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A MODIFIED HYBRID ALGORITHM FOR SOLVING A COMPOSITE MINIMIZATION PROBLEM IN BANACH SPACES

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Abstract. The purpose of this paper is to use a modified hybrid algorithm for finding a minimizer of a non-smooth composite minimization problem in Banach spaces. Without the assumptions that the potential function is Fréchet differentiable and its gradient is L-Lipschitz continuous, we prove that the iterative sequence generated by the hybrid algorithm converges strongly to a minimizer of the composite optimization problem in Banach spaces.

Keywords. Banach space; Composite optimization; Convex minimization problem; Lower semi-continuous function.

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1. INTRODUCTION

Let X be a real Banach space. Denote by $\Gamma_0(X)$ the class of all proper convex and lower semi-continuous functions from X to $(-\infty,+\infty]$.

In this paper, we study the following non-smooth composite optimization problem: find $x^* \in X$ such that

$$f(x^*)+g(x^*)=\min_{x\in X}\{f(x)+g(x)\}, \tag{1.1}$$

where $f,g\in\Gamma_0(X)$. This problem has a typical scenario in linear inverse problems, and it has applications in image reconstruction, machine learning, data recovering and compressed sensing; see [2, 5, 6, 13, 18, 19, 20, 21] and the references therein.

In the case that X is a real Hilbert space H, problem (1.1) has been studied by many authors; see, for example, [7, 8, 10, 12, 19] and the references therein. In 2011, Xu [16] considered the following constrained convex minimization problem:

$$\min_{x\in C} f(x), \tag{1.2}$$

where C is a nonempty closed and convex subset of H, $f\in\Gamma_0(H)$ is Fréchet differentiable and ∇f is L-Lipschitz continuous. By using the gradient-projection algorithm, Xu [16] proved the following result.

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Theorem 1.1. *Let f be Fréchet differentiable function and the gradient ∇f of f be L-Lipschitz continuous. Let $h : C \to C$ be a ρ-contraction with $\rho \in [0,1)$. Let $\{x_n\}$ be a sequence generated by the following gradient projection algorithm: for any given $x_0 \in H$*

$$x_{n+1} = \theta_n h(x_n) + (1-\theta_n) Proj_C(x_n - \gamma_n \nabla f(x_n)),\ n = 0,1,2,\cdots, \tag{1.3}$$

where $Proj_C$ is the projection from H onto C. If the set S of solutions of minimization problem (1.2) is nonempty, and the following conditions are satisfied:

(i) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < \frac{2}{L}$;

(ii) $\theta_n \to 0$; $\sum_{n=0}^{\infty} \theta_n = \infty$; $\sum_{n=0}^{\infty} |\theta_{n+1} - \theta_n| < \infty$;

(iii) $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,

then sequence $\{x_n\}$ converges strongly to a minimizer of problem (1.2).

Recently, Guo and Cui [8] considered the following composite optimization problem in Hilbert space H:

$$\min_{x\in X} \{f(x) + g(x)\}, \tag{1.4}$$

where $f, g \in \Gamma_0(H)$, f is differentiable, and ∇f is L-Lipschitz continuous. Let $\{x_n\}$ be a sequence generated by the following gradient projection algorithm: for given $x_0 \in H$

$$x_{n+1} = t_n h(x_n) + (1-t_n) Prox_{\alpha_n g}(I - \alpha_n \nabla f(x_n)) + e(x_n),\ n \ge 0 \tag{1.5}$$

where $Prox_{\lambda g}$ is a proximal operator defined by

$$Prox_{\alpha g}(x) := \arg\min_{y\in H} \{\frac{||y-x||^2}{2\alpha} + g(y)\},\ x \in H,\ \alpha \ge 0,$$

and $e : H \to H$ is a perturbation operator. They proved the following results.

Theorem 1.2. *Let S be the set of minimizers of problem (1.4). If $S \neq \emptyset$ and the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} ||e(x_n)|| < \infty$;

(ii) $0 < a = \inf_n \alpha_n \le \alpha_n < \frac{2}{L}$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\{t_n\} \subset (0,1)$, $\lim_{n\to\infty} t_n = 0$; *and* $\sum_{n=0}^{\infty} t_n = \infty$; $\sum_{n=0}^{\infty} |t_{n+1} - t_n| < \infty$;

Then the sequence $\{x_n\}$ generated in (1.5) converges strongly to a point $x^* \in S$.

