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EVENLY CONVEX SETS, AND EVENLY QUASICONVEX FUNCTIONS, REVISITED

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Abstract. Since its appearance, even convexity has become a remarkable notion in convex analysis. In the fifties, W. Fenchel introduced the evenly convex sets as those sets solving linear systems containing strict inequalities. Later on, in the eighties, evenly quasiconvex functions were introduced as those whose sublevel sets are evenly convex. The significance of even convexity relies on the different areas where it enjoys applications, ranging from convex optimization to microeconomics. In this paper, we review some of the main properties of evenly convex sets and evenly quasiconvex functions, provide further characterizations of evenly convex sets, and present some new results for evenly quasiconvex functions.

Keywords. Evenly convex set; Separation; Sublevel set; Evenly quasiconvex function.

1. Introduction

To the authors' knowledge, this is the first paper that reviews one of the favorite research topics of Juan-Enrique Martínez-Legaz, to whom this special issue is dedicated in occasion of his 70th birthday: the sets which are intersections of open halfspaces and those functions whose sublevel sets belong to this family of sets, which are called *evenly convex* sets and *evenly quasiconvex* functions, respectively.

The concept of evenly convex set was coined by Werner Fenchel [5], one of the fathers of convex analysis, in his attempt to extend the double polar theorem to a class of sets larger than the one of closed convex sets. The concept of evenly quasiconvex function, in turn, was introduced in 1981 by Martínez-Legaz in his PhD thesis on generalized conjugation [16], under the name of *normal quasiconvex* function. It was also used in his 1983 paper [17], while the name of evenly quasiconvex function was introduced a year later by Passy and Prisman [24] in a paper on conjugacy in quasiconvex programming (which was followed by a sequel on the same subject [25]) written in an independent way. Since then, many papers have been published on both subjects.

On the one hand, concerning the evenly convex sets, Schröder used them in the seventies to obtain his linear range-domain implications ([32, 33]). In the eighties and the nineties, the evenly convex sets were largely applied in quasiconvex programming ([17, 18, 20, 24, 25, 26]). Linear systems containing strict inequalities and their solutions sets, in turn, naturally arise in

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convex optimization ([30, 37]), separation problems ([1, 14]), stability analysis ([9, Chapter 6]), and probability measures ([2]), among other fields of mathematics and computer sciences. New characterizations and properties of the evenly convex sets were given along the aughts in [4, 8, 10, 15]. On the other hand, regarding evenly quasiconvex functions, they play a relevant role in duality theory, since they are the regular functions in all usual quasiconvex conjugation schemes developed in the eighties (see, e.g., [17, 24, 18, 26, 25]). A stream of works published during the last three decades show that these functions also play a role in mathematical economy, as the indirect utility functions arising in consumer theory are characterized as the non-increasing evenly quasiconvex functions that satisfy an additional mild regularity condition (see, e.g., [19, 21, 27, 28]), decision theory and risk measures (see [6] and the references therein).

The convenience of a review on this topic comes from the following facts:

(1) As shown in the last two paragraphs, results on these families of sets and functions are spread in the literature since the publication of the seminal paper of Fenchel in 1952.
(2) Most papers published before the celebrated book of Rockafellar [29] (where the standard terminology and notation of convex analysis was established) are hardly readable for today's readers. This is particularly true regarding the mentioned papers of Fenchel [5], on evenly polars, and of Klee [14], on separation theorems for (mainly) evenly convex sets.
(3) The existence of some interesting characterizations of the evenly convex sets which were privately communicated by Martínez-Legaz to Reinhard in 1997 and to the second author of this paper in 2000 and still remain unpublished.
(4) The methodology employed in recent results on evenly convex functions (those whose epigraph is evenly convex, introduced in [31] and [22]) can be applied to derive old and new results concerning the geometry of evenly quasiconvex functions.

