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NONSMOOTH DYNAMICS OF GENERALIZED NASH GAMES

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Abstract. The generalized Nash equilibrium problem (GNEP) is an N-player noncooperative game, where each player has to solve a nonlinear optimization problem whose objective function and constraints depend on the choices of the other players. As in the case of classic Nash games, where other players' choices only impact a player's objective function, a natural question arises as to how players might evolve their strategies over time, and whether or not this evolution would allow them to reach a Nash equilibrium strategy. The approach in classical Nash games is that of introducing some form of differential equations/systems whose stable points are exactly the Nash strategies of the game. This approach leads to considering projected dynamical systems and sweeping processes. In this paper, we show that these dynamical system approaches can be extended to the case of the GNEP. We present dynamical systems that are useful in this context and discuss the new difficulties introduced by this more complex game. Finally, we show how to exploit the existence proof to build numerical methods and solve GNEP problems from the literature.

Keywords. Generalized Nash equilibrium problem; Quasi-variational inequality; Projected dynamical system; Sweeping process.

1. Introduction

We consider an N-player noncooperative game in which the strategy sets of the players, as well as their objective functions, are mutually dependent. The problem of finding equilibria of this game is called the generalized Nash equilibrium problem (GNEP). Nash [47] introduced a notion of equilibrium for games, the well-known Nash equilibrium, where only the payoff function of each player depends on the others' strategies. Later on, Arrow and Debreu [4] extended this notion to the generalized Nash equilibrium. Initially motivated by economic applications, the notion of equilibrium in games has received a vivid interest thanks to its various applications in social sciences, biology, computer science or energy problems to cite a few among others. These applications have motivated the evolution of the Nash equilibrium concept, and its use, to complex games that now require a deep understanding of theoretical and computational mathematics used for identifying, computing and analyzing (all) the equilibrium strategy(ies) of a given game.

In the optimization and variational analysis literatures, the GNEP has become an active subject during the past two decades. We refer readers to the survey papers [25] and [28] for a complete overview of the state of the art of theoretical results and numerical methods for GNEP.

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The GNEP is popular as a modeling tool, though the theory of numerical algorithms to solve it is still in its infancy. One particular path for obtaining its solutions is to "bridge" the gap between the GNEP and several variational tools well-known from the literature, and then use this well-developed theory to solve it. Examples of such variational tools are variational inequalities and quasi-variational inequalities.

The variational inequality is an inequality involving a mapping that has to be satisfied for all values over a (convex) set and is now a well-established modeling tool in economics [17, 44], optimization [26], and game theory [7, 16, 30]. Its extension to a variational inequality with a parametric set, the quasi-variational inequality [10, 42], is now receiving increasing attention due to the need for modeling more complex problems. For instance, under classical convexity assumptions, the GNEP can be reformulated as a quasi-variational inequality [9, 51].

One approach to study (classical) Nash equilibrium problems consists in studying a differential equation/inclusion whose equilibria are the equilibria of the game. This approach leads one to consider some of the following dynamics to describe the time evolution of players' strategies: a projected dynamical system [19, 23, 24], a sweeping process [37, 41], or a replicator dynamics [57], to cite some of the most popular. The latter emerged in the context of evolutionary game theory, and we refer the interested reader to [56] for more on this topic. The projected dynamics arose in the course of considering time evolution of solutions to traffic and network equilibrium problems (see also [17, 46, 61] and the references therein), vaccination games and population games [15, 16, 38], oligopolistic market equilibrium problems [7, 61] and supply chain networks models. A projected dynamics is different from differential games where, at each time, a game has to be solved, see for instance [8], however, it can be used in the context of differential games when appropriate [18]. Yet other examples of dynamical systems arise in the theories of differential [5, 52] and evolutionary [17, 22] variational inequalities.

Dynamical systems have been mentioned in the GNEP literature in [14, 45]. However, in both cases, the study was restricted to computing the variational equilibrium, a specific type of equilibrium for shared constraints games that can be computed via a variational inequality. The difficulty in the generalized Nash equilibrium problem is the parametric constraint set. This moving set introduces an additional nonsmootheness in the solution of the dynamical system. Motivated by recent results in the literature, we propose a new application of nonsmooth dynamical systems for the generalized Nash equilibrium problem. We study a projected dynamical system and a sweeping process, both with moving sets, whose steady states are equivalent to the equilibria of the game. A further motivation for our work here is that following the path describing the solution of the differential equation/inclusion leads to an easily implementable method that relies on the evaluation of gradients and projections on the moving set. To the best of our knowledge this approach is new for the generalized Nash equilibrium problem. Additionally, it allows us to study the GNEP in an infinite-dimensional setting, thus extending the traditional Euclidean space as in [11, 34], and consider a generic constraint set. Last but not least, this work brings new incentives in the study of these nonsmooth dynamical systems and offers some insights into existence of solutions state-of-the-art and the need for generalizations.

The rest of the paper is organized as follows. In Section 2, we introduce some classical definitions related to variational inequalities and dynamical systems. In Section 3, we link the GNEP to two nonsmooth dynamical systems, whose stable states are solutions of the GNEP. In Section 4, we provide illustrations on the difficulty implied by the moving set introduction, and the assumptions used. In Section 5, we prove the existence of a solution of the PDS on a moving set. This result leads to two classes of algorithms that are presented in Section 6 and applied to classical problems from game theory. Finally, in Section 7 we discuss perspectives and future challenges motivated by this paper.

2. Variational Inequalities and Dynamical Systems

In this section, we recall a few key facts related to variational inequalities and their associated dynamical systems. We start first with some key classic definitions, listed below for ease of reading.

Notations. Let X be a separable Hilbert space endowed with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. Let Ω be a non-empty subset of X. A mapping $F : \Omega \to X$ is said to be Lipschitz continuous with modulus L on Ω if there exists $L \geq 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in \Omega.$$

For a given $x \in X$, the distance from x to Ω is defined by $\operatorname{dist}_\Omega(x) := \inf_{y\in\Omega} \|y - x\|$, and the projection of x onto Ω is defined by $\operatorname{proj}_\Omega(x) := \{y \in \Omega \ : \ \|y - x\| = \operatorname{dist}_\Omega(x)\}$. If Ω is non-empty closed and convex, then $\operatorname{proj}_\Omega$ is single-valued and Lipschitz continuous with modulus $L = 1$. Another classical measure of distance is the Hausdorff distance between two sets Ω and Ω' defined as

$$\operatorname{haus}(\Omega, \Omega') := \max \left\{ \sup_{x\in\Omega} \operatorname{dist}_{\Omega'}(x), \ \sup_{x\in\Omega'} \operatorname{dist}_\Omega(x) \right\}.$$

Given a non-empty closed convex set $K \subset X$ and a mapping $F : K \to X$, the variational inequality problem (VI) consists of finding $x \in K$ such that

$$\langle F(x), y - x \rangle \geq 0, \ \forall y \in K. \tag{2.1}$$

It is classical from convex analysis to rewrite this problem as the following inclusion

$$0 \in F(x) + N_K(x),$$

where

$$N_K(x) := \{v : \langle v, d \rangle \leq 0, \ \forall d \in T_K(x)\},$$

is the normal cone of K at x and

$$T_K(x) := \{d : \exists \{x^k\} \subset K, \{t_k\} \downarrow 0, x^k \to x \text{ and } \lim_{k\to\infty} \frac{x^k - x}{t_k} = d\},$$

is the tangent cone of K at x.

A popular approach to compute solutions of a VI problem of type (2.1) is to compute critical points (or 0-solutions) of a dynamical system whose equilibria are solutions of (2.1). We now present some definitions connected to nonsmooth dynamical systems. A comprehensive introduction on this subject can be found, for instance, in [1]. Denote the directional derivative of the projection by

$$\Pi_K(x, v) := \lim_{\delta\to 0^+} \frac{\operatorname{proj}(K; x + \delta v) - x}{\delta},$$

moreover it can be shown that $\Pi_K(x, v) = \operatorname{proj}_{T_K(x)}(v)$, as by [60, Lemma 4.5] it holds that

$$\operatorname{proj}_K(x + h) = x + \operatorname{proj}_{T_K(x)}(h) + o(h),$$

where $o(h)/h \to 0$.