Very recently, Sabach and Shtern [14] introduced and studied the following bi-level optimization problem in Euclidean space $\mathbf{R}^n$. The outer level is given by the following constraint minimization problem
.

$$\min_{x\in S} \omega(x), \tag{1.6}$$

where S is the set of minimizers of the following inner level minimization problem:

$$\min_{x\in \mathbf{R}^n} \{f(x) + g(x)\}. \tag{1.7}$$

Assume the following conditions are satisfied:

(1) $f : \mathbf{R}^n \to \mathbf{R}$ is convex and continuously differentiable and its gradient ∇f is Lipschitz with constant $L_f > 0$; $g : \mathbf{R}^n \to (-\infty, \infty]$ is proper, lower semicontinuous and convex, and S is nonempty;

(2) $\omega : \mathbf{R}^n \to \mathbf{R}$ is strongly convex with parameter $\sigma > 0$, i.e., there exists $\sigma > 0$ such that for all $x, y \in \mathbf{R}^n$, and $\alpha \in (0,1)$

$$\omega(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\frac{\sigma}{2}||x-y||^2 \le \alpha\omega(x) + (1-\alpha)\omega(y).$$

Assume further that ω is continuously differentiable and $\nabla\omega$ is L_ω-Lipschitz continuous;

(3) $\{\alpha_k\}$ is a sequence in (0, 1], $\alpha_k \to 0$, $\sum_{k=0}^{\infty}\alpha_k = \infty$ and $\lim_{k\to\infty}\frac{\alpha_{k+1}}{\alpha k} = 1$.

If $t \in (0, \frac{1}{L_f}]$, and $s \in (0, \frac{2}{(L_\omega+\sigma)}]$, then the sequence $\{x_n\}$ generated via the following manner: $x_0 \in \mathbf{R}^n$,

$$\begin{cases} y_k = Prox_{tg}(x_{k-1} - t\nabla f(x_{k-1})), \\ z_k = x_{k-1} - s\nabla\omega(x_{k-1}), \qquad k = 1, 2, \cdots \\ x_k = \alpha_k z_k + (1-\alpha_k)y_k \end{cases} \tag{1.8}$$

converges strongly to a solution of problem (1.6).

Motivated and inspired by the results going on in this direction, the purpose of this paper is to use a modified hybrid algorithm for solving non-smooth composite minimization problem (1.1) in Banach spaces without the assumptions that f is Fréchet differentiable and ∇f is L-Lipschitz continuous. Our results improve and extend the corresponding results in [7, 8, 11, 12, 16, 17].

2. Preliminaries

In order to prove the main results, we need the following basic concepts, notations and lemmas.

In the sequel, we assume that X is a real smooth, strictly convex, and reflexive Banach space (the definitions and properties, see, e.g. [4]), and X^* is the dual of X.

- A mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2\},\ x \in X \tag{2.1}$$

is called the normalized duality mapping. By Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in X$. $J : X \to 2^{X^*}$ is single-valued, one-to-one, and onto (see [4]).

- Let X be a smooth, strictly convex and reflexive Banach space, and let C be a nonempty closed and convex subset of X. Let $\phi : X \times X \to \mathbf{R}^+$ be the Lyapunov functional which is defined by

$$\phi(x,y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \quad \forall x, y \in X. \tag{2.2}$$

It is obvious from the definition of ϕ that

$$(||x|| - ||y||)^2 \le \phi(x,y) \le (||x|| + ||y||)^2, \quad \forall x, y \in X. \tag{2.3}$$

- Following Alber [1], the generalized projection $\Pi_C : X \to C$ is defined by

$$\Pi_C(x) = \inf_{y\in C}\phi(y,x), \quad x \in X. \tag{2.4}$$

Lemma 2.1. [1] *Let X be a smooth, strictly convex and reflexive Banach space, and let C be a nonempty closed and convex subset of X. Then*

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y)$, $\forall x \in C, y \in X$.

(2) Let $x \in X$ and $z \in C$, then

$$z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \ge 0, \quad \forall y \in C. \tag{2.5}$$

Remark 2.1. (i) If X is a real Hilbert space H, then $\phi(x,y) = ||x-y||^2$ and Π_C is the metric projection P_C of H onto C.

(ii) If X is a smooth, strictly convex and reflexive Banach space, then for $x,y \in X, \phi(x,y) = 0$ if and only if $x = y$ (see [4]).

Lemma 2.2. [9] *Let X be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n,y_n) = 0$, then $\lim_{n\to\infty} = ||x_n - y_n|| = 0$.*

Now we consider the following non-smooth composite optimization problem: find a point $x^* \in C$ such that

$$f(x^*) + g(x^*) = \min_{x\in C}\{f(x) + g(x)\}, \tag{2.6}$$

where C is a nonempty closed convex bset in a real Banach space X and $f, g : C \to (-\infty, +\infty]$ are proper convex and lower semi-continuous. If we set

$$\Theta(x,y) := f(y) - f(x), \tag{2.7}$$

then problem (2.6) is equivalent to the problem of finding $x^* \in C$ such that

$$\Theta(x^*,y) + g(y) - g(x^*) \geq 0, \quad \forall y \in C. \tag{2.8}$$

Setting

$$F(x,y) := \Theta(x,y) + g(y) - g(x), \quad \forall x,y \in C. \tag{2.9}$$

we show that the function $F(x,y) : C \times C \to (-\infty, +\infty]$ has the following properties:

(A1) $F(x,x) = 0$, $\forall x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;

(A3) $\limsup_{t\downarrow 0} F(x + t(z-x), y) \leq F(x,y)$, $\forall x,y,z \in C$;

(A4) The function $y \mapsto F(x,y)$ is convex and lower semi-continuous.