This paper collects in a systematic way results on evenly convex sets and evenly quasiconvex functions, gives references to the corresponding sources where they have been published, tries to provide a uniform terminology and notation, and finally provides (more or less detailed) proofs for the unpublished results or for those published results which were written in an anachronistic language. It is worth to say that, although the concept of evenly convex set was introduced in a finite-dimensional real space, it also makes sense in any separated locally convex space. As a matter of fact, some authors work in a finite-dimensional setting due to the nature of the tools they use (see, e.g., [8] and [10]) while in other cases even convexity is considered in Banach spaces ([4]) or in more general spaces (see, e.g., [22] and [36]). In order to avoid to lengthen the discussion on each result, we shall consider $\mathbb{R}^n$ as our framework, being aware that some results may apply also in a more general setting.

The layout of the paper is as follows. In Section 2, we present the class of evenly convex sets together with its main properties and characterizations. Section 3 introduces the evenly convex hull of a given set. Classic separation theorems involving evenly convex sets are given in Section 4. Section 5 summarizes well-known results on evenly quasiconvex functions. Finally, in Section 6 we present some results for evenly quasiconvex functions and the link with subdifferentiability.

2. Evenly convex sets

We start this section by introducing some of the main definitions used ahead. We shall employ the standard notation of convex analysis (see, e.g., [12, 29]). We consider $\mathbb{R}^n$ equipped with the $\langle\cdot,\cdot\rangle$ standard inner product and the Euclidean norm $\|x\| = \sqrt{\langle x,x\rangle}$. Given a set $X \subset \mathbb{R}^n$, we denote by $\operatorname{int} X$, $\operatorname{rint} X$, $\operatorname{cl} X$, $\operatorname{bd} X$, and $\operatorname{rbd} X$ the interior, the relative interior, the closure, the boundary, and the relative boundary of X, respectively. X is said to be *relatively open* if $\operatorname{rint} X = X$ (thus, $\emptyset$ and $\mathbb{R}^n$ are relatively open sets). Moreover, $\operatorname{conv} X$ stands for the convex hull of X, whereas $\operatorname{cone} X := \mathbb{R}_+ \operatorname{conv} X$ means the convex conical hull of $X \cup \{0_n\}$, where 0_n denotes the null vector of $\mathbb{R}^n$. When $\emptyset \neq X \subset \mathbb{R}^n$, we denote by $\operatorname{aff} X$ the affine span of X, and by

$$0^+X := \{d \in \mathbb{R}^n : x + td \in X, \forall t \geq 0, \forall x \in X\}$$

the *recession cone* of X. The *lineality space* of a convex cone K is $\operatorname{lin} K := K \cap (-K)$.

Additionally, if C is a nonempty convex subset of $\mathbb{R}^n$, $\dim C$ denotes the dimension of C (defined as the dimension of $\operatorname{aff} C$) and, for $\bar{x} \in C$, the *cone of feasible directions* of C at $\bar{x}$ is

$$D(C,\bar{x}) := \{v \in \mathbb{R}^n : \bar{x} + \alpha v \in C \text{ for some } \alpha > 0\} = \mathbb{R}_+(C - \bar{x}).$$

The halfline $\{x + \lambda y : \lambda \geq 0\}$ is a *tangent ray* for the convex set C if $x \in \operatorname{rbd} C$, $y \in \operatorname{cl} D(\operatorname{cl} C, x)$, and $\{x + \lambda y : \lambda \geq 0\} \cap \operatorname{rint} C = \emptyset$. If $\bar{x} \in \operatorname{cl} C$, the *tangent cone* to C at $\bar{x}$ is $T_C(\bar{x}) := \operatorname{cl} D(\operatorname{cl} C, \bar{x})$.

Definition 2.1. A set $C \subset \mathbb{R}^n$ is said to be *evenly convex* [5] if it is the intersection of some family, possibly empty, of open halfspaces.

