J. Nonlinear Var. Anal. 3 (2019), No. 3, pp. 305-316
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.3.2019.3.06

INFINITELY MANY HOMOCLINIC SOLUTIONS FOR FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH LOCALLY DEFINED POTENTIALS

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Abstract. In this paper, we are concerned with the existence of infinitely many homoclinic solutions for a class of fourth-order differential equations

$$u^{(4)}(x)+\omega u''(x)+a(x)u(x)=f(x,u(x)), \quad \forall x\in\mathbb{R},$$

where the function $a\in C(\mathbb{R},\mathbb{R})$ may be negative on a bounded interval and the potential $F(x,u)=\int_0^u f(x,t)dt$ is only locally defined near the origin with respect to the second variable. Some recent results in the literature are generalized and improved.
Keywords. Fourth-order differential equations; Homoclinic solutions; Local conditions; Variational methods.
2010 Mathematics Subject Classification. 58E05, 34C37, 70H05.

1. Introduction

Consider the following fourth-order differential equation

$$(\mathscr{F}) \qquad u^{(4)}(x)+\omega u''(x)+a(x)u(x)=f(x,u(x)), \quad \forall x\in\mathbb{R},$$

where ω is a constant, $a\in C(\mathbb{R},\mathbb{R})$ and $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ are two given functions. Here as usual, we say that a solution u of $(\mathscr{F})$ is homoclinic (to 0) if $u\in C^4(\mathbb{R},\mathbb{R})$ and $u(x)\longrightarrow 0$ as $|x|\longrightarrow\infty$. In addition, if $u\neq 0$, then u is called a nontrivial homoclinic solution. This equation arises from some problems associated to mathematical models for the study of pattern formation in physics and mechanics, for example, the well-known extended Fisher-Kolmogorov equation proposed by Coulet, Elphick and Repeaux [4], in the study of phase transitions, the fourth-order elastic beam equation in describing a large class of elastic deflection [13], the Swift-Hohenberg equation which is a general model for pattern-forming process derived in [5] to describe vandom thermal fluctuations in the Boussinesque equation and the propagation of lazers [7].

With the aid of variational methods and the critical point theory, the existence and the multiplicity of homoclinic solutions for $(\mathscr{F})$ have been extensively investigated in the literature over the past years, see, e.g., [1, 2, 3, 8, 9, 10, 15, 11, 12, 14, 16, 17, 18, 19, 20] and the references therein. Many results are on the existence of homoclinic solutions to equation $(\mathscr{F})$ when $a(x)$ and $f(x,u)$ are independent of x or periodic in x; see [1, 2, 3, 8, 14] and the references cited therein. Compared to the periodic case, the problem is quite different in nature for the nonperiodic case due to the lack of compactness of the Sobolev embedding. After the work of Li [9], there are some results concerning the nonperiodic case;

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Received March 29, 2019; Accepted July 1, 2019.

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see [9, 10, 15, 11, 12, 16, 17, 18, 19, 20] and the references therein. For this case, function a plays an important role. Most of these papers consider the following condition
$(\mathscr{A})$ There exists a constant $a_0 > 0$ such that

$$a_0 \leq a(x) \longrightarrow +\infty \ as \ |x| \longrightarrow \infty$$

and

$$\omega \leq 2\sqrt{a_0},$$

which is used to establish the corresponding compact embedding lemmas on suitable functional spaces. However, if there exists the function a satisfying $a \leq 0$ in some finite interval I of $\mathbb{R}$, the condition $(\mathscr{A})$ does not hold. Besides, we also note that all these papers required $F(x,u)$ to satisfy some kinds of growth conditions at infinity with respect to u, such as, superquadratic, asymptotically quadratic or subquadratic growth.

In the present paper, we will study the existence of infinitely many homoclinic solutions for $(\mathscr{F})$ when a is unnecessarily positive in all $\mathbb{R}$ and the nonlinearity $F(x,u) = \int_0^u f(x,t)dt$ is still only locally defined near the origin with respect to u. More precisely, let $f : \mathbb{R} \times I_\delta(0) \longrightarrow \mathbb{R}$ be a continuous function, where δ is a positive constant and $I_\delta(0)$ is the open interval in $\mathbb{R}$ centered at zero with radius δ. We make the following assumptions:
$(\mathscr{A}_\sigma)$ There exists a constant $\sigma < 0$ such that

$$|x|^{\sigma-1} a(x) \longrightarrow +\infty \ as \ |x| \longrightarrow \infty;$$

(F_1) There exist constants $\nu \in]0,1[$, $\alpha \in [\frac{1}{\nu}, \infty[$, $b \geq 0$ and $p \in L^\alpha(\mathbb{R}, \mathbb{R}^+)$ such that

$$|\nabla F(x,u)| \leq p(x) + b\,|u|^\nu, \ \forall (x,u) \in \mathbb{R} \times I_\delta(0);$$

(F_2)
$$F(x,-u) = F(x,u), \ \forall (x,u) \in \mathbb{R} \times I_\delta(0);$$

(F_3)
$$\lim_{|u| \to 0} \frac{|F(x,u)|}{|u|^2} = +\infty, \ uniformly \ for \ all \ x \in \mathbb{R}.$$

Our main result reads as follows.