Denoting $\dot{x}(t) = \frac{dx(t)}{dt}$, the Projected Dynamical System (PDS), introduced in [21, 31] and later extended in [19, 23, 24], is defined as

$$\dot{x}(t) = \Pi_K(x(t), -F(x(t))). \tag{2.2}$$

Another approach consists of a differential inclusion involving the normal cone, the so-called sweeping process (MSP). The autonomous MSP is defined as

$$-\dot{x}(t) \in F(x(t)) + N_K(x(t)). \tag{2.3}$$

In its original form, introduced by Moreau, the set K is a point to set mapping varying in time. The slow solutions of (2.3) are equivalent to the solutions of (2.2), see [1].

Definition 2.1. Given $x_0 \in K$ and $T \in \mathbb{R}_+$. Then, a function $x : [0,T] \to X$ is a solution of (2.2) (resp. (2.3)) if x is an absolutely continuous function such that $x(0) = x_0$ and (2.2) (resp. (2.3)) is satisfied for almost every $t \in [0,T]$.

In particular, previous definitions imply that any solution x is such that $x(t) \in K$ for all $t \in [0,T]$, since the normal cone is non-empty if and only if $x(t) \in K$. In the special case where K is a cone, then (2.3) can be rewritten as a complementarity system (see [13] for the link between these). In the context of Nash equilibrium problems, it frequently holds that $X = \mathbb{R}^n$ and K is a bounded set, in which case the solution will even be Lipschitz continuous.

An important interest for (2.2) and (2.3) is that the steady states of both dynamical systems are solutions of (2.1) [19, Theorem 2.2]. In this study, we focus on the PDS and MSP, which arise most naturally and hold our interest from the perspective of deriving numerical approaches. We leave the generalization of other dynamical systems for the GNEP for further research.

3. The GNEP as a Dynamical System

The Generalized Nash Equilibrium Problem (GNEP) is characterized by a finite set of N players, each of whom controls a variable $x^\nu \in X_\nu$ and has an objective function $\theta_\nu : \prod_{\nu=1}^N X_\nu \to \mathbb{R}$. We denote by $x^{-\nu}$ the vector formed by all the players' decision variables except those of player ν. We sometimes write $(x^\nu, x^{-\nu})$ instead of x, which does not mean that the block components of x are reordered. The goal of each player is to minimize their objective function subject to some constraint $x^\nu \in K_\nu(x^{-\nu})$. The key feature here is that each K_ν depends on variables beyond Player ν's control. A solution, or an equilibrium, is any strategy $x^* = (x^{*,1}, \dots, x^{*,N})$ where no player can lower their objective function by unilaterally altering their strategy. Stated otherwise, an N-player GNEP consists in N optimization problems with the player ν solving:

$$\min_{x^\nu \in X_\nu} \theta_\nu(x^\nu, x^{-\nu}) \text{ s.t. } x^\nu \in K_\nu(x^{-\nu}). \tag{3.1}$$

We denote by $X := \prod_{\nu=1}^N X_\nu$, and $X_{-\nu} := \prod_{\mu=1,\mu\neq\nu}^N X_\mu$. A typical choice of set K_ν is described by inequality constraints, for instance,

$$K_\nu(x^{-\nu}) := \{x^\nu \in X_\nu : g_\nu(x^\nu, x^{-\nu}) \leq 0\}, \tag{3.2}$$

where $g_\nu : X \to \mathbb{R}^{m_\nu}$. In the special case where there exists a closed set K such that for all ν the set of constraints of the ν-th player is given by

$$K_\nu(x^{-\nu}) = \{x^\nu : (x^\nu, x^{-\nu}) \in K\}, \tag{3.3}$$

we say that the GNEP has shared constraints, and we denote this game GNSC. This class of problems was first studied by Rosen [55]. It often arises in applications and is numerically more tractable. Assuming that K is convex, and $\theta_\nu(.,x^{-\nu})$ is convex and differentiable, any solution of the following variational inequality is a solution of the GNSC

$$\langle (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^N, y-x\rangle \geq 0, \qquad \forall y \in K. \tag{3.4}$$

However, the converse is false in general. We refer to the solutions of (3.4) as *variational equilibria*.

A classical assumption on the GNEP that we will use throughout the paper is the following.

Assumption 3.1. There exists N non-empty, convex and strongly compact sets $S_\nu \subset X_\nu$ such that for every $x \in S := \prod_{\nu=1}^N S_\nu$, $K_\nu(x^{-\nu}) \subseteq S_\nu$ is nonempty, closed and convex. For every player ν and every $x^{-\nu} \in S_{-\nu} := \prod_{\mu=1,\mu\neq\nu}^N S_\mu$, the objective function $\theta_\nu(\cdot,x^{-\nu})$ is convex and C^1.

In particular, assuming further continuity on K, this assumption ensures the existence of an equilibrium due to a classical result from Ishiichi [32], see also [25, Section 4.1].

Although VI's are very useful, a powerful tool called a Quasi-Variational Inequality (QVI) is used in the GNEP context. Given a point to set mapping $K : X \to 2^X$ and a mapping $F : X \to X$, the QVI strives to find a solution $x \in K(x)$ to the inequality [26]:

$$\langle F(x), y-x\rangle \geq 0, \qquad \forall y \in K(x). \tag{3.5}$$

Considering that (2.2) and (2.3) have been successfully used for the VI and the Nash equilibrium problem, we extend their definitions to the QVI and the generalized Nash equilibrium problem. Hence, the PDS in (2.2) and MSP in (2.3) can be extended to the case of a moving set as follows

$$-\dot{x}(t) \in F(x(t)) + N_{K(x(t))}(x(t)), \tag{3.6}$$
$$\dot{x}(t) = \Pi_{K(x(t))}(x(t), -F(x(t))). \tag{3.7}$$

The first motivation to analyze these nonsmooth dynamics is the strong connection between the QVI and the steady states of both previous dynamical systems. Thus, in the case where $K(x) = \prod_{\nu=1}^N K_\nu(x^{-\nu})$ and $F(x) = (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^N$, the following results show the strong connection between these systems and the GNEP.

Theorem 3.1. *Let Assumption 3.1 hold. Assume that there exists an absolutely continuous path $x(t)$ satisfying (3.6) for almost every $t \in [0,T]$ and $x(0) = x_0$. Then the steady state of (3.6) with $K(x) = \prod_{\nu=1}^N K_\nu(x^{-\nu})$ and $F(x) = (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^N$ are equivalent to the generalized Nash equilibria.*

Proof. Due to [9], under Assumption 3.1, the solutions of (3.1) are equivalently solutions of (3.5) with $K(x) = \prod_{\nu=1}^N K_\nu(x^{-\nu})$ and $F(x) = (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^N$. By definition of the normal cone, the QVI (3.5) is equivalent to the following inclusion

$$-F(x) \in N_{K(x)}(x).$$

This inclusion corresponds to the steady state, $\dot{x}(t) = 0$, of (3.6). □

The specificity of the GNEP is the product set structure, which allows a sort of decomposition of the problem. By [54, Proposition 6.41] the normal cone of the product set is equal to the product of the normal cones. Hence, the differential inclusion (3.6) is the concatenation of N dynamical systems

$$-\dot{x}^{\nu}(t) - \nabla_{x^{\nu}} \theta_{\nu}(x(t)) \in N_{K_{\nu}(x^{-\nu}(t))}(x^{\nu}(t)), \tag{3.8}$$

and similarly

$$\dot{x}^{\nu}(t) = \Pi_{K_{\nu}(x^{-\nu}(t))}(x^{\nu}(t), -\nabla_{x^{\nu}} \theta_{\nu}(x(t))). \tag{3.9}$$

We can show that the slow solutions of MSP are equivalent to the solutions of the PDS. A direct consequence is that Theorem 3.1 extends as well to (3.7).

