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SECOND-ORDER EFFICIENCY CONDITIONS FOR $C^{1,1}$-VECTOR EQUILIBRIUM PROBLEMS IN TERMS OF CONTINGENT DERIVATIVES AND APPLICATIONS

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Abstract. In this paper, we study the Fritz John and Kuhn-Tucker second-order necessary and sufficient optimality conditions for $C^{1,1}$-vector equilibrium problems in terms of contingent derivatives. By applying the strong separation theorem of disjoint convex sets in convex analysis, we establish the Fritz John necessary optimality conditions for a local weakly efficient solution of VEPC. We also propose the Kurcyusz-Robinson-Zowe constraint qualification in order to obtain the Kuhn-Tucker necessary optimality conditions. By making use of both the second-order contingent derivatives and the second-order asymptotic contingent derivatives for the class of locally Lipschitz functions in which its derivatives are locally Lipschitz, we obtain a second-order sufficient optimality condition for the problem considered above. As an application, we derive Fritz John and Kuhn-Tucker second-order necessary and sufficient optimality conditions for constrained vector variational inequalities and constrained vector optimization problems.

Keywords. Vector equilibrium problem; second-order optimality conditions; Second-order contingent derivatives; constraint qualification.

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1. INTRODUCTION

Vector equilibrium problems have widely investigated in recent years by many researchers, see [4, 5, 10, 11, 12, 13, 14, 18] and the references therein. They provide a unified mathematical model including vector complementarity problems, vector saddle point problems, vector optimization problems and vector variational inequality problems as special cases. Nowadays, the optimality condition for efficient solutions of vector equilibrium problems play an important role in nonlinear and variational analysis. Results in second-order conditions of vector equilibrium problems have taken a main part in the achievements for optimality conditions, see [2, 7, 8, 9, 11, 15] and the references therein.

There are many papers dealing with optimality conditions for vector equilibrium problems with the $C^{0,1}$ and $C^{1,1}$ data. Guerraggio and Luc [4, 5] established the necessary and sufficient second-order optimality conditions for the weak efficient solutions of constrained multiobjective programming problems CP in terms of second-order subdifferentiables in finite-dimensional spaces. They used the first-order Bouligand tangent cone and the second-order tangent cone to a set at an optimal point given to obtain the optimality conditions for weak efficient solutions of problem CP. Recently, by using the concepts of second-order contingent derivatives, second-order asymptotic contingent derivatives and second-order composed contingent derivatives, Khanh and Tung [16] received the Karush-Kuhn-Tucker second-order

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optimality conditions for nonsmooth set-valued optimization problems with attention to the envelope-like effect, Su [19, 20] obtained second-order optimality conditions for vector equilibrium problems in terms of contingent derivatives and epiderivatives with stable functions, and Luu [14] established second-order necessary optimality conditions for nonsmooth vector equilibrium problems using the Palés-Zeidan type second-order directional derivatives.

In this paper, we establish the second-order optimality conditions for $C^{1,1}$-constrained vector equilibrium problem based on the main tools of second-order contingent derivatives, second-order asymptotic contingent derivatives and metric regularity. Moreover, as an application, we also receive the second-order optimality condition results for local weakly efficient solutions of constrained vector variational inequalities and constrained vector optimization problems with the $C^{1,1}$ data.

The rest of this paper is organized as follows. After some preliminaries and definitions in Section 2, we provide the Fritz John and Kuhn-Tucker second-order necessary optimality conditions for local weakly efficient solutions of VEPC in terms of second-order contingent derivatives with the objective functions of $C^{1,1}$ in Section 3. Section 4 is devoted to establish the second-order sufficient optimality conditions based on both second-order contingent derivatives and second-order asymptotic contingent derivatives. As applications, in Section 5, we obtain the Fritz John and Kuhn-Tucker second-order necessary and sufficient conditions for vector variational inequalities and vector optimization problems.

2. Preliminaries

Throughout this paper, let X,Y,Z and W be real Banach spaces. Let C be a nonempty subset in X. Let $Q \subset Y$ be a closed convex cone, which defines a partial order on Y (note that cone Q is not necessarily pointed). Let S be a convex cone in Z with its interior nonempty, which defines a partial order on Z. Given a vector bifunction $F : X \times X \to Y$ such that $F(x,x) = 0$ for all $x \in X$, and the objective functions $g : X \to Z,\ h : X \to W$. We set $K = \{x \in C :\ g(x) \in -S,\ h(x) = 0\}$. Then the vector equilibrium problem with constraints (to short, VEPC) is defined as follows: Finding a vector $\bar{x} \in K$ such that

$$F(\bar{x},x) \notin -intQ\ (\forall x \in K).$$

Vector $\bar{x} \in K$ is called a weakly efficient solution to the VEPC. If there exists a neighborhood U of $\bar{x}$ in X such that $F(\bar{x},x) \notin -intQ$ for all $x \in K \cap U$, then one says that $\bar{x}$ is a local weakly efficient solution to the VEPC. Let $L(X,Y)$ be the space of all bounded linear mappings from X to Y, and let $l : X \to Y$ be a vector-valued mapping. Denote by $\langle T,x\rangle$ the value of $T \in L(X,Y)$ at x. Assume that $Tx \in L(X,Y)$ for all $x \in X$. If $F(x,y) = \langle Tx, y-x\rangle,\ x,y \in X$ (resp., $F(x,y) = l(y) - l(x),\ x,y \in X$) and if $\bar{x} \in K$ is a local weakly efficient solution to the VEPC, then $\bar{x}$ is called a local weakly efficient solution to the constrained vector variational inequality problem, say VVIC (resp., constrained vector optimization problem, say VOPC). Let Y^* be the topological dual space of Y and let Q^+ be the dual cone of Q defined by

$$Q^+ = \{\xi \in Y^* \mid \langle \xi, q\rangle \geq 0\ \forall\, q \in Q\},$$

where $\langle .\, , .\rangle$ denotes the canonical pairing of any pair of dual spaces.

We use $\mathbb{N}$ (resp., $\mathbb{R}$) to denote the set of natural numbers (resp., real numbers). For each $A \subset X$, denote by $intA, clA, bdA$, and $cone(A)$ the interior, the closure, the boundary and the cone generated by A, where

$$coneA = \{ta : t \geq 0,\ a \in A\}.$$

We denote B_X instead of the open unit ball of X. For $x \in X$ and $\delta > 0$, $B_X(x,\delta)$ denotes the open ball centered at x and radius $\delta > 0$ in X. Let $F : X \to 2^Y$ be a set-valued mapping from X into 2^Y. The effective domain, the graph and the epigraph of F are given respectively as

$$\mathrm{dom}(F) = \{x \in X \,|\, F(x) \neq \emptyset\},$$

$$\mathrm{graph}(F) = \{(x,y) \in X \times Y \,|\, y \in F(x)\},$$

and

$$\mathrm{epi}(F) = \{(x,y) \in X \times Y \,:\, x \in \mathrm{dom}(F),\; y \in F(x) + Q\}.$$

We denote by $F(A) = \bigcup_{a \in A} F(a)$ and $F_+(x) = F(x) + Q$ for any $x \in X$. Note that F_+ is called a profile set-valued mapping. If F is a single-valued mapping, then we write f instead of F. Furthermore, let us denote by

$$(f_+, g_+)(x) = (f(x) + Q) \times (g(x) + S)$$

and $(f,g)(x) = (f(x), g(x))$ for every $x \in X$. Let us denote by $S_{g(\bar{x})} := cone(S + g(\bar{x}))$. It is not difficult to see that

$$\left(S_{g(\bar{x})}\right)^+ = \{\eta \in S^+ \,:\, \langle \xi, g(\bar{x}) \rangle = 0\}.$$

We set $\ker \nabla h(\bar{x}) = \{u \in X \,:\, \nabla h(\bar{x})(u) = 0\}$, $\nabla h(\bar{x}) : X \to W$, $\nabla^2 h(\bar{x}) : X \times X \to W$ indicate the first-order derivative and the second-order derivative of h at $\bar{x}$, respectively. The first and second order derivatives for f and g are similarly illustrated. Finally, we use $t_n \to 0^+$ to stand for a sequence of positive numbers with limit 0, and $(t_n, x_n, y_n) \to (0^+, x, y)$ indicates $t_n \to 0^+$, $x_n \to x$, $y_n \to y$.

