

Conversely, assume that (3.8) holds. We first show that $\Gamma_\alpha(\varepsilon)$ is closed for all $\varepsilon > 0$. Let $x_n \in \Gamma_\alpha(\varepsilon)$ and $x_n \to x^*$ as $n \to +\infty$. Then

$$d(x_n,K) \leq \varepsilon,$$

$$\langle F(x_n), x_n - y\rangle - \frac{\alpha}{2}\eta\|x_n - y\|^2 - \varepsilon\eta \notin \text{int}C, \ \forall y \in \Omega_\alpha(\varepsilon),$$

where $\Omega_\alpha(\varepsilon)$ is the solution set of the lower-level regularization ε-mixed vector variational inequality

$$\langle G(x_n), x_n - z\rangle + \varphi(x_n) - \varphi(z) - \frac{\alpha}{2}\xi\|x_n - z\|^2 - \varepsilon\xi \notin \text{int}P, \ \forall z \in K.$$

Since F,G,φ are continuous, we obtain from Lemma 2.2 and $Z \setminus \text{int}C$ that

$$d(x^*,K) \leq \varepsilon,$$

$$\langle F(x^*), x^* - y\rangle - \frac{\alpha}{2}\eta\|x^* - y\|^2 - \varepsilon_n\eta \in Z \setminus \text{int}C, \ \forall y \in \Omega_\alpha(\varepsilon),$$

where $\Omega_\alpha(\varepsilon)$ is the solution set of the lower-level regularization ε-mixed vector variational inequality

$$\langle G(x^*), x^* - z\rangle + \varphi(x^*) - \varphi(z) - \frac{\alpha}{2}\xi\|x^* - z\|^2 - \varepsilon\xi \in Z\setminus \mathrm{int}P,\ \forall z \in K.$$

Hence,

$$d(x^*, K) \leq \varepsilon,$$

$$\langle F(x^*), x^* - y\rangle - \frac{\alpha}{2}\eta\|x^* - y\|^2 - \varepsilon\eta \notin \mathrm{int}C,\ \forall y \in \Omega_\alpha(\varepsilon),$$

where $\Omega_\alpha(\varepsilon)$ is the solution set of the lower-level regularization ε-mixed vector variational inequality

$$\langle G(x^*), x^* - z\rangle + \varphi(x^*) - \varphi(z) - \frac{\alpha}{2}\xi\|x^* - z\|^2 - \varepsilon\xi \notin \mathrm{int}P, \quad \forall z \in K,$$

which means that $x^* \in \Gamma_\alpha(\varepsilon)$. Observe that $\Gamma = \cap_{\varepsilon>0}\Gamma_\alpha(\varepsilon)$. From (3.8), we conclude from the theorem in [15] that Γ is nonempty, compact and

$$e(\Gamma_\alpha(\varepsilon_n), \Gamma) = H(\Gamma_\alpha(\varepsilon_n), \Gamma) \to 0 \ as\ \varepsilon_n \to 0.$$

Let $(x_n)_{n\in N}$ be a LP approximating sequence for BVVI. Then there exists $0 < \varepsilon_n \to 0$ such that

$$d(x_n, K) \leq \varepsilon_n,$$

$$\langle F(x_n), x_n - y\rangle - \frac{\alpha}{2}\eta\|x_n - y\|^2 - \varepsilon_n\eta \notin \mathrm{int}C,\ \forall y \in \Omega_\alpha(\varepsilon_n),$$

where $\Omega_\alpha(\varepsilon_n)$ is the solution set of the lower-level regularization ε_n-mixed vector variational inequality

$$\langle G(x_n), x_n - z\rangle + \varphi(x_n) - \varphi(z) - \frac{\alpha}{2}\xi\|x_n - z\|^2 - \varepsilon_n\xi \notin \mathrm{int}P, \forall z \in K.$$

It follows that $x_n \in \Gamma_\alpha(\varepsilon_n)$. Hence, $\exists u_n \in \Gamma$ such that

$$\|x_n - u_n\| = d(x_n, \Gamma) \leq e(\Gamma_\alpha(\varepsilon_n), \Gamma) = H(\Gamma_\alpha(\varepsilon_n), \Gamma) \to 0.$$

Since Γ is compact, there exists $x^* \in \Gamma$ such that

$$u_{n_j} \to x^*, \tag{3.12}$$

where $(u_{n_j})_{j\in N} \subset (u_n)_{n\in N}$. Therefore, there exists $(x_{n_j})_{j\in N} \subset (x_n)_{n\in N}$ such that x_{n_j} converges to $x^* \in \Gamma$. Hence, BVVI is LP well-posed in the generalized sense. This completes the proof. □

As an application of Theorem 3.2, we immediately obtain the following results, which shows the LP well-posedness for MVVI.

Corollary 3.2. *Let X and Z be real Banach spaces and let K is a nonempty, closed and convex subset of X. Let $P \subset Z$ be a nonempty proper closed convex pointed cone with the topological interior, intP. Let $G : K \to L(X,Z)$ be a continuous and P-monotone operator and let $\varphi : K \to Z$ be a continuous P-convex vector-valued function. Assume that there exists $s \in intP^*$ such that $\langle s, z\rangle \geq 0$ for all $z \notin -intP$. Then, MVVI is LP well-posed in the generalized sense if and only if*

$$\Omega_\alpha(\varepsilon) \neq \emptyset, \forall\varepsilon > 0 \ and\ \mu(\Omega_\alpha(\varepsilon)) \to 0 \ as\ \varepsilon \to 0.$$

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