Theorem 3.2. *Let $x : [0,T] \to X$ be an absolutely continuous function such that $x(0) = x_0$. Then, x is a solution of* (3.7) *if and only if x is a slow solution of* (3.6)*.*

Proof. By the comment above it is sufficient to compare (3.8) and (3.9). Consider a selection of (3.8), that is,

$$\mathscr{F}(x) \in -\nabla_{x^{\nu}} \theta_{\nu}(x) - N_{K_{\nu}(x^{-\nu})}(x^{\nu}).$$

The slow solutions are

$$\mathrm{proj}_{\mathscr{F}(x)}(0) = \arg\min\{\|w^{\nu}\| \text{ s.t. } w^{\nu} \in -\nabla_{x^{\nu}} \theta_{\nu}(x) - N_{K_{\nu}(x^{-\nu})}(x^{\nu})\}.$$

Stated otherwise

$$\mathrm{proj}_{\mathscr{F}(x)}(0) = -\nabla_{x^{\nu}} \theta_{\nu}(x) - \mathrm{proj}_{N_{K_{\nu}(x^{-\nu})}(x^{\nu})}(-\nabla_{x^{\nu}} \theta_{\nu}(x)).$$

Finally, using Moreau's decomposition theorem [1, Theorem 1.1.4] yields

$$\mathrm{proj}_{\mathscr{F}(x)}(0) = \mathrm{proj}_{T_{K_{\nu}(x^{-\nu})}(x^{\nu})}(-\nabla_{x^{\nu}} \theta_{\nu}(x)).$$

□

The rest of the paper focus on the study of (3.6) and (3.7) in relation with the GNEP.

4. Regarding the Moving Set

One of the main difficulties in extending the theory of PDS and MSP to (3.6) and (3.7) is the moving set $K(x(t))$. In this section, we give an example illustrating the new difficulties, and show two results connecting the GNEP to a classical assumption from the literature.

A new difficulty in dealing with functions $x : [0,T] \to X$ solutions of (3.6) or (3.7) is that, for any t, x must satisfy

$$x(t) \in K(x(t)).$$

An implicit assumption here is that $x_0 \in K(x_0)$. In particular, one necessary condition for the existence of a solution of the GNEP is that the set-valued map K possesses a fixed point. Assuming that there exists a non-empty strongly compact convex set S, $K : S \mapsto S$ is upper hemicontinuous, and for all $x \in S$ $K(x)$ is a non-empty convex set, then Kakutani-Fan-Glicksberg theorem guarantees the existence of a fixed point. Note that the assumption that the codomain of K is a subset of S is essential, as discussed in [6], where the authors define the so-called projected solutions when this assumption is not satisfied.

We know study an example giving more insights of the difficulties induced by the moving set.

4.1. **An illustrative example.** The following example is a game adaptation of [36, Example 3.1] and illustrates some of the difficulties one can encounter when studying the existence of (3.6).

Example 4.1. Let $L > 1$ be a constant. Let us consider the following 3-player GNEP

$$\begin{array}{ll}\min\limits_{x} & -x \\ \text{s.t.} & \begin{cases} x \in \{[Lz,1] \text{ if } Lz \leq 1, [1,Lz] \text{ otherwise }\}, \\ 2 \geq x \geq -2, \end{cases}\end{array} \qquad \begin{array}{ll}\min\limits_{y} & \frac{1}{2}(y+x)^2 \\ \text{s.t.} & \begin{cases} 2 \geq y \geq 0, \end{cases}\end{array} \qquad \begin{array}{ll}\min\limits_{z} & 0 \\ \text{s.t.} & \begin{cases} z = x. \end{cases}\end{array}$$

The solutions of the game must satisfy $z = x$ by the third player's problem. Hence, an equilibrium must satisfy either $Lx \leq x \leq 1$ or $1 \leq x \leq Lx$. Since $L > 1$, it holds that $x \in]-2,0] \cup [1,2]$ and therefore the first player has one solution satisfying $z = x$ that is $x = 2$. Therefore, the unique generalized Nash equilibrium is $\{2,0,2\}$.

The MSP, (3.6), associated to this game is

$$-\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} \in \begin{pmatrix} -1 \\ y(t)+x(t) \\ 0 \end{pmatrix} + N_{K(x,y,z)}(x,y,z).$$

Consider $x_0 = (0,0,0)^T$. Direct computation shows that $K(0,0,0) = [0,1] \times [0,2] \times \{0\}$. Hence

$$N_{K(0,0,0)}(0,0,0) =]-\infty,0] \times]-\infty,0] \times \mathbb{R}.$$

The differential inclusion starting at x_0 gives $-\dot{x}(t) \in]-\infty,0]-1 =]-\infty,-1]$, so $\dot{x}(0) > 0$. Hence there exists a δ such that $x(t) \geq 0$ for $t \in [0,\delta]$, which however contradicts $(x(t),y(t),z(t)) \in K(x(t),y(t),z(t))$ as we previously noted that K is empty for $x \in]0,1[$. Therefore, there is no solution to this differential inclusion.

This example does not violate the assumptions of Kakutani-Fan-Glicksberg theorem, since for the compact set $S = [0,-2] \times [2,0] \times [0,-2]$, the application K has non-empty convex values, and the graph of K on S is closed. Following [36], one assumption to avoid this difficulty is

$$\text{haus}(K(u),K(v)) \leq L\|u-v\| \text{ with } L < 1. \tag{4.1}$$

Hence, this assumption is violated in Example 4.1 as L is the haus-Lipschitz modulus.

4.2. **Properties of the moving distance.** Without loss of generality, we can consider an additional time-dependence in the moving set, so that the solution of (3.6) or (3.7) satisfies

$$x(t) \in K(x(t),t), \ \forall t.$$

The time dependence can also be interpreted as a regularization of the set and might ease the computation in some cases. Following [36], a classical assumption from the MSP literature that we will use later on is the following. Assume K to be Lipschitz continuous with modulus $0 < L_1 < 1$ and $L_2 \geq 0$, i.e. for all $t,s \in [0,T]$ and $x,u,v \in X$, we have

$$|\text{dist}_{K(u,t)}(x) - \text{dist}_{K(v,s)}(x)| \leq L_1\|u-v\| + L_2|t-s|. \tag{4.2}$$

This type of Lipschitz-property can also be found in the literature in terms of the Hausdorff distance. Hence, a different version of (4.2) becomes

$$\text{haus}(K(u,t),K(v,s)) \leq L_1\|u-v\| + L_2|t-s|. \tag{4.3}$$

In the evolutionary case where $K(x,t) = K(t)$, this assumption is not restrictive. Assuming the set is described by inequalities [2, Proposition 5.1] gives a practical sufficient condition. The rest of this section gives two conditions under which previous conditions are satisfied for: (i) the case of a translated set, and, (ii) the case of a set described by inequalities.

First, we study the special case where the moving set is defined by a translation, that is

$$K(x,t) = c(x,t) + C_0, \tag{4.4}$$

where $c : X \times [0,T] \to X$. This set was, for instance, studied in the context of the QVI in [20, 48] or more recently in [49]. The following result gives a sufficient condition to satisfy (4.2).

Proposition 4.1. *For all ν, let $C_{0,\nu} \subset X_\nu$. Assume that there exists two constants $\alpha \in \mathbb{R}_+, \beta \in [0,1[$ such that $c : X \times [0,T] \to X$ satisfies:*

$$\|c(u,t) - c(v,s)\| \leq \alpha|s-t| + \beta\|u-v\| \ \forall u,v \in X, \forall s,t \in [0,T], \tag{4.5}$$

with $c(x,t) = (c_\nu(x^{-\nu},t))_{\nu=1}^N$. Then, $K(x,t) = \prod_{\nu=1}^N K_\nu(x^{-\nu},t)$ defined for all ν as

$$K_\nu(x^{-\nu},t) = c_\nu(x^{-\nu},t) + C_{0,\nu} \tag{4.6}$$

satisfies (4.2).