Theorem 1.1. *Assume that* $(\mathscr{A}_\sigma)$ *and* $(F_1) - (F_3)$ *are satisfied. Then* $(\mathscr{F})$ *possesses infinitely many nontrivial orbits* (u_k) *such that* $\max_{x \in \mathbb{R}} |u_k(x)| \longrightarrow 0$ *as* $k \longrightarrow \infty$.

Example 1.1. Let $0 < \nu < 1$ and $b \geq 0$ be some given constants, and define

$$F(x,u) = (\frac{1}{x^2+1})^{\frac{\nu}{2}} ln(1+|u|^2) + \frac{b}{\nu+1}\,|u|^{\nu+1}, \ x \in \mathbb{R}, \ |u| < 1.$$

It is clear that F satisfies (F_2) and (F_3) with $\delta = 1$. An easy computation shows that

$$|f(x,u)| \leq p(x) + b\,|u|^\nu, \ \forall x \in \mathbb{R}, \ |u| < 1$$

where

$$p(x) = (\frac{1}{x^2+1})^{\frac{\nu}{2}}.$$

Moreover, for $\alpha = \frac{2}{\nu} > \frac{1}{\nu}$, we have

$$\int_{\mathbb{R}} (p(x))^\alpha = \int_{\mathbb{R}} \frac{1}{x^2+1} < \infty.$$

Hence assumption (F_1) is satisfied. Therefore by Theorem 1.1, the corresponding system $(\mathscr{F})$ possesses infinitely many homoclinic orbits.

2. Preliminaries

In this section, we will assume that function a satisfies the following condition
$(\widetilde{\mathscr{A}_\sigma})$ There exist two constants $\sigma < 0$ and $a_0 > 0$ such that

$$|x|^{\sigma-1}a(x) \longrightarrow +\infty \ as \ |x| \longrightarrow \infty;$$

and

$$\omega \leq 2\sqrt{a_0},\ a_0 \leq a(x),\ \forall x \in \mathbb{R}.$$

In the following, we shall use $\|.\|_{L^s}$ to denote the norm of $L^s(\mathbb{R})$ for any $s \in [1,\infty]$. Let $H^2(\mathbb{R})$ be the Sobolev space with inner product and norm given respectively by

$$< u,v >_{H^2}= \int_{\mathbb{R}} [u''(x)v''(x) + u'(x)v'(x) + u(x)v(x)]dx$$

and

$$\|u\|_{H^2} =< u,u >_{H^2}^{\frac{1}{2}}$$

for all $u,v \in H^2(\mathbb{R})$.

Lemma 2.1. [3] *Assume that there exists $a_0 > 0$ such that $\omega \leq 2\sqrt{a_0}$, $a_0 \leq a(x)$, $\forall x \in \mathbb{R}$. Then, there exists a constant $c_0 > 0$ such that*

$$\int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx \geq c_0 \|u\|_{H^2}^2,\ \forall u \in H^2(\mathbb{R}).$$

By Lemma 2.1, we define the Hilbert space

$$E = \left\{ u \in H^2(\mathbb{R}) / \int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx < \infty \right\}$$

with the inner product

$$< u,v >= \int_{\mathbb{R}} [u''(x)v''(x) - \omega u'(x)v'(x) + a(x)u(x)v(x)]dx$$

and the corresponding norm

$$\|u\| = (\int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx)^{\frac{1}{2}}.$$

In order to prove our result, the following compactness result is necessary.

Lemma 2.2. *Assume that $(\widetilde{\mathscr{A}_\sigma})$ is satisfied. Then E is compactly embedded in $L^s(\mathbb{R})$ for all $s \in [1,\infty]$. Moreover, for all $s \in [1,\infty]$, there exists $\eta_s > 0$ such that*

$$\|u\|_{L^s(\mathbb{R})} \leq \eta_s \|u\|,\ \forall u \in E. \tag{2.1}$$

Proof. Define

$$\beta(R) = \inf_{|x|\geq R} a(x),\ R \in [0,\infty[.$$

Since $a(x) \longrightarrow +\infty$ as $|x| \longrightarrow \infty$, we have that $\beta(R)$ increases and tends to $+\infty$ as $R \longrightarrow +\infty$. Let $(u_k) \subset E$ be a sequence such that $u_k \rightharpoonup u$ in E. The Banach-Steinhauss theorem implies that

$$M = \sup_{k\in\mathbb{N}} \|u_k - u\| < \infty.$$

By Lemma 2.1, we have, for all $k \in \mathbb{N}$,

$$\|u_k - u\|_{L^2}^2 \le \|u_k - u\|_{H^2}^2 \le \frac{1}{c_0}\|u_k - u\|^2 \le \frac{M^2}{c_0}.$$