Since any linear equation can be replaced by two linear inequalities and any closed halfspace $\{x \in \mathbb{R}^n : \langle a,x\rangle \leq b\}$, with $a \in \mathbb{R}^n \setminus \{0_n\}$ and $b \in \mathbb{R}$, can be written as the intersection of the open halfspaces $\left\{x \in \mathbb{R}^n : \langle a,x\rangle < b + \frac{1}{r}\right\}$, with $r \in \mathbb{N}$, a set is evenly convex if and only if it is the solution set of some (linear) *system* of the form

$$\sigma = \{\langle a_t, x\rangle \leq b_t, t \in W; \langle a_t, x\rangle < b_t, t \in S\}, \tag{2.1}$$

where W and S are disjoint index sets, $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$ for all $t \in T := W \cup S \neq \emptyset$. In particular, $\mathbb{R}^n$ and $\emptyset$ are evenly convex sets.

The system σ in (2.1) is said to be *ordinary* whenever $S = \emptyset$. The solution sets of ordinary systems are intersections of closed halfspaces and so, they are closed convex sets. The converse holds as a consequence of the well-known separation theorem of a closed convex set from any outside point (see, e.g., [12, Theorem 4.1.1]). Consequently, any closed convex set is evenly convex.

The first result in this paper gathers together five characterizations of evenly convex sets given in the literature.

Theorem 2.1 (Characterizations of evenly convex sets)**.** *Let $C \subset \mathbb{R}^n$ be such that $\emptyset \neq C \neq \mathbb{R}^n$. Then, the following statements are equivalent:*

- (i) *C is evenly convex.*
- (ii) *C is the result of eliminating from a closed convex set (precisely, $\operatorname{cl} C$) the union of a certain family of its exposed faces.*
- (iii) *C is a convex set and for each $x \in \mathbb{R}^n \setminus C$ there exists a hyperplane H such that $x \in H$ and $H \cap C = \emptyset$.*

(iv) *C is connected and through every point not in C there is some hyperplane H such that* $H \cap C = \emptyset$.
(v) *C is a convex set and* $x \in C$ *for all* $x \in \operatorname{rbd} C$ *such that* $\{x - \lambda y : \lambda \geq 0\} \cap C \neq \emptyset$ *for some tangent ray* $\{x + \lambda y : \lambda \geq 0\}$.
(vi) *C is a convex set such that* $(x + \operatorname{lin} T_C(x)) \cap C = \emptyset$, *for any* $x \in (\operatorname{cl} C) \setminus C$.

Sources. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ can be found in [8, Proposition 3.1], $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ in [5, 3.2 and 3.4], and $(i) \Leftrightarrow (vi)$ in [4, Theorem 5].

The following characterizations for evenly convex sets were conjectured by Martínez-Legaz in a private communication.

Theorem 2.2 (Further characterizations of evenly convex sets)**.** *Let* $C \subset \mathbb{R}^n$ *be such that* $\emptyset \neq C \neq \mathbb{R}^n$. *Then, the following statements are equivalent:*

(i) *C is evenly convex.*
(ii) *C is the intersection of a nonempty collection of nonempty open convex sets.*
(iii) *C is a convex set and is the intersection of a collection of complements of hyperplanes.*
(iv) *C is a convex set and for any convex set K contained in* $(\operatorname{cl} C) \setminus C$, *there exists a hyperplane containing K and not intersecting C.*
(v) *C is a convex set and for any convex set* $K \subset (\operatorname{cl} C) \setminus C$, *the minimal exposed face (in* $\operatorname{cl} C$*) containing K does not intersect C.*
(vi) *C is a convex set and for any* $x \in (\operatorname{cl} C) \setminus C$, *the minimal exposed face (in* $\operatorname{cl} C$*) containing x does not intersect C.*
(vii) *C is a convex set and for any* $x \in (\operatorname{cl} C) \setminus C$, *there exists a supporting hyperplane of* $\operatorname{cl} C$ *at x not intersecting C.*

Proof. $(i) \Leftrightarrow (ii)$ By definition, since any evenly convex set C such that $\emptyset \neq C \neq \mathbb{R}^n$ is the intersection of some nonempty family of open halfspaces, and any halfspace is convex, then C satisfies (ii). On the other hand, if C is the intersection of a nonempty collection of nonempty open convex sets, then C is an open convex set and satisfies condition (iii) in Theorem 2.1, meaning that C is evenly convex.