Proof. Since K is defined as a translation set, the following classical property holds

$$\mathrm{proj}_{c(x,t)+C_0}(v) = c(x,t) + \mathrm{proj}_{C_0}(v - c(x,t)). \tag{4.7}$$

Therefore, by (4.5), it follows that (4.2) holds true [48, Lemma 1]. □

We now consider the case where the constraint set is described by inequalities as in (3.2). As noted above, the Lipschitz property for the time dependence is covered in [2, Proposition 5.1], so we focus on the dependency in the state.

Proposition 4.2. *Assumption 3.1 holds. Denote r_ν the radius of a ball containing S_ν. Moreover, assume that*

(1) For all ν, K_ν is described as in (3.2), where $g_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n-n_\nu} \to \mathbb{R}^{m_\nu}$ is Lipschitz continuous (with modulus $L_{g,\nu}$) convex w.r.t. the first variable and Lipschitz continuous (with modulus $L_{g,-\nu}$) w.r.t. the second variable.

(2) There exists $x_0 := (x_0^\nu)_{\nu=1}^N$ and $\delta > 0$, such that for all ν, and for all $x^{-\nu} \in S_{-\nu}$, K_ν satisfies the Slater constraint qualification, i.e. $g_\nu(x_0^\nu, x^{-\nu}) \leq -\delta$.

(3) Assume that $\min_\nu L_{g,\nu}\delta \leq \max_\nu r_\nu + \|x_0^\nu\|$.

Then, there exists a constant $L = L_{g,-\nu} \max_\nu \frac{r_\nu + \|x_0^\nu\|}{\delta}$ such that for all $u,v \in S_{-\nu}$:

$$|\mathrm{dist}_{K(u)}(x) - \mathrm{dist}_{K(v)}(x)| \leq L\|u-v\| \tag{4.8}$$

Moreover, if $L < 1$, then (4.2) *is satisfied.*

Before moving to the proof, let us recall the following result due to Robinson [53].

Theorem 4.1. *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \infty$ be a closed proper convex function and let $S := g^{-1}(-\infty, 0]$. Assume that S satisfies Slater CQ ($\exists x^0 \in int(S)$ that is $g(x_0) < 0$) and S is bounded (i.e. $\exists r > 0$ such that $S \subset \mathbb{B}(r,0)$), then*

$$\mathrm{dist}(x,S) \leq \gamma g(x)_+ \textit{ for all } x \in \mathbb{R}^n, \tag{4.9}$$

for any $\gamma \geq \frac{r + \|x_0\|}{-g(x_0)}$.

We can now proceed to the proof of Proposition 4.2.

Proof. Denote by $\gamma := \max_\nu \frac{r_\nu + \|x_0^\nu\|}{\delta}$ and by $\alpha := \min_\nu 1/L_{g,\nu}$. Fix $\nu \in \{1,\dots,N\}$ and let $x^\nu \in S_\nu$ and $y^{-\nu}, z^{-\nu} \in S_{-\nu}$. Using Theorem 4.1 and the Lipschitz continuity of g, it holds true that

$$\alpha\|g_\nu(x^\nu, y^{-\nu})_+\| \le \mathrm{dist}(x^\nu, K_\nu(y^{-\nu})) \le \gamma\|g_\nu(x^\nu, y^{-\nu})_+\|.$$

Hence, it follows

$$\alpha\|g_\nu(x^\nu, z^{-\nu})_+\| \le \mathrm{dist}(x^\nu, K_\nu(z^{-\nu})) \le \gamma\|g_\nu(x^\nu, z^{-\nu})_+\|.$$

Combining the two inequalities yields

$$\begin{aligned}\alpha\|g_\nu(x^\nu, y^{-\nu})_+\| - \gamma\|g_\nu(x^\nu, z^{-\nu})_+\| &\le \mathrm{dist}(x^\nu, K_\nu(y^{-\nu})) - \mathrm{dist}(x^\nu, K_\nu(z^{-\nu}))\\ &\le \gamma\|g_\nu(x^\nu, y^{-\nu})_+\| - \alpha\|g_\nu(x^\nu, z^{-\nu})_+\|.\end{aligned}$$

Using $\gamma - \alpha \le 0$, we get

$$\begin{aligned}\gamma\|g_\nu(x^\nu, y^{-\nu})_+\| - \alpha\|g_\nu(x^\nu, z^{-\nu})_+\| &= \gamma(\|g_\nu(x^\nu, y^{-\nu})_+\| - \|g_\nu(x^\nu, z^{-\nu})_+\|)\\ &\quad + \|g_\nu(x^\nu, z^{-\nu})_+\|(\gamma - \alpha),\\ &\le \gamma L_{g,-\nu}\|y^{-\nu} - z^{-\nu}\|.\end{aligned}$$

Proceeding in the same way, we get

$$\begin{aligned}-\gamma L_{g,-\nu}\|y^{-\nu} - z^{-\nu}\| &\le \mathrm{dist}(x^\nu, K_\nu(y^{-\nu})) - \mathrm{dist}(x^\nu, K_\nu(z^{-\nu}))\\ &\le \gamma L_{g,-\nu}\|y^{-\nu} - z^{-\nu}\|.\end{aligned}$$

□

5. Existence of a path

In this section, we show that there exists a solution to the perturbed state-dependent sweeping process (3.6) connected with the GNEP. Our first existence result relies on recent progress on nonsmooth dynamical system [29, 43, 58].

Theorem 5.1. *Assumption 3.1 holds. Moreover, assume that*

$\mathscr{H}_1$ *K_ν is Lipschitz continuous with modulus $0 < L_1 < 1$ and $L_2 \ge 0$, i.e. for all $t, s \in [0, T]$ and $x, u, v \in S$ we have*

$$|\mathrm{dist}_{K(u,t)}(x) - \mathrm{dist}_{K(v,s)}(x)| \le L_1\|u - v\| + L_2|t - s|;$$

$\mathscr{H}_2$ *Assume also that $F(x) := (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^N$ is continuous and has linear growth, that is, there exists $L > 0$ such that $F(x) \le L(1 + \|x\|)$ for all $x \in S$.*

Then, for any $x_0 \in K(x_0, 0)$, there exists at least one Lipschitz solution of the differential inclusion

$$\begin{cases} -\dot{x}(t) \in N_{K(x(t),t)}(x(t)) + F(x(t)) \text{ a.e. on } [0,T],\\ x(t) \in K(x(t), t) \text{ for all } t \in [0,T],\\ x(0) = x_0 \in K(x_0, 0).\end{cases} \tag{5.1}$$

Proof. The proof is a direct consequence of [29, Theorem 3.1] for the case where K has convex values and the perturbation mapping is autonomous and single-valued. □

We illustrated in the previous section, in Example 4.1, that the Lipschitz assumption cannot be reduced to any L_1. However, to the best of our knowledge, it is an open question whether this result can be extended to $L_1 = 1$ or the condition $L_1 < 1$ can be replaced by a constraint qualification. The extension of this result to the case where $\mathscr{H}_1$ is replaced by (4.3) has been studied in [43]. We illustrated in Proposition 4.1 and Proposition 4.2 two sufficient conditions ensuring that $\mathscr{H}_1$ is satisfied.

Remark 5.1. We pointed out earlier than solutions of the privileged VI (3.4) are also solutions of the GNEP with shared constraints. Hence, one may wonder if solutions of the PDS associated to this VI may also be solutions of (3.6) or (3.7).

- The following example, Example 5.1, shows that the PDS associated with the variational equilibrium is not necessarily a solution of (3.7).
- On the other hand, Theorem 5.2 shows that the PDS associated with the variational equilibrium is a solution of (3.6).