It is easy to prove that F has the properties $(A1)-(A3)$. Now we only prove that $F(x,.)$ has the property $(A4)$ for each $x \in C$. In fact, for each $x \in C$, we have

$$\begin{aligned} F(x, ty + (1-t)z) &= f(ty + (1-t)z) - f(x) + g(ty + (1-t)z) - g(x) \\ &\leq t f(y) + (1-t) f(z) - f(x) + t g(y) + (1-t) g(z) - g(x) \\ &= t(\Theta(x,y) + g(y) - g(x)) + (1-t)(\Theta(x,z) + g(z) - g(x)) \\ &= tF(x,y) + (1-t)F(x,z). \end{aligned}$$

The conclusion is proved.

Lemma 2.3. [3, 15] *Let X be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty closed and convex subset of X and let $F : C \times C \to \mathbf{R}$ be the bifunction defined by (2.9). Let $r > 0$ and $x \in X$.*

(i) There exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z,\, Jz - Jx\rangle \geq 0,\ y \in C; \tag{2.10}$$

(ii) Define a mapping $T_r : X \to C$ by

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z,\, Jz - Jx\rangle \geq 0,\, \forall y \in C\},\, x \in X. \tag{2.11}$$

Then, the following conclusions hold:

(a) T_r is single-valued;

(b) T_r is a firmly nonexpansive-type mapping, i.e., $\forall z, y \in X$,

$$\langle T_r z - T_r y,\, JT_r z - JT_r y\rangle \leq \langle T_r z - T_r y,\, Jz - Jy\rangle;$$

(c) $Fix(T_r) = \Omega$, where Ω is the solution set of problem (2.8), (i.e., the set of minimizers of problem (2.6)) and Ω is closed and convex.

(d) $\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$, $\forall q \in Fix(T_r)$.

3. Main results

We are in a position to give the following main result.

Theorem 3.1. *Let X be a real uniformly smooth and uniformly convex Banach space. Let C be a nonempty closed and convex subset of X. Let $f, g : C \to (-\infty, +\infty]$ be two proper convex and lower semi-continuous functions, and let $F : C \times C \to (-\infty, +\infty]$ be the bifunction defined by (2.9). Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C, \\ C_0 = C, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n,\, Ju_n - Jx_n\rangle \geq 0, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $J : E \to E^*$ is the normalized duality mapping, $\{r_n\}$ is a sequence of positive numbers with $r_n \geq r > 0$. If Ω (the set of minimizers of problem (2.6)) is nonempty, then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$, which is a minimizer of problem (2.6).

Proof. We divide the proof into six parts.

(I) We prove that C_n, $\forall n \geq 0$, is a closed and convex subset of C.

In fact, since $C_0 = C$, we assert that C_0 is closed convex. By induction, if for some $n \geq 1$, C_n is closed and convex. We now prove that C_{n+1} is also closed and convex. Indeed,

$$\phi(v, u_n) \leq \phi(v, x_n)\} \Leftrightarrow 2\langle v, Jx_n - Ju_n\rangle \leq ||u_n||^2 + ||x_n||^2\}.$$

This implies that C_{n+1} is closed and convex.

(II) We prove that $\Omega \subset C_n$ for each $n \geq 0$.

In fact, it is obvious that $\Omega \subset C_0$. If for some $n \geq 1$, we have $\Omega \subset C_n$. We next prove that $\Omega \subset C_{n+1}$.

Indeed, it follows from Lemma 2.3 (ii) (a)-(d) and (3.1) that $u_n = T_{r_n}(x_n)$ and $Fix(T_{r_n}) = \Omega$. Therefore, for each $u \in \Omega \subset C_n$, we have

$$\phi(u, u_n) = \phi(u, T_{r_n}(x_n)) \leq \phi(u, x_n),$$

i.e., $u \in C_{n+1}$. This implies that $\Omega \subset C_{n+1}$. Summing up the above arguments, $\{C_n\}_{n=0}^{\infty}$ is a sequence of nonempty closed and convex subsets in C. Therefore the sequence $\{x_n\}$ is well defined.