Next, Let us provide the definitions on the tangent sets, which will be used in this paper.

Definition 2.1. [1, 6, 16] Let M be a subset of X and let $\bar{x}, u \in X$.

(i) The contingent cone (resp., adjacent cone and interior tangent cone) of M at $\bar{x}$ is defined as

$$T(M,\bar{x}) = \{x \in X : \exists t_n \to 0^+, \exists x_n \to x \text{ such that } \bar{x} + t_n x_n \in M \;\forall n \in \mathbb{N}\}$$

$$\Big(\text{resp., } A(M,\bar{x}) = \{x \in X : \forall t_n \to 0^+, \exists x_n \to x \text{ such that } \bar{x} + t_n x_n \in M \,\forall n \in \mathbb{N}\},$$

$$IT(M,\bar{x}) = \{x \in X : \forall t_n \to 0^+, \forall x_n \to x \text{ such that } \bar{x} + t_n x_n \in M \;\forall n \text{ large}\}\Big).$$

(ii) The second-order contingent set (resp., adjacent set and interior tangent set) of M at $(\bar{x}, u)$ is defined as

$$T^2(M,\bar{x},u) = \{x \in X : \exists t_n \to 0^+, \exists x_n \to x \text{ such that } \bar{x} + t_n u + \frac{1}{2} t_n^2 x_n \in M \;\forall n \in \mathbb{N}\}$$

$$\Big(\text{resp., } A^2(M,\bar{x},u) = \{x \in X : \forall\, t_n \to 0^+, \exists x_n \to x \text{ such that } \bar{x} + t_n u + \frac{1}{2} t_n^2 x_n \in M \,\forall n \in \mathbb{N}\},$$

$$IT^2(M,\bar{x},u) = \{x \in X : \forall\, t_n \to 0^+, \forall x_n \to x \text{ such that } \bar{x} + t_n u + \frac{1}{2} t_n^2 x_n \in M \;\forall n \text{ large}\}\Big).$$

(iii) The asymptotic second-order contingent cone (resp., adjacent cone) of M at $(\bar{x},u)$ is defined as

$$T^{''}(M,\bar{x},u)=\Big\{x:\exists(t_n,r_n)\to(0^+,0^+),\ \frac{t_n}{r_n}\to 0^+,\ \exists x_n\to x\ \text{ such that}\quad \bar{x}+t_nu+\frac{1}{2}t_nr_nx_n\in M\,\forall n\in\mathbb{N}\Big\}$$

$$\Big(\text{resp.,} A^{''}(M,\bar{x},u)=\Big\{x:\forall(t_n,r_n)\to(0^+,0^+),\ \frac{t_n}{r_n}\to 0^+,\ \exists x_n\to x\ \text{such that}\quad \bar{x}+t_nu+\frac{1}{2}t_nr_nx_n\in M\,\forall n\in\mathbb{N}\Big\}\Big).$$

Note that u is called the direction of second-order contingent sets and $\bar{x}\in clM$. In fact, from the definitions, it is not difficult to check that if $\bar{x}\notin clM$, then all the above tangent cones are null. In addition, if $u\notin T(M,\bar{x})$, then all the above second-order tangent sets are empty. Hence, we always have an assumption that $\bar{x}\in clM$, $u\in T(M,\bar{x})$ in all the statements.

Definition 2.2. [1, 16] Let $f:X\to Y$ be a single-valued mapping and let $\bar{x}\in X$, $(u,v)\in X\times Y$.

(i) The contingent derivative of f (resp., f_+) at point $\bar{x}$ is defined by

$$\text{graph}\big(D_cf(\bar{x})\big)=T(\text{graph}(f),(\bar{x},f(\bar{x})))$$
$$\Big(resp.,\ \text{graph}\big(D_cf_+(\bar{x},f(\bar{x}))\big)=T(\text{epi}(f),(\bar{x},f(\bar{x}))\Big).$$

(ii) The second-order contingent derivative of f (resp., f_+) at point $\bar{x}$ in direction (u,v) is defined by

$$\text{graph}\big(D_c^2f(\bar{x},f(\bar{x}),u,v)\big)=T^2(\text{graph}(f),(\bar{x},f(\bar{x})),(u,v))$$
$$\Big(\text{resp., graph}\big(D_c^2f_+(\bar{x},f(\bar{x}),u,v)\big)=T^2(\text{epi}(f),(\bar{x},f(\bar{x})),(u,v))\Big).$$

(iii) The second-order asymptotic contingent derivative of f (resp., f_+) at point $\bar{x}$ in direction (u,v) is defined by

$$\text{graph}\big(D_c^{''}f(\bar{x},f(\bar{x}),u,v)\big)=T^{''}(\text{graph}(f),(\bar{x},f(\bar{x})),(u,v))$$
$$\Big(\text{resp., graph}\big(D_c^{''}(f_+)(\bar{x},f(\bar{x}),u,v)\big)=T^{''}(\text{epi}(f),(\bar{x},f(\bar{x})),(u,v))\Big).$$

Definition 2.3. [8] A mapping $f:X\to Y$ is said to be stable at $\bar{x}$ if there exist a neighborhood U of $\bar{x}$ and $L>0$ such that

$$\|f(x)-f(\bar{x})\|\le L\|x-\bar{x}\|\quad\forall x\in U.$$

If, in addition,

$$\|f(x)-f(x')\|\le L\|x-x'\|\quad\forall x,x'\in U,$$

then one says that f is Lipschitz around $\bar{x}$. If for each $\bar{x}\in X$, there exists a neighborhood U of $\bar{x}$ such that f is Lipschitz around $\bar{x}$, we say that f is locally Lipschitz on X. Notice that if f Lipschitz around $\bar{x}$, then f is stable at $\bar{x}$, which leads to f must be continuous at that point.

3. First-and Second-order Necessary Conditions

We recall (see [4, 5]) that f is of class $C^{0,1}$ if it is locally Lipschitz on X, and f is of class $C^{1,1}$ if it is a differentiable vector function from X to Y whose its derivative is $C^{0,1}$. For each $M\subset Z$, the normal cone of M at $z\in Z$ is given as (see [8])

$$N(M,z)=-(T(M,z))^+.$$

In order to obtain the Fritz John and Kuhn-Tucker second-order optimality conditions for local weakly efficient solutions of VEPC, the following directionally metrically regular will be needed in the paper.

Definition 3.1. [17] Let $\bar{x}, u \in X$ with $u \neq 0$, $T \subset W$ and $k : X \longrightarrow W$. We say that k is directionally metrically regular at $(\bar{x}, u)$ with respect to T, if there exist $\mu > 0$, $\rho > 0$ such that, for every $t \in (0, \rho)$ and $v \in B_X(u, \rho)$,

$$d(\bar{x} + tv,\, k^{-1}(T)) \leq \mu\, d(k(\bar{x} + tv), T),$$

where

$$d(x, A) := \inf_{a \in A} \|x - a\|$$

and $\|\,.\,\|$ stands for a norm in real Banach spaces.

From now on, if not otherwise stated, for each $\bar{x} \in X$, we put $f = F(\bar{x}, .) : X \to Y$ such that $f(\bar{x}) = 0$. A Fritz John and Kuhn-Tucker necessary optimality condition for local weakly efficient solutions of VEPC can be stated as follows.