Therefore, it holds

$$\begin{aligned}\int_{\mathbb{R}} a(x)\,|u_k-u|^2\,dx &\le \int_{\mathbb{R}}[\left|u_k''-u''\right|^2 - \omega\left|u_k'-u'\right|^2 + a(x)\,|u_k-u|^2]dx\\ &\quad+\omega\int_{\mathbb{R}}\left|u_k'-u'\right|^2dx \le \|u_k-u\|^2 + |\omega|\,\|u_k-u\|_{H^2}^2\\ &\le M^2 + |\omega|\frac{M^2}{c_0} = \overline{M}.\end{aligned}$$

-Step 1. For any $\varepsilon > 0$, we take $R_0 > 0$ large enough such that

$$\frac{\overline{M}}{\beta(R_0)} < \frac{\varepsilon^2}{2}. \tag{2.2}$$

By the Sobolev embedding theorem, $H^2(-R_0,R_0)$ is compactly embedded in $L^2(-R_0,R_0)$. So by going to a subsequence if necessary, there exists k_0 such that, for all $k \ge k_0$,

$$\int_{|x|\le R_0} |u_k(x)-u(x)|^2\,dx < \frac{\varepsilon^2}{2}. \tag{2.3}$$

Combining (2.2) and (2.3) yields

$$\begin{aligned}\|u_k-u\|_{L^2}^2 &= \int_{|x|\le R_0}|u_k(x)-u(x)|^2\,dx + \int_{|x|\ge R_0}|u_k(x)-u(x)|^2\,dx\\ &\le \frac{\varepsilon^2}{2} + \int_{|x|\ge R_0}\frac{a(x)\,|u_k(x)-u(x)|^2}{\beta(R_0)}dx\\ &\le \frac{\varepsilon^2}{2} + \frac{\overline{M}}{\beta(R_0)} < \varepsilon^2.\end{aligned}$$

Consequently, $u_k \longrightarrow u$ in $L^2(\mathbb{R})$ and then E is compactly embedded in $L^2(\mathbb{R})$.

-Step 2. Since for any $n \in \mathbb{N}$, $v \in E$ and $\tau \in \mathbb{R}$, we have

$$\begin{aligned}v(x) &= \int_\tau^x \Big[v'(s)\frac{(s-\tau)^{n+1}}{(x-\tau)^n} + v(s)\frac{(n+1)(s-\tau)^n}{(x-\tau)^n}\Big]ds\\ &\quad+\int_x^{\tau+1}\Big[-v'(s)\frac{(\tau+1-s)^{n+1}}{(\tau+1-x)^n} + v(s)\frac{(n+1)(\tau+1-s)^n}{(\tau+1-x)^n}\Big]ds.\end{aligned} \tag{2.4}$$

By the Hölder's inequality, (2.4) implies, for all $\tau \le x \le \tau+1$,

$$|v(x)|^2 \le \frac{1}{2n+3}\int_\tau^{\tau+1}\left|v'(s)\right|^2 ds + \frac{(n+1)^2}{2n+1}\int_\tau^{\tau+1}|v(s)|^2\,ds.$$

In particular, for any $k \in \mathbb{N}$, $R > 0$ and $|x| \ge R$, one has

$$\begin{aligned}|u_k(x)-u(x)|^2 &\le \frac{1}{\sqrt{2n+3}}\Big(\int_{|s|\ge R}\left|u_k'(s)-u'(s)\right|^2 ds\Big)^{\frac{1}{2}}\\ &\quad+\frac{n+1}{\sqrt{2n+1}}\Big(\int_{|s|\ge R}|u_k(s)-u(s)|^2\,ds\Big)^{\frac{1}{2}}\\ &\le \frac{M}{c_0\sqrt{2n+3}} + \frac{n+1}{\sqrt{2n+1}}\Big(\int_{|s|\ge R}a(s)\frac{|u_k(s)-u(s)|^2}{\beta(R)}ds\Big)^{\frac{1}{2}}.\end{aligned} \tag{2.5}$$

For any $\varepsilon > 0$, taking first n large enough such that

$$\frac{M}{\sqrt{c_0}\sqrt{2n+3}} < \frac{\varepsilon^2}{8}$$

and then R_0 large enough satisfying

$$\frac{(n+1)\overline{M}}{\sqrt{2n+1}\beta(R_0)} < \frac{\varepsilon^2}{8},$$

we see from (2.5) that

$$\max_{|x|\geq R_0} |u_k(x) - u(x)| < \frac{\varepsilon}{2}, \tag{2.6}$$

for all $k \in \mathbb{R}$. Again by the Sobolev compact embedding theorem, there exists an integer k_0 such that, for all $k \geq k_0$,

$$\max_{|x|\leq R_0} |u_k(x) - u(x)| < \frac{\varepsilon}{2},$$

which together with (2.6) sends to $\|u_k - u\|_{L^\infty(\mathbb{R})} < \varepsilon$. Hence $u_k \longrightarrow u$ in $L^\infty(\mathbb{R})$ and E is compactly embedded in $L^\infty(\mathbb{R})$.