Example 5.1. Let us consider the following shared constraint game:

$$\begin{array}{lll} \min\limits_{x\in\mathbb{R}} & -x & \\ \text{s.t.} & \left\{x \geq 0, x+y = 15,\right. \end{array} \qquad \begin{array}{lll} \min\limits_{y\in\mathbb{R}} & \frac{1}{2}y & \\ \text{s.t.} & \left\{y \geq 0, x+y = 15.\right. \end{array}$$

In this case, the map K is given by

$$K(\bar{x},\bar{y}) = \{x \,:\, 0 \leq x = 15-\bar{y}\} \times \{y \,:\, 0 \leq y = 15-\bar{x}\}.$$

We can see that any feasible point is a GNE. The unique variational equilibrium is $(0,15)$. Considering $x_0 = (15,0)^T$. Hence, x_0 is a GNE, and the solution of (3.7) is x_0 for all t. However, the solution of the PDS associated with the VI (3.4), with $K = \{x,y : x,y \geq 0, x+y = 15\}$, is

$$x(t) = \begin{cases} (15-\frac{3}{4}t, \frac{3}{4}t)^T \text{ for } t \in [0,20], \\ (0,0)^T \text{ for } t > 20. \end{cases}$$

Additionally, note that $x(t)$ is a solution of (3.6), but not the slow solution as for the GNSC, with (3.3), it holds that $N_K(x) \subseteq N_{K(x)}(x)$. This observation will be used in the following result.

Theorem 5.2. *Let Assumption 3.1 hold. Assume that $F(x) := (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^N$ is continuous. Moreover, assume that for all $t \in [0,T]$, and for all ν, there exists a closed convex set $K_\nu(t) \subseteq S_\nu$ such that the set of constraints of the ν-th player is given by*

$$K_\nu(x^{-\nu},t) = \{x^\nu : (x^\nu, x^{-\nu}) \in K_\nu(t)\}, \tag{5.2}$$

and there exists a constant $L_2 \geq 0$ such that for all $u \in S$

$$|\mathrm{dist}_{K(u,t)}(x) - \mathrm{dist}_{K(u,s)}(x)| \leq L_2|t-s|.$$

Furthermore, assume that, for all t, and for all $x \in S$, K_ν satisfies

$$N_{\cap_\nu K_\nu(t)}(x(t)) \subseteq N_{\prod_\nu K_\nu(x(t),t)}(x(t)). \tag{5.3}$$

Then, for any $x_0 \in K(x_0,0)$, there exists at least one Lipschitz solution of (5.1).

Proof. Consider the following MSP:

$$-\dot{x}(t) \in N_{\cap_\nu K_\nu(t)}(x(t)) + F(x(t)) \text{ a.e. on } [0,T]. \tag{5.4}$$

Using that $\cap_\nu K_\nu(t) \subseteq S$ is a closed strongly compact convex set, and the continuity of F, there exists a Lipschitz continuous function solution of (5.4) by [19]. Furthermore, since $K(x) := \prod_\nu K_\nu(x^{-\nu}) \subseteq \cap_\nu K_\nu$, and, by assumption, it holds that $N_K(x(t)) \subseteq N_{K(x)}(x(t))$, the solution of (5.4) is also a solution of (5.1). □

Assumptions (5.2) and (5.3) may seem restrictive although they are clearly satisfied by the GNSC (3.3). The former was recently used for decomposition methods for the GNEP in [39].

Note that the technique used in the previous theorem cannot be extended to PDS, as shown in Example 5.1. The assumption that K_ν is a convex set can be relaxed to a non-convex setting, in particular, a prox-regular set, as discussed in Section 7. Another way of handling a nonconvex set is by way of implicit projected dynamics as in [20].

6. Two classes of algorithms to compute the path

The existence theorem presented in the previous section is that its proof (and very often alternative proofs) are constructive and suggest an algorithm to build the path. Then, by tracking the steady states of the path, we can find a generalized Nash equilibrium. In this section, we will present two classes of algorithms: the Moreau-Yosida regularization and the catching-up algorithm.

Recall that in our context of the GNEP, the mapping F is the pseudo-gradient given by $(\nabla_{x^\nu}\theta_\nu)_{\nu=1}^N$ and $K(x) := \prod_\nu K_\nu(x^{-\nu})$.

6.1. Moreau-Yosida regularizations.

In [58], the author presented an alternative proof relying on the following regularization of the MSP

$$-\dot{x}_\lambda(t) - F(x_\lambda(t)) \in \frac{1}{2\lambda}\partial \mathrm{dist}^2_{K(x_\lambda(t),t)}(x_\lambda(t)).$$

The system (3.6) can then be recovered as $\lambda \to 0^+$. An alternative regularization used in [33] is the following

$$-\dot{x}_\lambda(t) \in \frac{1}{2\lambda}\partial \mathrm{dist}^2_{K(x_\lambda(t),t)}(x_\lambda(t) + \lambda F(x_\lambda(t))).$$

Recall that whenever the projection is well-defined and K convex it holds, see [58, Prop. 4.6], that

$$\partial \mathrm{dist}^2_{K(x,t)}(x) = 2(x - \mathrm{proj}_{K(x,t)}(x)).$$

Therefore, both previous inclusions are differential equations in our context.

The benefit of this approach, compared to the catching-up, is the possibility to use existing ode solver, for instance, with adaptive time-stepping. However, evaluating the implicit projection remains non-trivial, and therefore we focus on a second approach.

6.2. Catching-up algorithms.

Building on the constructive proof of the existence in [29, 36, 43, 50], we consider the following algorithm.

Catching-up algorithm For all ν, let $\alpha_\nu \in [0,1]$. For some integer n, $\mu_n = T/2^n$, we choose by induction

- $\bar{x}_0 = x_0 \in K(x_0, 0)$;
- $0 \leq i \leq 2^n - 1, 1 \leq \nu \leq N$:

- $\bar{x}_{i+1}^{\nu} = \text{proj}_{K(\alpha_\nu \bar{x}_i^{-\nu} + (1-\alpha_\nu)\bar{x}_{i+1}^{-\nu}, t_{i+1})}(\bar{x}_i^{\nu} - \mu_n \nabla_{x^\nu}\theta_\nu(\alpha_\nu \bar{x}_i^{-\nu} + (1-\alpha_\nu)\bar{x}_{i+1}^{-\nu}))$.

Fixing, for all ν, $\alpha_\nu = 1$ yields to the semi-implicit discretization scheme used in [29, Theorem 3.1]. Note that in this case, the N projections are independent from each others. Thus, considering a large number of players, one could exploit parallel computing to improve the performance. However, the drawback with this approach is that, if the contraction assumption (4.2) is not satisfied, the iterates will likely be infeasible, i.e. $\bar{x}_i \notin K(\bar{x}_i, t_i)$ for some i.

On the other side, fixing, for all ν, $\alpha_\nu = 0$ yields to the implicit discretization scheme used in [50]. In this case, the N projections are no longer independent from each other, and computing the projection implicitly might be a computationally difficult task.

A third approach consists in exploiting the known information for the ν-th player by computing the projection on the following set:

$$K_\nu(x_{i+1}^1, \dots, x_{i+1}^{\nu-1}, x_i^{\nu+1}, \dots, x_i^N, t_{i+1}).$$

This type of discretization scheme has not been used in the sweeping process literature, but is used in the so-called Gauss-Seidel decomposition method for the GNEP [25]. We also refer the reader to [27] for a more detailed discussion on this approach and the theoretical difficulties.

When using this approach beyond the contraction assumption (4.2), we might run into the obvious problem that the iterates might not get closer to a fixed point of K. As a heuristic, we consider the semi-implicit scheme ($\alpha_\nu = 1$) combined with a line search to ensures the feasibility of each iterate.

Catching-up line search algorithm For some integer n, $\mu_n = T/2^n$, we choose by induction

- $\bar{x}_0 = x_0 \in K(x_0, 0)$;
- $0 \le i \le 2^n - 1, 1 \le \nu \le N:\ \bar{x}_{i+1}^{\nu} = \text{proj}_{K(t_{i+1}^n, \bar{x}_i^{-\nu})}(\bar{x}_i^{\nu} - \mu_n \nabla_{x^\nu}\theta_\nu(\bar{x}_i^{-\nu}))$;
- Find the largest $\beta \in (0,1)$ such that $\bar{x}_i + \beta(\bar{x}_{i+1} - \bar{x}_i) \in K(t_i, \bar{x}_i + \beta(\bar{x}_{i+1} - \bar{x}_i))$.