Theorem 3.1. *Let $\bar{x}$ be a local weakly efficient solution of VEPC. Assume, furthermore, that $u \in ker \nabla h(\bar{x}) \cap IT(C, \bar{x})$ and h is directionally metrically regular at $(\bar{x}, u)$ with respect to $T = \{0\}$ when $u \neq 0$. The following assertions hold.*

(i) For f and g of class $C^{0,1}$ with $\nabla f(\bar{x})(u)$ and $\nabla g(\bar{x})(u)$ exist, $\exists \lambda_0 \in Q^+$, $\exists \eta_0 \in N(-S, g(\bar{x}))$ with $(\lambda_0, \eta_0) \neq (0, 0)$ such that

$$\langle \lambda_0, \nabla f(\bar{x})(u) \rangle + \langle \eta_0, \nabla g(\bar{x})(u) \rangle \geq 0. \tag{3.1}$$

(ii) For f and g of class $C^{1,1}$ with $\nabla^2 f(\bar{x})(u, u)$ and $\nabla^2 g(\bar{x})(u, u)$ exist, $\exists\, (\lambda, \eta) \in Q^+ \times N(-S, g(\bar{x}))$, not all zero, such that $\forall x \in A^2(C, \bar{x}, u)$, $\forall v \in D_c f_+(\bar{x}, f(\bar{x}))(u) \cap (-bdQ)$ and $\forall w \in D_c g_+(\bar{x}, g(\bar{x}))(u) \cap (-cl\, S_{g(\bar{x})})$,

$$\begin{aligned} &\inf \Big\{ \langle \lambda, y \rangle + \langle \eta, z \rangle \mid (y, z) \in D^2_c(f_+, g_+)(\bar{x}, (f, g)(\bar{x}), u, (v, w))(x) \Big\} \\ \geq \quad &\big\langle \lambda, \nabla f(\bar{x})(x) + \nabla^2 f(\bar{x})(u, u) \big\rangle + \big\langle \eta, \nabla g(\bar{x})(x) + \nabla^2 g(\bar{x})(u, u) \big\rangle. \end{aligned} \tag{3.2}$$

In addition, $\lambda \neq 0$ if the following qualification condition of the KRZ type holds.

$$\begin{aligned} \Big\{ z \in Z : (y, z) \in cone\Big(&D^2_c(f_+, g_+)(\bar{x}, (f, g)(\bar{x}), u, (v, w))(A^2(C, \bar{x}, u)) \\ &- \{0\} \times \nabla g(\bar{x})(A^2(C, \bar{x}, u)) - \{(0, \nabla^2 g(\bar{x})(u, u))\} \Big) \Big\} + S_{g(\bar{x})} = Z. \end{aligned} \tag{KRZ}$$

Proof. Case (i). If $u = 0$, then there is nothing to prove. Next, let us consider the case $u \in ker \nabla h(\bar{x}) \cap IT(C, \bar{x}) \setminus \{0\}$ and prove that

$$(\nabla f(\bar{x})(u),\, \nabla g(\bar{x})(u)) \notin (-intQ) \times IT(-S, g(\bar{x})). \tag{3.3}$$

In fact, two cases can occur as follows:
Case I. If $\nabla g(\bar{x})(u) \notin IT(-S, g(\bar{x}))$, then obviously condition (3.3) is fulfilled.
Case II. If $\nabla g(\bar{x})(u) \in IT(-S, g(\bar{x}))$, then, for every $t_n \to 0^+$, $\frac{h(\bar{x}+t_n u) - h(\bar{x})}{t_n} \to \nabla h(\bar{x})(u)$, which yields that $\frac{h(\bar{x}+t_n u)}{t_n} \to 0$. By the metric regularity of h, there exist $\mu > 0$, $\rho > 0$ such that for every $t \in (0, \rho)$ and $v \in B_X(u, \rho)$, $d(\bar{x} + tv, H) \leq \mu\, \|h(\bar{x} + tv)\|$, where $H := h^{-1}(0)$. By virtue of the definition of infimum,

one finds $y_n \in H$ such that, for sufficiently large n, $\frac{\bar{x}+t_n u - y_n}{t_n} \to 0$. Setting $u_n = \frac{y_n - \bar{x}}{t_n} \quad (\forall n \geq 1)$, one sees that u_n converges to $u \in IT(C, \bar{x})$. Therefore, for n large enough, $\bar{x} + t_n u_n \in C$, which is equivalent to

$$\bar{x} + t_n u_n \in H \cap C \quad \text{for n large enough.} \tag{3.4}$$

Applying the Taylor expansion, we express

$$\frac{g(\bar{x}+t_n u_n) - g(\bar{x})}{t_n} \to \nabla g(\bar{x})(u) \in IT(-S, g(\bar{x})).$$

For the preceding sequences t_n and $\frac{g(\bar{x}+t_n u_n)-g(\bar{x})}{t_n}$, one gets

$$g(\bar{x}) + t_n \Big(\frac{g(\bar{x}+t_n u_n) - g(\bar{x})}{t_n} \Big) \in -S \quad \text{for n large enough,}$$

which is equivalent to

$$g(\bar{x}+t_n u_n) \in -S \quad \text{for n large enough.}$$

Using (3.4), we obtain that

$$\bar{x} + t_n u_n \in K \quad \text{for n large.} \tag{3.5}$$

On the other hand, by virtue of the definition of local weakly efficient solutions (see Section 2), there exists a neighborhood U of $\bar{x}$ such that

$$f(x) \notin -intQ \quad \forall x \in K \cap U.$$

It follows from (3.5) and the fact $\bar{x} + t_n u_n \to \bar{x} \in U$ that, for n large enough,

$$f(\bar{x}+t_n u_n) \notin -intQ.$$

Again applying the Taylor expansion, we express

$$\frac{f(\bar{x}+t_n u_n) - f(\bar{x})}{t_n} \to \nabla f(\bar{x})(u).$$

Because $-intQ$ is a convex cone, $t_n > 0$ for all n, and $f(\bar{x}) = 0$ by the initial hypotheses,

$$\frac{f(\bar{x}+t_n u_n) - f(\bar{x})}{t_n} \in Y \setminus (-intQ) \quad \forall n \text{ large enough.}$$

Consequently, $\nabla f(\bar{x})(u) \in Y \setminus (-intQ)$, which yields that (3.3) holds. By a separation theorem, one finds the pair $(\lambda_0, \eta_0) \in Q^+ \times N(-S, g(\bar{x})) \setminus \{(0,0)\}$ such that (3.1) is fulfilled.

Case (ii). As $f, g \in C^{1,1}$, $f, g \in C^{0,1}$. Making use of (i), one finds the pair (λ_0, η_0) satisfying (3.1). We put $(\lambda, \eta) = (\lambda_0, \eta_0)$ and then fix x, v, w as in the assumption (ii). For every $(y,z) \in Y \times Z$ such that

$$(y,z) \in D_c^2(f_+, g_+)(\bar{x}, (f,g)(\bar{x}), u, (v,w))(x),$$

one has

$$(x,y,z) \in T^2(epi(f,g), (\bar{x}, (f,g)(\bar{x})), u, (v,w)).$$

This leads to the facts that $\exists t_n \to 0^+$, $\exists x_n \to x$ and $\exists (y_n, z_n) \to (y,z)$ such that

$$f(\bar{x}) + t_n v + \frac{1}{2} t_n^2 y_n \in f(\bar{x} + t_n u + \frac{1}{2} t_n^2 x_n) + Q, \tag{3.6}$$

$$g(\bar{x}) + t_n w + \frac{1}{2} t_n^2 z_n \in g(\bar{x} + t_n u + \frac{1}{2} t_n^2 x_n) + S. \tag{3.7}$$

It follows from (3.6) that

$$y_n \in \frac{f(\bar{x}+t_n u+\frac{1}{2}t_n^2 x_n)-f(\bar{x})-t_n\nabla f(\bar{x})(u)+t_n(\nabla f(\bar{x})(u)-v)}{2^{-1}t_n^2}+Q \quad \forall n\geq 1.$$

From (3.6) and (3.7), one also has

$$z_n \in \frac{g(\bar{x}+t_n u+\frac{1}{2}t_n^2 x_n)-g(\bar{x})-t_n\nabla g(\bar{x})(u)+t_n(\nabla g(\bar{x})(u)-w)}{2^{-1}t_n^2}+S \quad \forall n\geq 1.$$

Consequently

$$\begin{aligned}\langle\lambda,y\rangle+\langle\eta,z\rangle \geq{}& \lim_{n\to\infty}\frac{\left\langle\lambda, f(\bar{x}+t_n u+\frac{1}{2}t_n^2 x_n)-f(\bar{x})-t_n\nabla f(\bar{x})(u)\right\rangle}{2^{-1}t_n^2}\\ &+\lim_{n\to\infty}\frac{\left\langle\eta, g(\bar{x}+t_n u+\frac{1}{2}t_n^2 x_n)-g(\bar{x})-t_n\nabla g(\bar{x})(u)\right\rangle}{2^{-1}t_n^2}.\end{aligned}\tag{3.8}$$

Since $v\in -Q$ and $\lambda\in Q^+$, it yields that $\langle\lambda,-v\rangle\geq 0$. It can be seen that $\eta\in\big(cone(S+g(\bar{x}))\big)^+$, and this along with the fact that $-w\in cl\,cone(S+g(\bar{x}))$, we deduce that $\langle\eta,-w\rangle\geq 0$. Consequently,

$$\langle\lambda,-v\rangle+\langle\eta,-w\rangle\geq 0.$$

This together with the obtained inequality in (3.1) deduces that the inequality of (3.8) holds. Again using the Taylor expansions, we express

$$\lim_{n\to\infty}\frac{\left\langle\lambda, f(\bar{x}+t_n u+\frac{1}{2}t_n^2 x_n)-f(\bar{x})-t_n\nabla f(\bar{x})(u)\right\rangle}{2^{-1}t_n^2}=\left\langle\lambda,\nabla f(\bar{x})(x)+\nabla^2 f(\bar{x})(u,u)\right\rangle$$

and

$$\lim_{n\to\infty}\frac{\left\langle\eta, g(\bar{x}+t_n u+\frac{1}{2}t_n^2 x_n)-g(\bar{x})-t_n\nabla g(\bar{x})(u)\right\rangle}{2^{-1}t_n^2}=\left\langle\eta,\nabla g(\bar{x})(x)+\nabla^2 g(\bar{x})(u,u)\right\rangle.$$

This combines with (3.8) yields the result (3.2) holds.