-Step 3. For any $s \in]2,\infty[$, since, for all $v \in E$,

$$\int_{\mathbb{R}} |v(x)|^s \, dx \leq \|u\|_{L^\infty}^{s-2} \|u\|_{L^2}^2,$$

one sees immediately by Steps 1 and 2 that $u_k \longrightarrow u$ in $L^s(\mathbb{R})$. Thus, E is compactly embedded in $L^s(\mathbb{R})$.

-Step 4. Finally, let $s \in [1,2[$. For any $R > 0$, let

$$\mu(R) = \inf_{|x|\geq R} |x|^{\sigma-1} a(x).$$

Then by $(\widetilde{\mathscr{A}_\sigma})$, $\mu(R) \longrightarrow +\infty$ as $R \longrightarrow \infty$. Set $\gamma = \frac{1-\sigma}{2-s}$. Then $s > \frac{2}{2-s}$ and $\gamma s > 1$. Note that, for any $R \geq 1$ and $v \in E$,

$$\begin{aligned}
\int_{|x|\geq R} |v(x)|^s \, dx &= \int_{|x|\geq R, |x|^\gamma |v(x)|\leq 1} |v(x)|^s \, dx + \int_{|x|\geq R, |x|^\gamma |v(x)|>1} |v(x)|^s \, dx \\
&\leq \int_{|x|\geq R} |x|^{-\gamma s} \, dx + \int_{|x|\geq R, |x|^\gamma |v(x)|>1} (|x|^\gamma |v(x)|)^s \, |x|^{-\gamma s} \, dx \\
&\leq \int_{|x|\geq R} |x|^{-1+\sigma} \, dx + \int_{|x|\geq R} |v(x)|^2 \, |x|^{(2-s)\gamma} \, dx \\
&\leq \int_{|x|\geq R} |x|^{-1+\sigma} \, dx + \int_{|x|\geq R} |v(x)|^2 \, |x|^{1-\sigma} \, dx \\
&\leq \int_{|x|\geq R} |x|^{-1+\sigma} \, dx + \frac{1}{\mu(R)} \int_{|x|\geq R} a(x) \, |v(x)|^2 \, dx \\
&= -2\frac{R^\sigma}{\sigma} + \frac{\|v\|^2}{\mu(R)}.
\end{aligned} \tag{2.7}$$

Set $\varepsilon > 0$. There is $R > 0$ such that

$$-2\frac{R^\sigma}{\sigma} + \frac{M^2}{\mu(R)} < \frac{\varepsilon}{2}.$$

It follows from (2.7) that

$$\int_{|x|\geq R} |u_k(x) - u(x)|^s \, dx < \frac{\varepsilon}{2}, \; \forall k \in \mathbb{N}. \tag{2.8}$$

Since $u_k \longrightarrow u$ uniformly on $[-R,R]$, there is $k_0 \in \mathbb{N}$ such that

$$\int_{|x|\leq R} |u_k(x)-u(x)|^s\,dx < \frac{\varepsilon}{2},\ \forall k \geq k_0. \tag{2.9}$$

Combining (2.8) and (2.9), we get $u_k \longrightarrow u$ in $L^s(\mathbb{R})$ and then E is compactly embedded in $L^s(\mathbb{R})$. The proof of Lemma 2.2 is completed. □

To study the critical points of the variational functional associated with $(\mathscr{F})$, we need to recall the following variant symmetric mountain pass lemma due to Kajikiya [6]. We will first recall the notion of genus. Let E be a Banach space and let A be a subset of E. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k for which there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If such a k does not exist, we define $\gamma(A) = +\infty$. Moreover, we set $\gamma(\phi) = 0$. Let

$$\Gamma_k = \{A \subset E / A \textit{ is a closed symmetric subset},\ 0 \notin A,\ \gamma(A) \geq k\}.$$

The properties of genus used in the proof of our main result are summarized as follows.

Lemma 2.3. [6] *Let A and B be closed symmetric subsets of E that do not contain the origin. Then the following hold.*
a) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
b) The $n-$dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam theorem.

Lemma 2.4. [6] *Let E be an infinite-dimensional Banach space and $\Phi \in C^1(E,\mathbb{R})$ satisfies the following*
(Φ_1) $\Phi(0) = 0$, Φ is even and bounded from below and Φ satisfies the $(PS)-$condition;
(Φ_2) For each $k \in \mathbb{N}$, there exists $A_k \subset \Gamma_k$ such that

$$\sup_{u\in A_k} \Phi(u) < 0.$$

Then Φ possesses a sequence of critical points (u_k) such that

$$\Phi(u_k) \leq 0,\ u_k \neq 0,\ \forall k \in \mathbb{N} \textit{ and } \lim_{k\longrightarrow\infty} u_k = 0.$$