One classical approximation to find β is to consider a backtraking technique that consists in taking the largest element in the sequence $1, \frac{1}{2}, \frac{1}{4}, \dots$ satisfying the desired condition (up to some precision).

6.3. **Applications to GNEP.** In this section, we present several illustrations of the *catching-up line search algorithm* on classical examples from the GNEP literature. We run a straightforward implementation of the algorithm in Julia. In all the examples, except for the first one, the contraction property is not satisfied. We choose $n = 9$, so the time step is approximatively $T \cdot 10^{-3}$, where T is the final time.

The most expensive step in this algorithm is clearly to compute the projection. In all these examples, the set is described by inequalities (3.2). Hence, we can use classical optimization solvers for nonlinear programming to find an approximate solution of the projection. Here, we used Ipopt [59].

Example 6.1. The following example is a 2-player GNEP built on an example from [30].

$$\begin{array}{lll} \min\limits_{x \in \mathbb{R}} & x^2 + 8/3xy - 34x \\ \text{s.t.} & \{x \ge 0, x + 0.9y \le 14.4, \end{array} \qquad \begin{array}{lll} \min\limits_{y \in \mathbb{R}} & y^2 + 5/4xy - 24.25y \\ \text{s.t.} & \{y \ge 0, 0.9x + y \le 14.1. \end{array}$$

This game satisfies the contraction property with $L_1 = 0.9$, thus we can apply the catching-up algorithm without line search, i.e. the algorithm always choose $\beta = 1$. Starting at the origin the solution path is presented in Figure 1 and attains the stable state $(5,9)^T$, which is an equilibrium of the game.

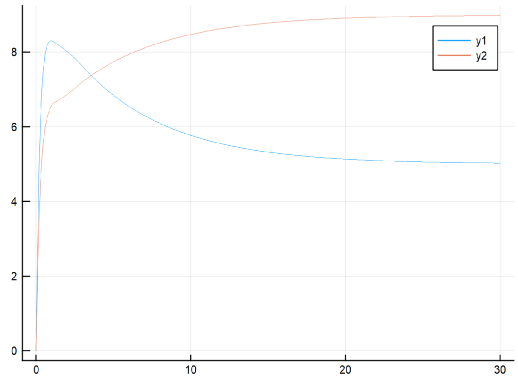

FIGURE 1. Catching-up on Example 6.1 with $x_0 = (0,0)^T$.

Example 6.2 (River basin pollution game)**.** Consider the 3-player river basin pollution game with two shared constraints studied in [35].

$$\begin{aligned}
&\min_{x_1\in\mathbb{R}} \quad (\alpha_1 x_1 + \beta(x_1+x_2+x_3) - \xi_1)x_1\\
&\text{s.t.} \quad \begin{cases} x_1 \geq 0, 3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100,\\ 2.29115x_1 + 1.5625x_2 + 2.8125x_3 \leq 100,\end{cases}\\
&\min_{x_2\in\mathbb{R}} \quad (\alpha_2 x_2 + \beta(x_1+x_2+x_3) - \xi_2)x_2\\
&\text{s.t.} \quad \begin{cases} x_2 \geq 0, 3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100,\\ 2.29115x_1 + 1.5625x_2 + 2.8125x_3 \leq 100,\end{cases}\\
&\min_{x_3\in\mathbb{R}} \quad (\alpha_3 x_3 + \beta(x_1+x_2+x_3) - \xi_3)x_3\\
&\text{s.t.} \quad \begin{cases} x_3 \geq 0, 3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100,\\ 2.29115x_1 + 1.5625x_2 + 2.8125x_3 \leq 100,\end{cases}
\end{aligned}$$

where

$$\begin{aligned}
&\alpha_1 = 0.01, \alpha_2 = 0.05, \alpha_3 = 0.01, \beta = 0.01,\\
&\xi_1 = 2.9, \xi_2 = 2.88, \xi_3 = 2.85.
\end{aligned}$$

The unique variational equilibrium is $x = (\frac{4673}{221}, \frac{5754}{359}, \frac{567}{208})^T \approx (21.14, 16.03, 2.73)^T$ (only the first constraint is active at this point). Starting at the origin the solution path is presented in Figure 2 and attains the stable state $(12.0004, 9.92694, 11.7794)^T$, which is an equilibrium of the game up to $\varepsilon = 10^{-3}$ where the first constraint is active.

Example 6.3. We now consider three variants of a game of environmental accords between $N = 5$ countries introduced in [12] and further extended in [40, Section 7.4]. Each player/country controls a strategy vector $x^\nu =\in \mathbb{R}^6$ whose first component is the greenhouse gas emissions of the ν-th country and the other components are investments in other countries. Overall, there are 30 variables.

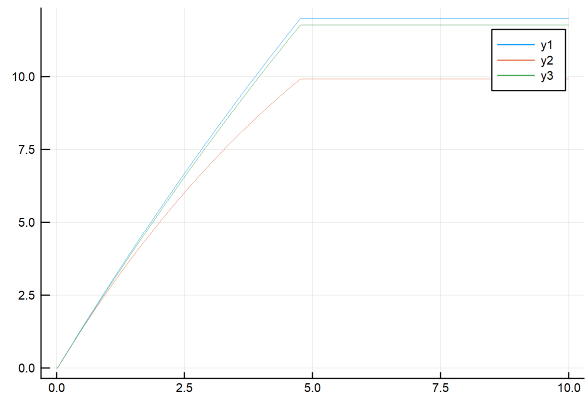

FIGURE 2. Catching-up with linesearch on Example 6.2 with $x_0 = (0,0,0)^T$.

Following [40], we study each country separately, which leads to the following point that we use as an initial guess:

$$x_0 = \begin{pmatrix} 97.5 & 2.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 97.95 & 0.0 & 2.45 & 0.0 & 0.0 & 0.0 \\ 99.0 & 0.0 & 0.0 & 2.0 & 0.0 & 0.0 \\ 99.5 & 0.0 & 0.0 & 0.0 & 1.5 & 0.0 \\ 99.60 & 0.0 & 0.0 & 0.0 & 0.0 & 1.35 \end{pmatrix}.$$

- The first variant is a shared constraint game [40, Eq. 18]. Starting from x_0 the *catching-up with linesearch* computed a path whose steady state is:

$$x_0 = \begin{pmatrix} 99.40 & 0.59 & 0.01 & 0.11 & 0.44 & 0.65 \\ 99.41 & 3.06e-5 & 0.69 & 0.10 & 0.41 & 0.62 \\ 99.46 & 2.91e-6 & 2.83e-6 & 1.06 & 0.26 & 0.45 \\ 99.55 & 1.01e-6 & 8.22e-7 & 7.65e-7 & 1.33 & 0.16 \\ 99.60 & 7.77e-7 & 6.32e-7 & 5.08e-7 & 2.16e-6 & 1.35 \end{pmatrix}.$$

  Note that this is the same solution given in [40, Appendix B], and it is an equilibrium of the game.
- The second variant is a game where players 3 and 4 share some additional restrictions [40, Eq. 20]. Starting from x_0, the algorithm computes a path whose stable point is:

$$x_0 = \begin{pmatrix} 99.38 & 0.61 & 0.01 & 2.98e-7 & 0.41 & 0.74 \\ 99.39 & 2.31e-5 & 0.72 & 9.24e-8 & 0.37 & 0.70 \\ 99.24 & 1.72e-7 & 1.82e-7 & 1.66 & 6.95e-6 & 0.26 \\ 99.52 & 1.92e-7 & 1.75e-7 & 4.59e-8 & 1.43 & 0.06 \\ 99.60 & 5.95e-7 & 5.12e-7 & 1.37e-7 & 5.31e-7 & 1.35 \end{pmatrix}.$$