If, furthermore, the qualification condition of (KRZ) type is fulfilled, then $\lambda\neq 0$. In fact, if it was not, then $\lambda=0$ and hence $\eta\neq 0$. Note that $\eta\in N(-S,g(\bar{x}))\setminus\{0\}$, which yields that $\eta\in(S_{g(\bar{x})})^+\setminus\{0\}$ and a consequence is $\langle\eta,z_2\rangle\geq 0$ for all $z_2\in S_{g(\bar{x})}$. Consequently, for all $z\in Z$, there exists

$$\begin{aligned}z_1\in\Big\{z_1\in Z:\ (y_1,z_1)\in cone\big(&D_c^2(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))(A^2(C,\bar{x},u))\\ &-\{0\}\times\nabla g(\bar{x})(A^2(C,\bar{x},u))-\{(0,\nabla^2 g(\bar{x})(u,u))\}\big)\Big\}\end{aligned}$$

such that

$$\langle\eta,z\rangle\geq\langle\eta,z_1\rangle.$$

In other words, one can find a real number $t\geq 0$, a direction $x\in A^2(C,\bar{x},u)$ and a pair $(y_2,z_2)\in D_c^2(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)$ such that

$$(y_1,z_1)=t(y_2,z_2-\nabla g(\bar{x})(x)+\nabla^2 g(\bar{x})(u,u)).$$

By initial hypotheses, it follows that

$$\langle\eta,z\rangle\geq\langle\eta,z_1\rangle=t\Big(\langle\eta,z_2\rangle-\left\langle\eta,\nabla g(\bar{x})(x)+\nabla^2 g(\bar{x})(u,u)\right\rangle\Big)\geq 0.$$

Since $z\in Z$ is arbitrary, one gets $\eta=0$, a contradiction. This completes the proof. □

Next, we give an example to illustrate Theorem 3.1 as follows.

Example 3.1. Let $X=\mathbb{R}$, $C=[0,1]$, $\bar{x}=0$, $Y=\mathbb{R}^2$, $Z=\mathbb{R}^3$, $Q=\mathbb{R}^2_+$, $W=\mathbb{R}$, $S=\{(x,y,z)\in\mathbb{R}^3 : x\leq 0,\, y\geq 0,\, z\geq 0\}$. Let us consider the objective functions $f=F(0,.):\mathbb{R}\to\mathbb{R}^2$, $g:\mathbb{R}\to\mathbb{R}^3$ and $h:\mathbb{R}\to\mathbb{R}$ be defined respectively by

$$f(x)=F(0,x)=(x^2-x,2x^2-x)\ \forall x\in\mathbb{R};$$
$$g(x)=(x+1,x^2-1,x^2-x),\quad h(x)=0\ \forall x\in\mathbb{R}.$$

It is easy to check that f and g are of class $C^{1,1}$, and its Fréchet derivatives at $\bar{x}$ are given respectively as

$$\nabla f(\bar{x})=\begin{pmatrix} -1 & -1 \end{pmatrix},\quad \nabla^2 f(\bar{x})(u,u)=\begin{pmatrix} 2u^2, & 4u^2 \end{pmatrix},$$
$$\nabla g(\bar{x})=\begin{pmatrix} 1 & 0 & -1 \end{pmatrix},\quad \nabla^2 g(\bar{x})(u,u)=\begin{pmatrix} 0, & 2u^2, & 2u^2 \end{pmatrix}.$$

It can easily be seen that h is directionally metrically regular at $(\bar{x},u)$ with respect to $T=\{0\}$. The feasible set K of VEPC (see Section 2) is given by

$$K=\{x\in C : g(x)\in -S,\, h(x)=0\}=C.$$

By directly calculating, one gets $ker\nabla h(\bar{x})\cap IT(C,\bar{x})=(0,+\infty)$, $Q^+=\mathbb{R}^2_+$, $S^+=\mathbb{R}_-\times\mathbb{R}_+\times\mathbb{R}_+$, $N(-S,g(\bar{x}))=\{0\}\times\{0\}\times\mathbb{R}_+$, $A^2(C,\bar{x},u)=\mathbb{R}$,

$$-bdQ=-(Q\setminus intQ)=\bigcup_{a\leq 0}\{(a,0)\}\bigcup\bigcup_{b\leq 0}\{(0,b)\},$$
$$-clS_{g(\bar{x})}=-cl\,cone(S+\{(1,-1,0)\})=\mathbb{R}\times\mathbb{R}\times\mathbb{R}_-.$$

For all $u\in ker\nabla h(\bar{x})\cap IT(C,\bar{x})$, it follows that $u>0$. Moreover,

$$D_c f_+(\bar{x},f(\bar{x}))(u)=\{(-u,-u)\}+\mathbb{R}^2_+,$$
$$D_c g_+(\bar{x},g(\bar{x}))(u)=\{(u,0,-u)\}+\mathbb{R}_-\times\mathbb{R}_+\times\mathbb{R}_+.$$

We pick $\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2_+$ and $\eta=(0,0,\eta_3)\in\{0\}\times\{0\}\times\mathbb{R}_+=N(-S,g(\bar{x}))$ with $(\lambda_1,\lambda_2,\eta_3)\neq(0,0,0)$. If we put $x=0$, $v=(0,0)$ and $w=(u,0,-u)$, then the right-hand side of (3.2) is equal to $2u^2(\lambda_1+2\lambda_2+\eta_3)$. Furthermore, one has

$$D^2_c(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)=\mathbb{R}\times\mathbb{R}\times\mathbb{R}_-\times[2u^2,+\infty)\times[2u^2,+\infty).$$

Therefore the left-hand side of (3.2) is less than $2u^2(\lambda_1+2\lambda_2+\eta_3)$. It is obvious that the (KRZ) is satisfied. Applying Theorem 3.1, $\bar{x}=0$ is not a local weakly efficient solution of VEPC.

Theorem 3.2. *Let $\bar{x}$ be a local weakly efficient solution of VEPC. Assume, furthermore, that $u\in ker\nabla h(\bar{x})\cap IT(C,\bar{x})$ and h is directionally metrically regular at $(\bar{x},u)$ with respect to $T=\{0\}$ when $u\neq 0$. Then the following assertion holds.*

(iii) For f and g of class $C^{1,1}$, $\exists\,(\lambda,\eta)\in Q^+\times N(-S,g(\bar{x}))\setminus\{(0,0)\}$ such that $\forall x\in A''(C,\bar{x},u)$, $\forall v\in D_c f_+(\bar{x},f(\bar{x}))(u)\cap(-bdQ)$, $\forall w\in D_c g_+(\bar{x},g(\bar{x}))(u)\cap(-cl\,S_{g(\bar{x})})$,

$$\inf\{\langle\lambda,y\rangle+\langle\eta,z\rangle\,|\,(y,z)\in D''_c(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)\}\geq\langle\lambda,\nabla f(\bar{x})(x)\rangle+\langle\eta,\nabla g(\bar{x})(x)\rangle.$$

Moreover, $\lambda\neq 0$ if the following qualification condition of the KRZ' type holds.

$$\Big\{z\in Z:(y,z)\in cone\big(D''_c(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))$$
$$(A''(C,\bar{x},u))-\{(0,\nabla g(\bar{x})(A''(C,\bar{x},u)))\}\big)\Big\}+S_{g(\bar{x})}=Z. \tag{KRZ'}$$