(III) We prove that $\{x_n\}$ is bounded

In fact, it follows from (3.1) that $x_n = \Pi_{C_n}(x_0)$, $\forall n \geq 0$. For any given $u \in \Omega$, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}(x_0), x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{C_n}(x_0)) \leq \phi(u, x_0),\ \forall n \geq 0. \tag{3.2}$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded. By virtue of (2.3), we have that $\{x_n\}$ is bounded.

(IV) We prove that $\{x_n\}$ is a Cauchy sequence.

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$ and $x_n = \Pi_{C_n} x_0$. From the definition of Π_{C_n}, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.$$

This shows that $\{\phi(x_n, x_0)\}$ is nondecreasing and bounded. Hence $\lim_{n\to\infty} \phi(x_n, x_0$ exists. By Lemma 2.1 (1), for any given positive integer m, we have

$$\begin{aligned}\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0)) \leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0)\\ &= \phi(x_{n+m}, x_0) - \phi(x_n, x_0),\ \forall n \geq 0.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0, \quad \forall m \geq 1.$$

By Remark 2.1, we have

$$\lim_{n\to\infty} ||x_{n+m} - x_n|| = 0, \quad \forall m \geq 1, \tag{3.3}$$

i.e., $\{x_n\}$ is a Cauchy sequence in C. Without loss of the generality, we may assume that

$$\lim_{n\to\infty} x_n = p \in C. \tag{3.4}$$

(V) We prove that $x_n \to p \in \Omega$.

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, it follows from the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n),\ \forall n \geq 0. \tag{3.5}$$

Since X is uniformly smooth and uniformly convex, it follows from (3.3)-(3.5) and Lemma 2.3 that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||x_{n+1} - u_n|| = \lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{3.6}$$

This implies that

$$\lim_{n\to\infty} \frac{||Ju_n - Jx_n||}{r_n} \leq \lim_{n\to\infty} \frac{||Ju_n - Jx_n||}{r} = 0. \tag{3.7}$$

Since $F(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ Ju_n - Jx_n\rangle \geq 0$, $\forall y \in C$, by condition (A1), we have

$$\frac{1}{r_n}\langle y - u_n,\ Ju_n - Jx_n\rangle \geq -F(u_n, y) \geq H(y, u_n),\ \forall y \in C. \tag{3.8}$$

Let $n \to \infty$ in (3.8). It follows from (3.7) and $y \mapsto F(x, y)$ is convex and lower semi-continuous that $H(y, p) \leq 0$, $\forall y \in C$. For each $t \in (0, 1]$ and each $y \in C$, let $y_t = ty + (1-t)p$. Hence $y_t \in C$ and $F(y_t, p) \leq 0$. By conditions (A1) and (A4), we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, p) \leq tF(y_t, y).$$

Dividing both sides by t, we have $F(y_t, y) \geq 0$, $\forall y \in C$. Letting $t \downarrow 0$, we have by condition (A3) that $F(p,y) \geq 0$, $\forall y \in C$, i.e., $p \in \Omega$.

(VI) We prove that $x_n \to p = \Pi_\Omega(x_0)$, i.e., p is the unique minimizer of Problem (2.6).

Letting $w = \Pi_\Omega(x_0)$, we have $w \in \Omega \subset C_{n+1}$, for each $n \geq 0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$. Therefore,

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0),\ \forall n \geq 0,\ i.e.,$$
$$\phi(p, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(w, x_0). \tag{3.9}$$

By the definition of $\Pi_\Omega(x_0)$ and (3.9) we have $w = p$, i.e., $\lim_{n\to\infty} x_n = p = \Pi_\Omega(x_0)$, which is a minimizes of problem (2.6). This completes the proof of Theorem 3.1. □

Now we consider the following composite optimization problem in Hilbert space H:

$$\min_{x\in C}\{f(x) + g(x)\}, \tag{3.10}$$

where C is a nonempty closed and convex subset of H and $f, g : C \to (-\infty, +\infty]$ are two proper convex and lower semi-continuous functions.

The following result can be obtained from Theorem 3.1 immediately.

Corollary 3.1. *Let H, C, f, g be the same as in Theorem 3.1. Let $F : C \times C \to (-\infty, +\infty]$ be the bifunction defined by (2.9). Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C,\ C_0 = C, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \\ C_{n+1} = \{v \in C_n : \|v - u_n\|^2 \leq \|v - x_n\|^2\}, \\ x_{n+1} = Proj_{C_{n+1}} x_0, \end{cases} \quad \forall n \geq 0$$

where $Proj_C : H \to C$ is the metric projection, $\{r_n\}$ is a sequence of positive numbers with $r_n \geq r > 0$. If Γ (the set of minimizers of problem (3.10)) is nonempty, then $\{x_n\}$ converges strongly to $Proj_\Gamma(x_0)$, which is a minimizer of problem (3.1).

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