  This point is an equilibrium of the game up to $\varepsilon = 10^{-3}$.
- Finally, the third variant is a GNEP, where the first player invests only in countries that are significantly reducing their emissions [40, Eq. 21]. Starting from x_0, the algorithm

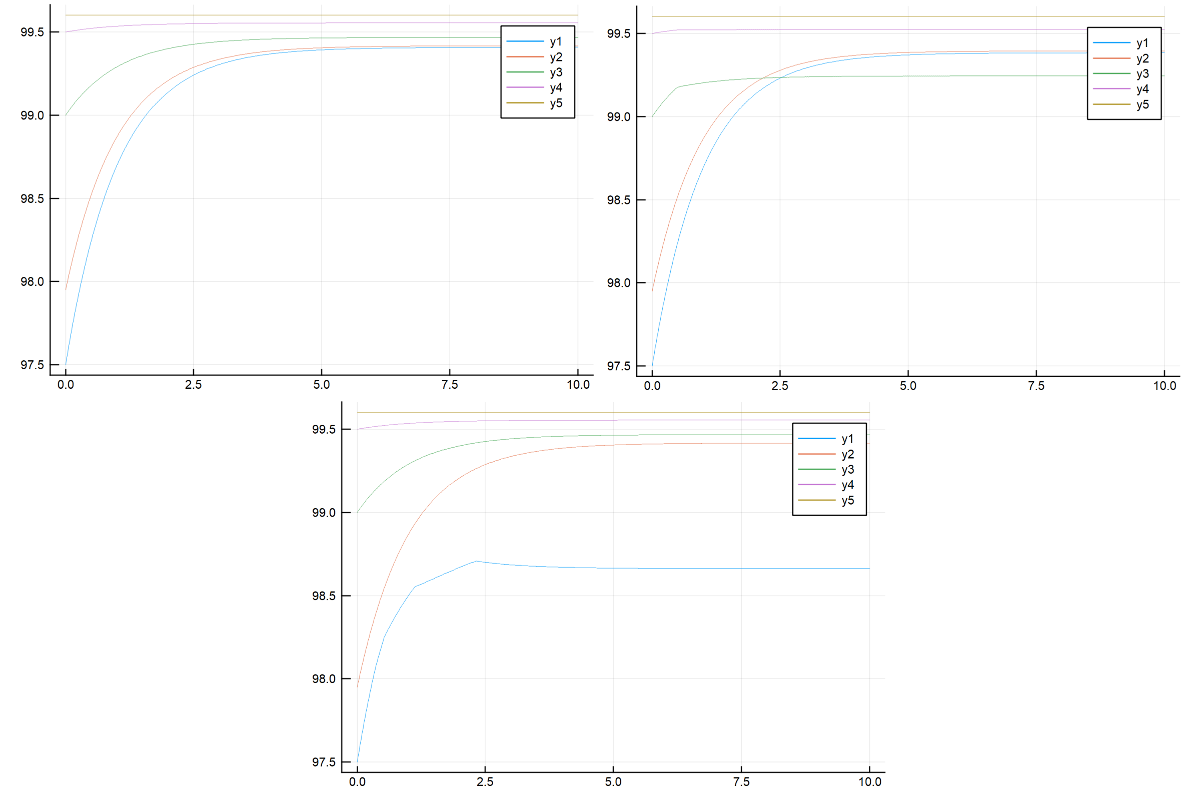

FIGURE 3. Gas emission of each of the five countries in Example 6.3. Top left: the 1st variant. Top right: 2nd variant. Bottom: 3rd variant.

computes a path whose stable point is:

$$x_0 = \begin{pmatrix} 98.66 & 1.33 & 0.27 & 0.25 & 0.23 & 0.22 \\ 99.41 & 1.80e-7 & 0.69 & 0.10 & 0.41 & 0.62 \\ 99.46 & 1.49e-7 & 2.74e-6 & 1.06 & 0.26 & 0.45 \\ 99.55 & 1.33e-7 & 8.08e-7 & 7.51e-7 & 1.33 & 0.16 \\ 99.60 & 1.26e-7 & 6.14e-7 & 4.94e-7 & 1.89e-6 & 1.35 \end{pmatrix}.$$

Once again, this point is an equilibrium of the game up to $\varepsilon = 10^{-3}$.

Overall, these experiments validate our approach as in every cases we manage to compute an equilibrium. We illustrate the paths computed in each variant in Figure 3 where only the first component of each player's strategy is plotted.

7. DISCUSSIONS

We introduced in this paper two nonsmooth dynamical systems connected to the GNEP. These systems are useful in analyzing and tracking the equilibria of the problem. We showed how recent results can be extended to prove the existence of a path, which very often helps in the design of numerical approaches. By mean of a modified catching-up algorithm, we illustrated in Section 6, practical applications of this study. Further improvements would be required to refine the numerical approaches and weaken the assumptions used. One advantage of the approach presented here is the global convergence, i.e. convergence from any initial point, without any specific structure on the constraint set. This type of result exists in the GNEP literature only when the set is described by inequalities.

This work reviews and extends the motivation behind the recent interest in MSP and PDS dynamics with moving sets. Here we made use of the contraction assumption ($L_2 < 1$), which has become a standard assumptions in several recent studies for MSP with moving sets [29, 36, 43, 58] and also for QVI [48, 49]. However, for GNEP, this assumption is restrictive, and further research will focus on finding new techniques or assumptions to achieve similar results.

We can also mention that there are two straightforward extensions of this work. Indeed, the results proposed in Section 5 and 6 can also be obtained assuming the values of K are prox-regular sets [3], which is a generalization of convex sets. In this case, the QVI is no longer equivalent to the GNEP but is a necessary condition. Moreover, these results can also be extended to the case where θ is not C^1, using the convex subdifferential. Last but not least, given that a dynamic is in place to describe the potential time-evolution of player strategies in a generalized Nash game, one can further ask questions about dynamic stability of generalized Nash equilibria, and whether or not we can tackle these questions with classic dynamical systems definitions tailored to equilibrium sets, rather than isolated equilibrium points.

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References

[1] S. Adly, A Variational Approach to Nonsmooth Dynamics: Applications in Unilateral Mechanics and Electronics, Springer, 2017.

[2] S. Adly, T. Haddad, Well-posedness of nonconvex degenerate sweeping process via unconstrained evolution problems, Nonlinear Anal. 36 (2020), 100832.

[3] S. Adly, F. Nacry, L. Thibault, Preservation of prox-regularity of sets with applications to constrained optimization SIAM J. Optim. 26 (2016), 448-473.

[4] K.J. Arrow , G. Debreu, Existence of an equilibrium for a competitive economy, Econometrica: J. Econometric Soc. (1954), 265-290.

[5] J.P. Aubin, A. Cellina, Differential inclusions: set-valued maps and viability theory, vol. 264, Springer Science & Business Media, 2012.

[6] D. Aussel, A. Sultana, V. Vetrivel, On the existence of projected solutions of quasi-variational inequalities and generalized Nash equilibrium problems, J. Optim. Theory Appl. 170 (2016), 818-837.

[7] A. Barbagallo, M.G. Cojocaru, Dynamic equilibrium formulation of the oligopolistic market problem, Math. Comput. Modelling 49 (2009), 966-976.

[8] T. Başar, G.J. Olsder, Dynamic Noncooperative Game Theory, vol. 23, SIAM, 1999.

[9] A. Bensoussan, Points de Nash dans le cas de fonctionnelles quadratiques et jeux différentiels linéaires à N personnes, SIAM J. Control 12 (1974), 460-499.

[10] A. Bensoussan, J.L. Lions, Impulse control and quasivariational inequalities, Gauthier-Villars, Montrouge; Heyden & Son, Inc, 1984.

[11] E. Börgens, C. Kanzow, A distributed regularized Jacobi-type ADMM-method for generalized Nash equilibrium problems in Hilbert spaces, Numer. Funct. Anal. Optim. 39 (2018), 1316-1349.