Proof. Argue similarly as for Theorem 3.1. Use the adjacent cone $A^{''}(C,\bar{x},u)$ instead of the adjacent set $A^2(C,\bar{x},u)$ and the second-order asymptotic contingent derivative $D_c^{''}(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)$ stands for the second-order contingent derivative $D_c^2(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)$. we note here that, for all $(t_n,r_n)\to(0^+,0^+)$, $\frac{t_n}{r_n}\to 0^+$ and $x_n\to x$,

$$\lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_nu+\frac{1}{2}t_nr_nx_n)-f(\bar{x})-t_n\nabla f(\bar{x})(u)}{2^{-1}t_nr_n}=\nabla f(\bar{x})(x)$$

and

$$\lim_{n\longrightarrow\infty}\frac{g(\bar{x}+t_nu+\frac{1}{2}t_nr_nx_n)-g(\bar{x})-t_n\nabla g(\bar{x})(u)}{2^{-1}t_nr_n}=\nabla g(\bar{x})(x),$$

which completes the proof. $\square$

Example 3.2. Let us consider problem VEPC in which $X,Y,Z,W,C,Q,S,\bar{x},F(\bar{x},.),g$ and h are given as in Example 3.1. By directly calculating, we obtain the following results

$$A''(C,\bar{x},u)=\mathbb{R},\ D_c^{''}(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(0,0),(u,0,-u))(0)=\mathbb{R}\times\mathbb{R}\times\mathbb{R}_-\times\mathbb{R}_+\times\mathbb{R}_+.$$

By choosing λ,η,x,v,w as in Example 3.1, we find that

$$\langle\lambda,\nabla f(\bar{x})(x)\rangle+\langle\eta,\nabla g(\bar{x})(x)\rangle=0,$$

$$\begin{aligned}&\inf\left\{\langle\lambda,y\rangle+\langle\eta,z\rangle\mid(y,z)\in D_c^{''}(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)\right\}\\&=\inf\left\{\langle(\lambda_1,\lambda_2,\eta_3),(y_1,y_2,z_3)\rangle\,:\,(y_1,y_2)\in\mathbb{R}^2,\ z_3\geq 0\right\}\\&=\inf\left\{\langle(\lambda_1,\lambda_2),(y_1,y_2)\rangle\,:\,(y_1,y_2)\in\mathbb{R}^2\right\}<0.\end{aligned}$$

On the other hand, by direct computation, it holds that

$$\bigcup_{x\in A''(C,\bar{x},u)}D_c^{''}(f_+,g_+)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)=\mathbb{R}^2\times\big(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\big),$$

$$\bigcup_{x\in A''(C,\bar{x},u)}(0,0,\nabla g(\bar{x})(x))=\{0\}\times\{0\}\times\mathbb{R}\times\{0\}\times\mathbb{R},\ S_{(1,-1,0)}=\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+.$$

Consequently, the qualification condition of (KRZ') type also is fulfilled. Making use of Theorem 3.2 to conclude that $\bar{x}=0$ is not a local weakly efficient solution to the VEPC.

4. First-and Second-Order Sufficient Conditions

In this section, we consider problem VEPC in which $dim(X)<+\infty$. A first-order sufficient optimality condition for local weakly efficient solutions of VEPC can be stated as follows.

Theorem 4.1. *Let $\bar{x}\in K$ and f, g and h are of class $C^{0,1}$. Suppose, in addition, that, for each $u\in ker\nabla h(\bar{x})\cap T(C,\bar{x})\setminus\{0\}$, there exist $\lambda_0\in Q^+$, $\eta_0\in N(-S,g(\bar{x}))$ with $(\lambda_0,\eta_0)\neq(0,0)$ such that*

$$\langle\lambda_0,\nabla f(\bar{x})(u)\rangle+\langle\eta_0,\nabla g(\bar{x})(u)\rangle>0. \tag{4.1}$$

Then $\bar{x}$ is a local weakly efficient solution of VEPC.

Proof. Assume to the contrary that, $\bar{x}$ is not a local weakly efficient solution of problem VEPC. Then there exists a sequence $x_n \in K \setminus \{\bar{x}\} \subset C \setminus \{\bar{x}\}$ with $x_n \to \bar{x}$ such that $f(x_n) \in -intQ \quad \forall n \geq 1$. Setting $t_n = \|x_n - \bar{x}\|$ and $u_n = \frac{x_n - \bar{x}}{t_n}$ for all $n \geq 1$, we find $t_n \to 0^+$. Taking a subsequence if necessary, we may assume that $u_n \to u$. According to [3], it follows that $u \in T(C, \bar{x})$ with the norm of u equals 1, i.e., $\|u\| = 1$. By using the Taylor expansions, one obtains

$$\frac{f(\bar{x}+t_n u_n) - f(\bar{x})}{t_n} \to \nabla f(\bar{x})(u),$$

$$\frac{g(\bar{x}+t_n u_n) - g(\bar{x})}{t_n} \to \nabla g(\bar{x})(u),$$

and

$$\frac{h(\bar{x}+t_n u_n) - h(\bar{x})}{t_n} \to \nabla h(\bar{x})(u).$$

Obviously, $\nabla f(\bar{x})(u) \in -cl\, Q = -Q$, $\nabla g(\bar{x})(u) \in -cl\, cone(S + g(\bar{x}))$, and $u \in ker \nabla h(\bar{x}) \cap T(C, \bar{x}) \setminus \{0\}$. By the initial hypotheses, there exist $\lambda_0 \in Q^+$, $\eta_0 \in N(-S, g(\bar{x}))$ with $(\lambda_0, \eta_0) \neq (0,0)$ such that (4.1) is fulfilled. From the fact $\eta_0 \in N(-S, g(\bar{x}))$, it follows that $\eta \in (cone(S + g(\bar{x})))^+$. Consequently

$$\langle \lambda_0, \nabla f(\bar{x})(u) \rangle + \langle \eta_0, \nabla g(\bar{x})(u) \rangle \leq 0,$$

which conflicts with (4.1) and the claim follows. □

Example 4.1. Let us consider problem VEPC given as in Example 3.1. Then, for every $\lambda_0 = (\lambda_1, \lambda_2) \in Q^+$ and $\eta_0 = (\eta_1, \eta_2, \eta_3) \in N(-S, g(\bar{x}))$,f $\lambda_1 \geq 0, \lambda_2 \geq 0$, $\eta_1 = \eta_2 = 0$, $\eta_3 \geq 0$. By picking u as in Example 3.1, we find that that $u > 0$ and

$$\langle \lambda_0, \nabla f(\bar{x})(u) \rangle + \langle \eta_0, \nabla g(\bar{x})(u) \rangle = -u(\lambda_1 + \lambda_2 + \eta_3) \leq 0.$$

It can be seen that $\bar{x}$ is not a local weakly efficient solution of VEPC, as was to be checked.

Theorem 4.2. *Let $\bar{x}$ be a feasible point of VEPC and f, g and h are of class $C^{1,1}$. Assume, furthermore, that the following conditions are fulfilled.*

(i) For all $u \in T(C, \bar{x}) \cap ker \nabla h(\bar{x})$, $D_c f(\bar{x})(u) \cap (-intQ) = \emptyset$ and $D_c g(\bar{x})(u) \cap T(-S, g(\bar{x})) \neq \emptyset$;

(ii) For all $u \in T(C, \bar{x}) \cap ker \nabla h(\bar{x}) \setminus \{0\}$, $v \in D_c f(\bar{x})(u) \cap (-bdQ)$ and $w \in D_c g(\bar{x})(u) \cap (-cl\, S_{g(\bar{x})})$, there exists $(\lambda, \eta) \in Q^+ \times N(-S, g(\bar{x}))$ such that

(a) For all $x \in T^2(C, \bar{x}, u)$, $(y, z) \in D_c^2(f, g)(\bar{x}, (f, g)(\bar{x}), u, (v, w))(x)$,

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \; > \; \left\langle \lambda, \nabla f(\bar{x})(x) + \nabla^2 f(\bar{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\bar{x})(x) + \nabla^2 g(\bar{x})(u,u) \right\rangle; \tag{4.2}$$

(b) For all $x \in T''(C, \bar{x}, u) \cap u^{\perp} \setminus \{0\}$, $(y, z) \in D_c''(f, g)(\bar{x}, (f, g)(\bar{x}), u, (v, w))(x)$,

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \; > \; \langle \lambda, \nabla f(\bar{x})(x) \rangle + \langle \eta, \nabla g(\bar{x})(x) \rangle, \tag{4.3}$$

where $u^{\perp}$ is the orthogonal complement of u in X. Then $\bar{x}$ is a local weakly efficient solution of VEPC.