3. Proof of Theorem 1.1

From $(\mathscr{A}_\sigma)$, we know that there exists a positive constant a_0 such that $a(x)+2a_0 \geq a_0$ for all $x \in \mathbb{R}$ and $\omega \leq 2\sqrt{a_0}$. Let

$$\overline{a}(x) = a(x) + 2a_0$$

and

$$\overline{f}(x,u) = f(x,u) + 2a_0 u.$$

Consider the following fourth-order differential equation

$$(\overline{\mathscr{F}}).\qquad u^{(4)}(x) + \omega u''(x) + \overline{a}(x)u(x) = \overline{f}(x,u(x)),\ \forall x \in \mathbb{R}$$

Then $(\overline{\mathscr{F}})$ is equivalent to $(\mathscr{F})$. Moreover, it is easy to check that the hypotheses $(F_1)-(F_3)$ still hold for $\overline{F}(x,u) = \int_0^u \overline{f}(x,t)dt$, with b in (F_1) is replaced by $\overline{b} = b + 2a_0\delta^{1-\nu}$, provided that those hold for F, and $\overline{a}$ satisfies the condition $(\widetilde{\mathscr{A}_\sigma})$ whenever a satisfies $(\mathscr{A}_\sigma)$. Hence, in what follows, we will always assume without loss of generality that a satisfies $(\overline{\mathscr{A}}_\sigma)$.

In order to prove our main result via critical point theory, we need to modify $f(x,u)$ for u outside a neighborhood of the origin to get $\widetilde{f}(x,u)$ and $\widetilde{F}(x,u) = \int_0^u \widetilde{f}(x,t)dt$ as follows. Choose a constant $r \in]0, \frac{\delta}{2}[$ and define a cut-off function $\chi \in C^1(\mathbb{R},\mathbb{R})$ such that $\chi(s) = 1$ for $0 \leq s \leq r$, $\chi(s) = 0$ for $s \geq 2r$ and

$$-\frac{2}{r} \leq \chi'(s) < 0$$

for $r < s < 2r$. Let

$$\widetilde{F}(x,u) = \chi(|u|)F(x,u),\ \forall (x,u) \in \mathbb{R}\times\mathbb{R}. \tag{3.1}$$

Combining (F_1), (F_2) and the definition of χ yields

$$\left|\widetilde{F}(x,u)\right| \leq p(x)\,|u| + b\,|u|^{\nu+1},\ \forall (x,u) \in \mathbb{R}\times\mathbb{R} \tag{3.2}$$

and

$$\left|\widetilde{f}(x,u)\right| \leq 5(p(x) + b\,|u|^{\nu}),\ \forall (x,u) \in \mathbb{R}\times\mathbb{R}. \tag{3.3}$$

Now, we introduce the following modified damped vibration system

$$(\widetilde{\mathscr{F}}) \qquad u^{(4)}(x) + \omega u''(x) + a(x)u(x) = \widetilde{f}(x,u(x)),\ \forall x \in \mathbb{R}$$

and define on E the variational functional Φ associated with $(\widetilde{\mathscr{F}})$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}}[u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2]dx - \int_{\mathbb{R}}\widetilde{F}(x,u(x))dx \\ &= \frac{1}{2}\|u\|^2 - \varphi(u)\end{aligned} \tag{3.4}$$

where

$$\varphi(u) = \int_{\mathbb{R}}\widetilde{F}(x,u)dx,\ u \in E.$$

To prove our result, we will apply Lemma 2.4 to the functional Φ on E.

Lemma 3.1. *Assume that $(\tilde{\mathscr{A}_\sigma})$ and (F_1) hold. If $u_n \rightharpoonup u$ in E, then $\widetilde{f}(x,u_n) \longrightarrow \widetilde{f}(x,u)$ in $L^{\alpha}(\mathbb{R})$.*

Proof. From Lemma 2.2, we may assume that there exists a subsequence (u_{n_k}) such that

$$u_{n_k} \longrightarrow u \text{ in } L^{\nu\alpha}(\mathbb{R}) \text{ and } u_{n_k} \longrightarrow u \text{ a.e. in } \mathbb{R} \text{ as } k \longrightarrow \infty \tag{3.5}$$

and

$$\int_{\mathbb{R}}\left|\widetilde{f}(x,u_{n_k}) - \widetilde{f}(x,u)\right|^{\alpha}dx \geq \varepsilon_0,\ \forall k \in \mathbb{N} \tag{3.6}$$