[12] M. Breton, G. Zaccour, M. Zahaf, A game-theoretic formulation of joint implementation of environmental projects, Eur. J. Oper. Res. 168 (2006), 221-239.

[13] B. Brogliato, A. Daniilidis, C. Lemarechal, V. Acary, On the equivalence between complementarity systems, projected systems and differential inclusions Sys. Control Lett. 55 (2006), 45-51.

[14] E. Cavazzuti, M. Pappalardo, M. Passacantando, Nash equilibria, variational inequalities, and dynamical systems, J. Optim. Theory Appl. 114 (2002), 491-506.

[15] M.G. Cojocaru, Dynamic equilibria of group vaccination strategies in a heterogeneous population, J. Global Optim. 40 (2008), 51-63.

[16] M.G. Cojocaru, C.T. Bauch, M.D. Johnston, Dynamics of vaccination strategies via projected dynamical systems, Bull. Math. Biol. 69 (2007), 1453-1476.

[17] M.G. Cojocaru, P. Daniele, A. Nagurney, Projected dynamical systems and evolutionary variational inequalities via Hilbert spaces with applications, J. Optim. Theory Appl. 127 (2005), 549-563.

[18] M.G. Cojocaru, S. Greenhalgh, Dynamic vaccination games and hybrid dynamical systems, Optim. Eng. 13 (2012), 505-517.

[19] M.G. Cojocaru, L. Jonker, Existence of solutions to projected differential equations in Hilbert spaces, Proc. Amer. Math. Soc. 132 (2004), 183-193.

[20] M.G. Cojocaru, S. Pia, Nonpivot and implicit projected dynamical systems on Hilbert spaces, J. Funct. Spaces Appl. 2012 (2012), 508570.

[21] B. Cornet, Existence of slow solutions for a class of differential inclusions, J. Math. Anal. Appl. 96 (1983), 130-147.

[22] P. Daniele, Dynamic networks and evolutionary variational inequalities, Edward Elgar Publishing, 2006.

[23] P. Dupuis, H. Ishii, On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications, Stochastics and Stochastic Reports, 35 (1991), 31-62.

[24] P. Dupuis, A. Nagurney, Dynamical systems and variational inequalities, Ann. Oper. Res. 44 (1993), 7-42.

[25] F. Facchinei, C. Kanzow, Generalized Nash equilibrium problems, Ann. Oper. Res. 175 (2010), 177-211.

[26] F. Facchinei, J.S. Pang, Finite-dimensional variational inequalities and complementarity problems, Springer Science & Business Media, 2007.

[27] F. Facchinei, V. Piccialli, M. Sciandrone, Decomposition algorithms for generalized potential games, Comput. Optim. Appl. 50 (2011), 237-262.

[28] A. Fischer, M. Herrich, K. Schönefeld, Generalized Nash equilibrium problems-recent advances and challenges, Pesquisa Operacional, 34 (2014), 521-558.

[29] T. Haddad, Nonconvex differential variational inequality and state-dependent sweeping process, J. Optim. Theory Appl. 159 (2013), 386-398.

[30] P.T. Harker, Generalized Nash games and quasi-variational inequalities, Eur. J. Oper. Res. 54 (1991), 81-94.

[31] C. Henry, An existence theorem for a class of differential equations with multivalued right-hand side, J. Math. Anal. Appl. 41 (1973), 179-186.

[32] T. Ichiishi, Game Theory for Economic Analysis, Elsevier, 2014.

[33] A. Jourani, E. Vilches, A differential equation approach to implicit sweeping processes, J. Differential Equations 266 (2019), 5168-5184.

[34] C. Kanzow, D. Steck, Quasi-variational inequalities in Banach spaces: theory and augmented Lagrangian methods, SIAM J. Optim. 29 (2019), 3174-3200.

[35] J.B. Krawczyk, S. Uryasev, Relaxation algorithms to find Nash equilibria with economic applications, Environmental Modeling & Assessment, 5 (2000), 63-73.

[36] M. Kunze, M. Monteiro-Marques, On parabolic quasi-variational inequalities and state-dependent sweeping processes, Topological Methods Nonlinear Anal. 12 (1998), 179-191.

[37] M. Kunze, M. Monteiro-Marques, An introduction to Moreau's sweeping process, In: Impacts in Mechanical Systems, p. 1-60, Springer, 2000.

[38] R. Lahkar, W.H. Sandholm, The projection dynamic and the geometry of population games, Games and Economic Behavior 64 (2008), 565-590.

[39] T. Migot, M.G. Cojocaru, A decomposition method for the convex generalized Nash equilibrium problem. submitted, 2019.

[40] T. Migot, M.G. Cojocaru, A parametrized variational inequality approach to track the solution set of a generalized Nash equilibrium problem, Eur. J. Oper. Res. 283 (2020), 1136-1147.

[41] J.J. Moreau, Numerical aspects of the sweeping process, Computer methods in applied mechanics and engineering, 177 (1999), 329-349.

[42] U. Mosco, Implicit variational problems and quasi variational inequalities, In: Nonlinear operators and the calculus of variations, pp. 83-156, Springer, 1976.
[43] F. Nacry, Truncated nonconvex state-dependent sweeping process: implicit and semi-implicit adapted Moreau's catching-up algorithms, J. Fixed Point Theory Appl. 20 (2018), 121.
[44] A. Nagurney, Network economics: A variational inequality approach, vol. 10, Springer Science & Business Media, 2013.
[45] A. Nagurney, P. Daniele, E.A. Flores, V. Caruso, A variational equilibrium network framework for humanitarian organizations in disaster relief: effective product delivery under competition for financial funds, pp. 109-133, Springer International Publishing, Cham, 2018.
[46] A. Nagurney, D. Zhang, Projected dynamical systems and variational inequalities with applications, Vol.2, Springer Science & Business Media, 2012.
[47] J.F. Nash, Equilibrium points in n-person games, Proc. Natl. Acad. Sci. 36 (1950), 48-49.
[48] Y. Nesterov, L. Scrimali, Solving strongly monotone variational and quasi-variational inequalities, Discrete & Continuous Dyn. Syst. A 31 (2011), 1383-1396.
[49] L.V. Nguyen, X. Qin, Some results on strongly pseudomonotone quasi-variational inequalities, Set-Valued Var. Anal. 2019, doi: 10.1007/s11228-019-00508-1.
[50] J. Noel, L. Thibault, Nonconvex sweeping process with a moving set depending on the state, Vietnam J. Math. 42 (2014), 595-612.
[51] J.S. Pang, M. Fukushima, Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games, Comput. Management Sci. 2 (2005), 21-56.
[52] J.S. Pang, D.E. Stewart, Differential variational inequalities, Math. Program. 113 (2008), 345-424.
[53] S.M. Robinson, Stability theory for systems of inequalities, part ii: Differentiable nonlinear systems, SIAM J. Numer. Anal. 13 (1976), 497-513.
[54] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, vol.317, Springer Science & Business Media, 2009.
[55] J.B. Rosen, Existence and uniqueness of equilibrium points for concave n-person games, Econometrica, 33 (1965), 520-534.
[56] W.H. Sandholm, Population games and evolutionary dynamics, MIT press, Massachusetts, 2010.
[57] P.D. Taylor, L.B. Jonker, Evolutionary stable strategies and game dynamics, Math. Biosci. 40 (1978), 145-156.
[58] E. Vilches, Regularization of perturbed state-dependent sweeping processes with nonregular sets, J. Nonlinear Convex Anal. 19 (2018), 633-651.
[59] A. Wächter, L.T. Biegler, On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming, Math. Program. 106 (2006), 25-57.
[60] E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory: Part i. projections on convex sets: Part ii. spectral theory. In: Contributions to nonlinear functional analysis, pp. 237–424, Elsevier, 1971.
[61] D. Zhang, A. Nagurney, Stability analysis of an adjustment process for oligopolistic market equilibrium model as a projected dynamical system, Optimization 36 (1996), 263-285.