Proof. We assume, to the contrary, that $\bar{x}$ is not a local weakly efficient solution of VEPC. By a similar argument as in the proof of Theorem 4.1, there exist sequences $(x_n)_{n \geq 1} \subset K \setminus \{\bar{x}\}$, $(t_n)_{n \geq 1} \subset (0, +\infty)$ with $x_n \to \bar{x}$, $t_n \to 0$, and $(u_n)_{n \geq 1} \subset cone(C - \bar{x})$ with $u_n \to u$. Clearly, $u \in T(C, \bar{x}) \cap ker \nabla h(\bar{x}) \setminus \{0\}$. By taking $(v, w) = (\nabla f(\bar{x})(u), \nabla g(\bar{x})(u))$, and making use of Definition 2.2, we deduce that $v \in D_c f(\bar{x})(u) \cap (-Q)$, $w \in D_c g(\bar{x})(u) \cap (-cl\, S_{g(\bar{x})})$. In other words, by hypotheses (i), it follows that $v \in D_c f(\bar{x})(u) \cap (-bdQ)$.

Again making use of assumption (ii), it follows that there exists $(\lambda,\eta) \in Q^+ \times N(-S, g(\bar{x}))$ such that (4.2) holds when any

$$(x,(y,z)) \in A^2(C,\bar{x},u) \times D_c^2(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x),$$

and (4.3) holds when any

$$(x,(y,z)) \in \left(T''(C,\bar{x},u) \cap u^{\perp} \setminus \{0\}\right) \times D_c''(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x).$$

For the sequence $(w_n)_{n\geq 1}$ is given by

$$w_n = \frac{x_n - \bar{x} - t_n u}{2^{-1}t_n^2}, \quad \forall n \geq 1.$$

Two cases can occur as follows:

(I) $(w_n)_{n\geq 1}$ **is bounded.** Note that $dim(X) < +\infty$. Hence there exists the limit of sequence $(w_n)_{n\geq 1}$ (taking a subsequence if necessary) and assume that $w_n \to x \in X$. It is easy to check that $x \in T^2(C,\bar{x},u)$ and $(y,z) \in D_c^2(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)$ if and only if

$$(y,z) = (\nabla f(\bar{x})(x) + \nabla^2 f(\bar{x})(u,u),\ \nabla g(\bar{x})(x) + \nabla^2 g(\bar{x})(u,u)),$$

which contradicts condition (4.2).

(II) $(w_n)_{n\geq 1}$ **is unbounded.** Let us may assume that $\|w_n\| \to +\infty$ and

$$W_n = \frac{w_n}{\|w_n\|} \to x_1 \in X \cap \{u \in X\ \|u\| = 1\}.$$

For the preceding sequence t_n, we consider sequence $r_n = t_n\|w_n\| \quad \forall n \geq 1$. It is not difficult to check that (see [6]) $r_n \to 0^+$, $\frac{t_n}{r_n} \to 0^+$ and $x_n = \bar{x} + t_n u + \frac{1}{2} r_n t_n W_n \quad \forall n \geq 1$. For the sequence

$$(y_n', z_n') := \frac{(f,g)(x_n) - (f,g)(\bar{x}) - t_n(v,w)}{2^{-1}t_n r_n} \to (\nabla f(\bar{x})(x_1),\ \nabla g(\bar{x})(x_1)).$$

In the similar way as in (I) (see the proof of case (ii) in Theorem 3 [6]), we conclude that $x_1 \in T''(C,\bar{x},u) \cap u^{\perp} \setminus \{0\}$, and

$$(\nabla f(\bar{x})(x_1),\ \nabla g(\bar{x})(x_1)) \in D_c''(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x_1),$$

which contradicts (4.3). This completes the proof of Theorem 4.2. □

To illustrate the above results, we give the following example.

Example 4.2. Let $X = Y = \mathbb{R}^2$, $Z = \mathbb{R}^3$, $W = \mathbb{R}$, $C = [0,1] \times [0,1]$, $Q = \mathbb{R}_+^2$, $S = \{y_1, y_2, y_3 \in \mathbb{R}^3 : y_2 y_3 \geq 2y_1^2, y_2 \leq 0\}$, $\bar{x} = (0,0)$ and the mappings $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$, $g = (g_1, g_2, g_3) : \mathbb{R}^2 \to \mathbb{R}^3$, $h : \mathbb{R}^2 \to \mathbb{R}$ be defined respectively by

$$f_1((x_1,x_2)) = \begin{cases} ax_1^2 - x_2 + c\sin(ln|x_1|)x_1^2 & \text{if } x_1 \neq 0, \\ x_2 & \text{if } x_1 = 0, \end{cases}$$

$$f_2((x_1,x_2)) = x_1^2 - x_2^2 \ \text{ for all } (x_1,x_2) \in \mathbb{R}^2,$$

$$g((x_1,x_2)) = (x_1, 1 + x_1^2, g_3((x_1,x_2))) \ \text{ for all } (x_1,x_2) \in \mathbb{R}^2,$$

$$h((x_1,x_2)) = x_1 - x_2 \ \text{ for all } (x_1,x_2) \in \mathbb{R}^2.$$

with a and c are real numbers such that $a > c \geq 0$,

$$g_3((x_1,x_2)) = \begin{cases} x_2 + exp\big(|x_1 - x_2|(\sin\frac{\pi}{x_1} - 2)\big) - 1, & \text{if } x_1 \neq 0, \\ 3x_2, & \text{if } x_1 = 0. \end{cases}$$

It is obvious that Q and S are closed convex cones with nonempty interiors, $g(\bar{x}) = (0,1,0)$, the mappings f,g,h are of class $C^{1,1}$ and the Fréchet derivatives of them at $\bar{x}$ are given respectively as

$$\nabla\mathbf{f}(\bar{\mathbf{x}}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \nabla\mathbf{g}(\bar{\mathbf{x}}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad \nabla\mathbf{h}(\bar{\mathbf{x}}) = \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

It is easy to see that $T(C,\bar{x}) = \mathbb{R}^2_+$, $ker\nabla h(\bar{x}) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = x_2\}$, which yields that $T(C,\bar{x}) \cap ker\nabla h(\bar{x}) = cone\{(1,1)\}$. For every $u = (u_0,u_0)$, where $u_0 \in \mathbb{R}_+$, we have that $D_c f(\bar{x})((u_0,u_0)) = \{(-u_0,0)\}$, $-intQ = -int(\mathbb{R}^2_+)$, $T(-S,g(\bar{x})) = -cl\,S_{g(\bar{x})} = \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$ and $D_c g(\bar{x})((u_0,u_0)) = \{(u_0,0,u_0)\}$. Thus, the assumption (i) is fulfilled.

On the other hand, by hypotheses (ii), it follows that $u = (u_0,u_0)$ $(\forall u_0 > 0)$, $v = (-u_0,0)$. Moreover, $w = (u_0,0,u_0)$. By directly calculating, it holds that $N(-S,g(\bar{x})) = cone(0,0,-1)$, $T^2(C,\bar{x},u) = T''(C,\bar{x},u) = \mathbb{R}^2$, and for every $x = (x_1,x_2) \in \mathbb{R}^2$,

$$\begin{aligned} D^2_c(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x_1,x_2) &= (2u_0^2[a-c,a+c]-x_2)\times\{0\} \\ &\times\{x_1\}\times\{2u_0^2\}\times([-3|x_1-x_2|,-|x_1-x_2|]+x_2), \end{aligned} \tag{4.4}$$

$$\begin{aligned} D''_c(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x_1,x_2) &= \{-x_2\}\times\{0\} \\ &\times\{x_1\}\times\{2u_0^2\}\times([-3|x_1-x_2|,-|x_1-x_2|]+x_2). \end{aligned} \tag{4.5}$$

Choose $(\lambda,\eta) = (1,0,0,0,-1) \in \mathbb{R}^2_+ \times cone\{(0,0,-1)\}$. For all $x = (x_1,x_2) \in T^2(C,\bar{x},u)$, $(y,z) := (y',y'',z''',z'',z') \in D^2_c(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)$, it follows from (4.4) that

$$\langle\lambda,y\rangle + \langle\eta,z\rangle = y' - z' \geq 2u_0^2(a-c) - 2x_2 + |x_2 - x_1|. \tag{4.6}$$

The right-hand side of (4.2) equals

$$x_2 - 3x_2 = -2x_2. \tag{4.7}$$

Combining (4.6) and (4.7) yields that (4.2) because $|x_2 - x_2| \geq 0$, $2u_0^2(a-c) > 0$.