for some positive constant ε_0. By (3.5) and up to a subsequence if necessary, we can assume that $\sum_{k=1}^{\infty}\|u_{n_k} - u\|_{L^{\nu\alpha}} < \infty$. Let $w(x) = \sum_{k=1}^{\infty}|u_{n_k}(x) - u(x)|$ for all $x \in \mathbb{R}$, then $w \in L^{\nu\alpha}(\mathbb{R})$. By (3.3), there holds for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\begin{aligned}
\left|\widetilde{f}(x,u_{n_k}) - \widetilde{f}(x,u)\right|^{\alpha} &\le \left(\left|\widetilde{f}(x,u_{n_k})\right| + \left|\widetilde{f}(x,u)\right|\right)^{\alpha} \\
&\le 2^{\alpha-1}(|f(x,u_{n_k})|^{\alpha} + |f(x,u)|^{\alpha}) \\
&\le 2^{\alpha-1}[(p(x)+b\,|u_{n_k}|^{\nu})^{\alpha} + (p(x)+b\,|u|^{\nu})^{\alpha}] \\
&\le 2^{2(\alpha-1)}[2(p(x))^{\alpha} + b^{\alpha}\,|u_{n_k}|^{\nu\alpha} + b^{\alpha}\,|u|^{\nu\alpha}] \\
&\le 2^{2(\alpha-1)}[2(p(x))^{\alpha} + b^{\alpha}(|u_{n_k}-u| + |u|)^{\nu\alpha} + b^{\alpha}\,|u|^{\nu\alpha}] \\
&\le 2^{2(\alpha-1)}[2(p(x))^{\alpha} + b^{\alpha}2^{\nu\alpha-1}(|u_{n_k}-u|^{\nu\alpha} + |u|^{\nu\alpha}) + b^{\alpha}\,|u|^{\nu\alpha}] \\
&\le 2^{2(\alpha-1)}[2(p(x))^{\alpha} + b^{\alpha}2^{\nu\alpha-1}(|u_{n_k}-u|^{\nu\alpha} + |u|^{\nu\alpha}) + b^{\alpha}\,|u|^{\nu\alpha}] \\
&\le c_1[(p(x))^{\alpha} + |w|^{\nu\alpha} + |u|^{\nu\alpha}] = g(x)
\end{aligned}$$

where c_1 is a positive constant. Since $g \in L^1(\mathbb{R})$, which together with (3.5), we find that the Lebesgue's Dominated Convergence Theorem implies

$$\lim_{k\longrightarrow\infty} \int_{\mathbb{R}} \left|\widetilde{f}(x,u_{n_k}) - \widetilde{f}(x,u)\right|^{\alpha} dx = 0$$

which contradicts (3.6). Hence $\widetilde{f}(x,u_{n_k}) \longrightarrow \widetilde{f}(x,u)$ in $L^{\alpha}(\mathbb{R})$ and the proof of Lemma 3.1 is completed. □

Lemma 3.2. *Under assumptions* $(\tilde{\mathscr{A}_\sigma})$ *and* (F_1)*, the functional* φ *is continuously differentiable on* E *and*

$$\varphi'(u)v = \int_{\mathbb{R}} \widetilde{f}(x,u)v\,dx,\ \forall u,v \in E. \tag{3.7}$$

Proof. For any $u \in E$, define an associated linear operator $K(u) : E \longrightarrow \mathbb{R}$ by

$$< K(u), v >= \int_{\mathbb{R}} \widetilde{f}(x,u)v\,dx,\ v \in E. \tag{3.8}$$

By (2.1), (3.3) and the Hölder's inequality, one has, for all $v \in E$,

$$\begin{aligned}
|< K(u), v >| &\le \int_{\mathbb{R}} \left|\widetilde{f}(x,u)\right| |v|\,dx \\
&\le 5\int_{\mathbb{R}} [p(x) + b\,|u|^{\nu}]\,|v|\,dx \\
&\le 5\int_{\mathbb{R}} p(x)\,|v|\,dx + 5b\int_{\mathbb{R}} |u|^{\nu}\,|v|\,dx \\
&\le 5\Big(\int_{\mathbb{R}} (p(x))^{\alpha}dx\Big)^{\frac{1}{\alpha}}\Big(\int_{\mathbb{R}} |v(x)|^{\frac{\alpha}{\alpha-1}}\,dt\Big)^{\frac{\alpha-1}{\alpha}} \\
&+ 5b\Big(\int_{\mathbb{R}} |u(x)|^{\nu+1}\,dx\Big)^{\frac{\nu}{\nu+1}}\Big(\int_{\mathbb{R}} |v(x)|^{\nu+1}\,dx\Big)^{\frac{1}{\nu+1}} \\
&\le 5\,\|p\|_{L^{\alpha}}\,\|v\|_{L^{\frac{\alpha}{\alpha-1}}} + 5b\,\|u\|_{L^{\nu+1}}^{\nu}\,\|v\|_{L^{\nu+1}} \\
&\le 5[\eta_{\frac{\alpha}{\alpha-1}}\,\|p\|_{L^{\alpha}} + b\eta_{\nu+1}^{\nu+1}\,\|u\|^{\nu}]\,\|v\|\,.
\end{aligned}$$

Hence $K(u)$ is bounded. By (3.3), for any $s \in [0,1]$, $t \in \mathbb{R}$ and $u, v \in E$, there holds

$$\begin{aligned}
\left|\widetilde{f}(x,u+sv)v\right| &\le 5[p(x)\,|v| + b\,|u+sv|^{\nu}\,|v|] \\
&\le 5[p(x)\,|v| + b(|u^{\nu}|\,|v| + |v|^{\nu+1})],
\end{aligned}$$

which is integrable in $\mathbb{R}$. Consequently, for all $u,v\in E$, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, there holds

$$\begin{aligned}\lim_{s\longrightarrow 0}\frac{\varphi(u+sv)-\varphi(u)}{s} &= \lim_{s\longrightarrow 0}\int_{\mathbb{R}}\int_0^1 \widetilde{f}(t,u+\theta sv)vd\theta dx \\ &= \int_{\mathbb{R}}\widetilde{f}(x,u)vdx =< K(u),v> .\end{aligned} \tag{3.9}$$

This implies that φ is Gâteaux differentiable on E and the Gâteaux derivative of φ at $u\in E$ is $K(u)$.

Next, we prove that K is weakly continuous. Suppose that $u_n \rightharpoonup u$ in E. Using Lemma 3.1, we have $\widetilde{f}(x,u_n)\longrightarrow \widetilde{f}(x,u)$ in $L^{\alpha}(\mathbb{R})$. By Hölder's inequality and (2.1), it holds

$$\begin{aligned}\|K(u_n)-K(u)\|_{E'} &= \sup_{\|v\|=1}\int_{\mathbb{R}}(\widetilde{f}(x,u_n)-\widetilde{f}(x,u)).vdx \\ &\le \sup_{\|v\|=1}\Big(\int_{\mathbb{R}}\Big|\widetilde{f}(x,u_n)-\widetilde{f}(x,u)\Big|^{\alpha}dx\Big)^{\frac{1}{\alpha}}\Big(\int_{\mathbb{R}}|v|^{\frac{\alpha}{\alpha-1}}dx\Big)^{\frac{\alpha-1}{\alpha}} \\ &\le \eta_{\frac{\alpha}{\alpha-1}}\Big(\int_{\mathbb{R}}\Big|\widetilde{f}(x,u_n)-\widetilde{x}(x,u)\Big|^{\alpha}dx\Big)^{\frac{1}{\alpha}}\longrightarrow 0 \; as \; n\longrightarrow\infty.\end{aligned}$$

This means that K is weakly continuous and then continuous. Thus $\varphi\in C^1(E,\mathbb{R})$ and (3.7) holds with $\varphi'=K$. In addition, due to the form of Φ, it is clear to see that $\Phi\in C^1(E,\mathbb{R})$ and, for all $u,v\in E$,

$$\Phi'(u)v=\frac{1}{2}\int_{\mathbb{R}}\Big[u''(x)v''(x)-\omega u'(x)v'(x)+a(x)u(x)v(x)\Big]dx-\int_{\mathbb{R}}\widetilde{f}(x,u(x))v(x)dx.$$

Finally, a standard argument shows that nontrivial critical points of Φ on E are homoclinic solutions of $(\widetilde{\mathscr{F}})$. The proof of Lemma 3.2 is completed. ◻

Lemma 3.3. *Assume that $(\widetilde{\mathscr{A}_\sigma})$, (F_1) and (F_2) are satisfied. Then Φ is bounded from below and satisfies the $(PS)-$condition.*

Proof. First, we prove that Φ is bounded from below. By (2.1), (3.2) and the Hölder's inequality, it holds, for all $u\in E$,

$$\begin{aligned}\int_{\mathbb{R}}\Big|\widetilde{F}(x,u)\Big|dt &\le \int_{\mathbb{R}}(p(x)|u|+b|u|^{\nu+1})dx \\ &\le \|p\|_{L^\alpha}\|u\|_{L^{\frac{\alpha}{\alpha-1}}}+b\|u\|_{L^{\nu+1}}^{\nu+1} \\ &\le \eta_{\frac{\alpha}{\alpha-1}}\|p\|_{L^\alpha}\|u\|+b\eta_{\nu+1}^{\nu+1}\|u\|^{\nu+1},\ \forall u\in E.\end{aligned}$$

Thus, for all $u\in E$,

$$\begin{aligned}\Phi(u) &\ge \frac{1}{2}\|u\|^2-\int_{\mathbb{R}}\Big|\widetilde{F}(x,u)\Big|dx \\ &\ge \frac{1}{2}\|u\|^2-\eta_{\frac{\alpha}{\alpha-1}}\|p\|_{L^\alpha}\|u\|-b\eta_{\nu+1}^{\nu+1}\|u\|^{\nu+1}.\end{aligned} \tag{3.10}$$

Since $\nu<1$, it follows that f is bounded from below.

Next, we show that Φ satisfies the (PS)-condition. Let $(u_n)\subset E$ be a (PS)-sequence, that is,

$$|\Phi(u_n)|\le M,\ \forall n\in\mathbb{N},\ \Phi'(u_n)\longrightarrow 0 \; as \; n\longrightarrow\infty \tag{3.11}$$

for some constant $M>0$. By (3.10) and (3.11), we get

$$M\ge\frac{1}{2}\|u_n\|^2-\eta_{\frac{\alpha}{\alpha-1}}\|p\|_{L^\alpha}\|u_n\|-b\eta_{\nu+1}^{\nu+1}\|u_n\|^{\nu+1},$$

which implies that (u_n) is bounded in E since $\nu < 1$. Hence, up to a subsequence if necessary, we can assume that

$$u_n \rightharpoonup u \text{ in } E \text{ and } u_n \longrightarrow u \text{ in } L^{\frac{\alpha}{\alpha-1}}(\mathbb{R}) \text{ as } n \longrightarrow \infty, \tag{3.12}$$

for some $u \in E$. We have

$$\|u_n - u\|^2 = (\Phi'(u_n) - \Phi'(u))(u_n - u) + \int_{\mathbb{R}} (\widetilde{f}(x,u_n) - \widetilde{f}(x,u))(u_n - u)dx. \tag{3.13}$$