For any $x = (x_1,x_2) \in T''(C,\bar{x},u) \cap u^\perp \setminus \{(0,0)\}$, one gets $x_1,x_2 \in \mathbb{R}$, $(x_1,x_2) \neq (0,0)$, $< x,u >= 0$, i.e., $x_1 + x_2 = 0$, $x_1 \neq 0, x_2 \neq 0$. Thus, $|x_1 - x_2| = 2|x_1| > 0$. For any $(y,z) := (y',y'',z''',z'',z') \in D''_c(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)$, (4.5) yields that

$$\langle\lambda,y\rangle + \langle\eta,z\rangle = y' - z' \geq -2x_2 + 2|x_1| > -2x_2. \tag{4.8}$$

It can easily be seen that

$$\langle\lambda,\nabla f(\bar{x})(x)\rangle + \langle\eta,\nabla g(\bar{z})(x)\rangle = -2x_2.$$

This combines with (4.8) to find that (4.3) holds. Theorem 4.2 leads to $\bar{x} = (0,0)$, which is a local weakly efficient solution of VEPC.

Corollary 4.1. *Let problem VEPC with X,Y,Z be finite-dimensional real Banach spaces and let $C \subset X$ be a convex subset. Let $\bar{x} \in K$ and the mappings f,g and h be of class $C^{1,1}$ such that $graph(f,g)$ is convex. Assume, furthermore, that the following conditions are fulfilled*

(i) *For all $u \in T(C,\bar{x}) \cap ker\nabla h(\bar{x})$, $D_c f(\bar{x})(u) \cap (-intQ) = \emptyset$ and $D_c g(\bar{x})(u) \cap T(-S,g(\bar{x})) \neq \emptyset$;*

(ii) *For all $u \in T(C,\bar{x}) \cap ker\nabla h(\bar{x}) \setminus \{0\}$, $v \in D_c f(\bar{x})(u) \cap (-bdQ)$, $w \in D_c g(\bar{x})(u) \cap (-cl\, S_{g(\bar{x})})$ with $T^{''}(C,\bar{x},u) \neq \emptyset$, $T^{''}(graph(f,g),(\bar{x},(f,g)(\bar{x})),u,(v,w)) \neq \emptyset$ and*

$$0 \in T^2(C,\bar{x},u) \cap T^2(graph(f,g),(\bar{x},(f,g)(\bar{x})),u,(v,w)),$$

there exists the pair $(\lambda,\eta) \in Q^+ \times N(-S,g(\bar{x}))$ satisfying, either (4.2) or (4.3) holds for all $x \in T^2(C,\bar{x},u)$, $(y,z) \in D_c^2(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x)$.

Then $\bar{x}$ is a local weakly efficient solution of VEPC.

Proof. Since $\dim(X) < +\infty$, we find that X is reflexible. By hypotheses $\dim(Y \times Z) < \infty$, we see that $Y \times Z$ is reflexible. It follows from assumption (ii) and the well known properties in [2, 6] that

$$T^{''}(C,\bar{x},u) = T^2(C,\bar{x},u),\ \ T^{''}(\text{graph}(f,g),(\bar{x},(f,g)(\bar{x})),u,(v,w)) = T^2(\text{graph}(f,g),(\bar{x},(f,g)(\bar{x})),u,(v,w)).$$

Therefore,

$$D_c^2(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x) = D_c^{''}(f,g)(\bar{x},(f,g)(\bar{x}),u,(v,w))(x).$$

Hence, the right-hand side of (4.2) equals the right-hand side of (4.3). The rest proof follows Theorem 4.2. This completes the proof. □

5. Applications to vector variational inequalities and vector optimization problem

In this subsection, we give some applications of Theorem 3.1, Theorem 3.2 and Theorem 4.2 to vector variational inequalities with constraints and vector optimization problems with constraints. From now on, on sufficient optimality conditions, we always assume that, for all $k \in C^{0,1}$, $\nabla k(\bar{x})(x)$ exists, and, for all $k \in C^{1,1}$, $\nabla^2 k(\bar{x})(u,u)$ exists.

Theorem 5.1. *Let $T : X \to L(X,Y)$ be a vector-valued mapping and let $\bar{x}$ be a local weakly efficient solution of VVIC. Suppose, in addition, that $u \in ker\nabla h(\bar{x}) \cap IT(C,\bar{x})$ and h is directionally metrically regular at $(\bar{x},u)$ with respect to $T = \{0\}$ when $u \neq 0$. The following assertions hold.*

(i) For g of class $C^{0,1}$ with $\nabla g(\bar{x})(u)$ exists, $\exists\lambda_0 \in Q^+$, $\exists\eta_0 \in N(-S,g(\bar{x}))$ with $(\lambda_0,\eta_0) \neq (0,0)$ satisfying

$$\langle\lambda_0,\langle T\bar{x},u\rangle\rangle + \langle\eta_0,\nabla g(\bar{x})(u)\rangle \geq 0.$$

(ii) For g of class $C^{1,1}$ with $\nabla^2 g(\bar{x})(u,u)$ exists, $\exists\lambda \in Q^+$, $\exists\eta \in N(-S,g(\bar{x}))$ with $(\lambda,\eta) \neq (0,0)$ satisfying

(a) *$\forall\, x \in A^2(C,\bar{x},u)$, $\forall\, v \in (\langle T\bar{x},u\rangle + Q) \cap (-bd\,Q)$, and $\forall\ w \in D_c g_+(\bar{x},g(\bar{x}))(u) \cap (-cl\, S_{g(\bar{x})})$,*

$$\begin{aligned} inf\Big\{\, \langle\lambda,y\rangle + \langle\eta,z\rangle \mid (y,z) \in D_c^2(T'_+,g_+)(\bar{x},(T',g)(\bar{x}),u,(v,w))(x)\Big\} \\ \geq \quad \langle\lambda,\langle T\bar{x},x\rangle\rangle + \big\langle\eta,\nabla g(\bar{x})(x) + \nabla^2 g(\bar{x})(u,u)\big\rangle. \end{aligned}$$

(b) *$\forall\, x \in A^{''}(C,\bar{x},u)$, $\forall\, v \in (\langle T\bar{x},u\rangle + Q) \cap (-bd\,Q)$, and $\forall\ w \in D_c g_+(\bar{x},g(\bar{x}))(u) \cap (-cl\, S_{g(\bar{x})})$,*

$$\begin{aligned} inf\Big\{\, \langle\lambda,y\rangle + \langle\eta,z\rangle \mid (y,z) \in D_c^{''}(T'_+,g_+)(\bar{x},(T',g)(\bar{x}),u,(v,w))(x)\Big\} \\ \geq \quad \langle\lambda,\langle T\bar{x},x\rangle\rangle + \langle\eta,\nabla g(\bar{x})(x)\rangle, \end{aligned}$$

where the mapping $T' = \langle T\bar{x},\, .\, - \bar{x}\rangle : X \to Y$.

Proof. Case (i). Repeat the proof of Theorem 3.1 with a note that $T\bar{x} \in C^{0,1}$ and $\nabla F(\bar{x},\bar{x})(u) = \langle T\bar{x}, u\rangle$, which implies that (i) holds.