It is clear that

$$(\Phi'(u_n) - \Phi'(u))(u_n - u) \longrightarrow 0 \text{ as as } n \longrightarrow \infty. \tag{3.14}$$

Thanks to Hölder's inequality, (2.1) and Lemma 3.1, we have

$$\left| \int_{\mathbb{R}} (\widetilde{f}(x,u_n) - \widetilde{f}(x,u))(u_n - u)dx \right| \leq \left| \widetilde{f}(x,u_{n_k}) - \widetilde{f}(x,u) \right\|_{L^\alpha} \|u_n - u\|_{L^{\frac{\alpha}{\alpha-1}}} \longrightarrow 0 \tag{3.15}$$

as $n \longrightarrow \infty$. Combining (3.13)-(3.15), we deduce that $u_n \longrightarrow u$ in E and the proof of Lemma 3.3 is completed. □

Lemma 3.4. *Suppose that $(\widetilde{\mathscr{A}_\sigma})$ and (F_3) are satisfied. Then, for each $k \in \mathbb{N}$, there exists an $A_k \subset E$ with genus $\gamma(A_k) = k$ such that $\sup_{u \in A_k} \Phi(u) < 0$.*

Proof. Let (e_n) be an orthonormal basis of E. For any $k \in \mathbb{N}$, let

$$E_k = \oplus_{m=1}^{k} X_m,\ X_m = span\{e_m\}.$$

Since E_k is with finite-dimensional, there exists a constant $\alpha_k > 0$ such that

$$\|u\| \leq \alpha_k \|u\|_{L^2},\ \forall u \in E_k. \tag{3.16}$$

By (F_3), there exists a constant $R > 0$ such that

$$\widetilde{F}(x,u) \geq \alpha_k^2 |u|^2,\ \forall x \in \mathbb{R},\ |u| \leq R. \tag{3.17}$$

Let $u \in E$ be such that

$$\|u\| \leq \frac{R}{\eta_\infty}.$$

By (2.1), we know that $|u(x)| \leq R$ for all $x \in \mathbb{R}$. It follows from (3.17) that

$$\widetilde{F}(x,u(x)) \geq \alpha_k^2 |u(x)|^2,\ \forall x \in \mathbb{R}. \tag{3.18}$$

Therefore, by (3.16) and (3.18), for all $u \in E_k$ with

$$0 < \|u\| = \tau_k = \min\{r,R\} \frac{1}{\eta_\infty},$$

where r is defined in the beginning of the proof of Theorem 1.1, one has

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \widetilde{F}(x,u)dx \\
&\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \alpha_k^2 |u(x)|^2 dx \\
&\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\tau_k^2,
\end{aligned}$$

which implies

$$\{u \in E_k / \|u\| = \tau_k\} \subset A_k = \left\{ u \in E / \Phi(u) \leq -\frac{1}{2}\tau_k^2 \right\}. \tag{3.19}$$

Thanks to Lemma 2.3, (3.19) implies

$$\gamma(A_k) \geq \gamma(\{u \in E_k / \|u\| = \tau_k\}) \geq k.$$

Hence, by the definition of Γ_k, we have $A_k \subset \Gamma_k$. Moreover, the definition of Γ_k implies

$$\sup_{u \in A_k} \Phi(u) \leq -\frac{1}{2}\tau_k^2 < 0,$$

which ends the proof of Lemma 3.4.

□

Finally, assumption (F_2) implies that $\Phi(0) = 0$ and Φ is even. It follows from this and Lemma 3.3 that the condition (Φ_1) of Lemma 2.4 is satisfied. Lemma 3.4 shows that Φ satisfies (Φ_2) of Lemma 2.4. Consequently, by Lemma 2.4, there exists a sequence of nontrivial critical points (u_k) for Φ satisfying $u_k \longrightarrow 0$ in E as $k \longrightarrow \infty$. By virtue of Lemma 3.2, we have that (u_k) is a sequence of homoclinic solutions of $(\widetilde{\mathscr{F}})$. By (2.1), it follows that $\sup_{x\in\mathbb{R}} |u_k(x)| \longrightarrow 0$ as $k \longrightarrow \infty$. Hence, there exists a positive constant k_0 such that, for all $k \geq k_0$, $\sup_{x\in\mathbb{R}} |u_k(x)| \leq r$, where r is defined above. Therefore, for all $k \geq k_0$, u_k is a solution of $(\mathscr{F})$. This completes the proof of Theorem 1.1.

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