Case (ii). It is not difficult to check that $T\bar{x} \in C^{1,1}$ and, as Q is closed, thus $D_c F(\bar{x}, .)_+(\bar{x}, 0)(u) = \langle T\bar{x}, u\rangle + Q$. Arguing similarly as in the proof of Theorem 3.1 and Theorem 3.2, we get the desired conclusion immediately. □

Theorem 5.2. *Let $T : X \to L(X,Y)$ be a vector-valued mapping, $\bar{x} \in K$, and the mappings g, h of class $C^{1,1}$. Suppose, in addition, that $\dim(X) < +\infty$ and the following conditions are fulfilled.*

(i) For all $u \in T(C,\bar{x}) \cap ker \nabla h(\bar{x})$, $\langle T\bar{x}, u\rangle \notin -intQ$ and $D_c g(\bar{x})(u) \cap T(-S, g(\bar{x})) \neq \emptyset$;

(ii) For all $u \in T(C,\bar{x}) \cap ker \nabla h(\bar{x}) \setminus \{0\}$, $\langle T\bar{x}, u\rangle \in -bdQ$ and $w \in D_c g(\bar{x})(u) \cap (-cl\, S_{g(\bar{x})})$, there exists $\eta \in N(-S, g(\bar{x}))$ such that

(a) For all $x \in T^2(C,\bar{x},u)$, $z \in D_c^2 g(\bar{x}, g(\bar{x}), u, w)(x)$ satisfying $\langle \eta, z\rangle > \left\langle \eta, \nabla g(\bar{x})(x) + \nabla^2 g(\bar{x})(u,u)\right\rangle$;

(b) For all $x \in T^{''}(C,\bar{x},u) \cap u^{\perp} \setminus \{0\}$, $z \in D_c^{''} g(\bar{x}, g(\bar{x}), u, w)(x)$ satisfying $\langle \eta, z\rangle > \langle \eta, \nabla g(\bar{x})(x)\rangle$.

Then $\bar{x}$ is a local weakly efficient solution of VVIC.

Proof. Assume that (i) and (ii) of Theorem 5.2 are fulfilled. It is easy to see that $D_c F(\bar{x},\bar{x})(u) = \{\langle T\bar{x}, u\rangle\}$. Taking $v = \langle T\bar{x}, u\rangle$, and setting $T' = \langle T\bar{x}, . - \bar{x}\rangle : X \to Y$, one obtains the results similar to Proposition 3.3 (ii) in [8] as follows:

$$D_c^2(T', g)(\bar{x}, (T', g)(\bar{x}), u, (v, w))(x) = \{\langle T\bar{x}, x\rangle\} \times D_c^2 g(\bar{x}, g(\bar{x}), u, w)(x),$$

$$D_c^{''}(T', g)(\bar{x}, (T', g)(\bar{x}), u, (v, w))(x) = \{\langle T\bar{x}, x\rangle\} \times D_c^{''} g(\bar{x}, g(\bar{x}), u, w)(x).$$

In other words, for any $\lambda \in Q^+$, $x \in T^2(C,\bar{x},u)$, and $(y,z) \in D_c^2(T', g)(\bar{x}, (T', g)(\bar{x}), u, (v,w))(x)$, one gets

$$\langle \lambda, y\rangle + \langle \eta, z\rangle > \langle \lambda, \langle T\bar{x}, x\rangle\rangle + \left\langle \eta, \nabla g(\bar{x})(x) + \nabla^2 g(\bar{x})(u,u)\right\rangle,$$

and for any $\lambda \in Q^+$, $x \in T^{''}(C,\bar{x},u) \cap u^{\perp} \setminus \{0\}$, $(y,z) \in D_c^{''}(T', g)(\bar{x}, (f,g)(\bar{x}), u, (v,w))(x)$, one also has

$$\langle \lambda, y\rangle + \langle \eta, z\rangle > \langle \lambda, \langle T\bar{x}, x\rangle\rangle + \langle \eta, \nabla g(\bar{x})(x)\rangle.$$

Therefore the proof of Theorem 5.2 follows from Theorem 4.2, which completes the proof. □

Theorem 5.3. *Let $l : X \to Y$ be a vector-valued mapping and let $\bar{x}$ be a local weakly efficient solution of VOPC. Suppose, in addition, that $u \in ker \nabla h(\bar{x}) \cap IT(C,\bar{x})$ and h is directionally metrically regular at $(\bar{x}, u)$ with respect to $T = \{0\}$ when $u \neq 0$. The following assertions hold.*

(i) For l and g of class $C^{0,1}$ with $\nabla l(\bar{x})(u)$ and $\nabla g(\bar{x})(u)$ exist, $\exists \lambda_0 \in Q^+$, $\exists \eta_0 \in N(-S, g(\bar{x}))$ with $(\lambda_0, \eta_0) \neq (0,0)$ satisfying $\langle \lambda_0, \nabla l(\bar{x})(u)\rangle + \langle \eta_0, \nabla g(\bar{x})(u)\rangle \geq 0$.

(ii) For l and g of class $C^{1,1}$ with $\nabla^2 l(\bar{x})(u,u)$ and $\nabla^2 g(\bar{x})(u,u)$ exist, $\exists \lambda \in Q^+$, $\exists \eta \in N(-S, g(\bar{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

(a) $\forall\, x \in A^2(C,\bar{x},u)$, $\forall\, v \in D_c l_+(\bar{x}, l(\bar{x}))(u) \cap (-bd\, Q)$, *and* $\forall\, w \in D_c g_+(\bar{x}, g(\bar{x}))(u) \cap (-cl\, S_{g(\bar{x})})$,

$$inf\left\{\langle \lambda, y\rangle + \langle \eta, z\rangle \mid (y,z) \in D_c^2(l_+, g_+)(\bar{x}, (l,g)(\bar{x}), u, (v,w))(x)\right\}$$
$$\geq \left\langle \lambda, \nabla l(\bar{x})(x) + \nabla^2 l(\bar{x})(u,u)\right\rangle + \left\langle \eta, \nabla g(\bar{x})(x) + \nabla^2 g(\bar{x})(u,u)\right\rangle.$$

(b) $\forall\, x \in A^{''}(C,\bar{x},u)$, $\forall\, v \in D_c l_+(\bar{x}, l(\bar{x}))(u) \cap (-bd\, Q)$, *and* $\forall\, w \in D_c g_+(\bar{x}, g(\bar{x}))(u) \cap (-cl\, S_{g(\bar{x})})$,

$$inf\left\{\langle \lambda, y\rangle + \langle \eta, z\rangle \mid (y,z) \in D_c^{''}(l_+, g_+)(\bar{x}, (l,g)(\bar{x}), u, (v,w))(x)\right\} \geq \langle \lambda, \nabla l(\bar{x})(x)\rangle + \langle \eta, \nabla g(\bar{x})(x)\rangle.$$

Proof. Argue similarly as for Theorems 3.1 and 3.2, with l replacing $F(\bar{x}, .)$. This obtains the claim. □

Theorem 5.4. *Let $l : X \to Y$ be a vector-valued mapping, $\bar{x} \in K$ and assume that l, g and h of class $C^{1,1}$. Assume, furthermore, that $dimX < +\infty$ and the following conditions are fulfilled.*

(i) For all $u \in T(C,\bar{x}) \cap ker \nabla h(\bar{x})$, $D_c l(\bar{x})(u) \cap (-intQ) = \emptyset$ and $D_c g(\bar{x})(u) \cap T(-S, g(\bar{x})) \neq \emptyset$;

(ii) For all $u \in T(C,\bar{x}) \cap ker \nabla h(\bar{x}) \setminus \{0\}$, $v \in D_c l(\bar{x})(u) \cap (-bdQ)$ and $w \in D_c g(\bar{x})(u) \cap (-cl\, S_{g(\bar{x})})$, there exist $(\lambda, \eta) \in Q^+ \times N(-S, g(\bar{x}))$ such that

(a) For all $x \in T^2(C,\bar{x},u)$, $(y,z) \in D_c^2(l,g)(\bar{x},(l,g)(\bar{x}),u,(v,w))(x)$,

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \; > \; \big\langle \lambda, \nabla l(\bar{x})(x) + \nabla^2 l(\bar{x})(u,u)\big\rangle + \big\langle \eta, \nabla g(\bar{x})(x) + \nabla^2 g(\bar{x})(u,u)\big\rangle;$$

(b) For all $x \in T''(C,\bar{x},u) \cap u^{\perp} \setminus \{0\}$, $(y,z) \in D_c''(l,g)(\bar{x},(l,g)(\bar{x}),u,(v,w))(x)$,

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \; > \; \langle \lambda, \nabla l(\bar{x})(x)\rangle \, + \, \langle \eta, \nabla g(\bar{x})(x)\rangle .$$

Then $\bar{x}$ is a local weakly efficient solution of VOPC.

Proof. It is an immediately consequence from Theorem 4.2. □

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