$(i) \Leftrightarrow (iii)$ If C is evenly convex, it satisfies condition (iii) in Theorem 2.1, and then, given $t \in T := \mathbb{R}^n \setminus C$, there exists a hyperplane H_t such that $t \in H_t$ and $H_t \cap C = \emptyset$. Therefore,

$$C \subset \bigcap_{t \in T} (\mathbb{R}^n \setminus H_t) \tag{2.2}$$

and

$$\mathbb{R}^n \setminus C = T \subset \bigcup_{t \in T} H_t. \tag{2.3}$$

By applying De Morgan's laws to (2.3), we get the equality in (2.2), so C satisfies (iii). Now, assume that C satisfies (iii) and let $C = \bigcap_{t \in T} (\mathbb{R}^n \setminus H_t)$, with $H_t = \{x \in \mathbb{R}^n \mid \langle a_t, x \rangle = b_t\}$, $a_t \in \mathbb{R}^n \setminus \{0_n\}$ and $b_t \in \mathbb{R}$, for all $t \in T$. Since C is a convex set and $C \subset \mathbb{R}^n \setminus H_t$ for each $t \in T$, we have that C is contained in one of the two open halfspaces determined by H_t. Then, we can assume without loss of generality that

$$C \subset \bigcap_{t \in T} \{x \in \mathbb{R}^n \mid \langle a_t, x \rangle > b_t\}. \tag{2.4}$$

On the other hand, if $\bar{x} \notin C$, there exists $s \in T$ such that $\bar{x} \notin \mathbb{R}^n \backslash H_s$ or, equivalently, $\langle a_s, \bar{x} \rangle = b_s$. Therefore, $\bar{x} \notin \{x \in \mathbb{R}^n \mid \langle a_s, x \rangle > b_s\}$ and we obtain the equality in (2.4). Thus, C is evenly convex.

Finally, we prove $(i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (i)$.

$(i) \Rightarrow (iv)$ If C is evenly convex, it satisfies condition (ii) in Theorem 2.1 and then, there exists a family $\{X_t,\, t \in S\}$ of exposed faces of $\operatorname{cl} C$ such that

$$C = (\operatorname{cl} C) \backslash \left[\underset{t \in S}{\cup} X_t \right].$$

Let $K \subset (\operatorname{cl} C) \backslash C = \underset{t \in S}{\cup} X_t$ be a nonempty convex set (if $K = \emptyset$, any hyperplane not intersecting C contains K) and let $\bar{x} \in \operatorname{rint} K$. Then, there exists $t \in S$ such that $\bar{x} \in X_t$, so that X_t is a face of $\operatorname{cl} C$ intersecting $\operatorname{rint} K$ and, by [29, Theorem 18.1], $K \subset X_t$. Since X_t is an exposed face of $\operatorname{cl} C$, there exists a hyperplane H such that $X_t = H \cap \operatorname{cl} C$. Therefore, $K \subset H$ and

$$H \cap C = H \cap [(\operatorname{cl} C) \cap C] = X_t \cap C = \emptyset.$$

$(iv) \Rightarrow (v)$ Let $K \subset (\operatorname{cl} C) \backslash C$ be a convex set and let X be the minimal exposed face (in $\operatorname{cl} C$) such that $K \subset X$. By (iv), there exists a hyperplane H such that $K \subset H$ and $H \cap C = \emptyset$. If we take $Y := H \cap \operatorname{cl} C \neq \emptyset$ (since $K \subset Y$), then Y is an exposed face containing K such that

$$Y \cap C = (H \cap \operatorname{cl} C) \cap C = H \cap C = \emptyset. \tag{2.5}$$

Since X is the minimal exposed face containing K, we have $X \subset Y$. From (2.5), we have $X \cap C = \emptyset$.

$(v) \Rightarrow (vi)$ It is trivial because (vi) is a particular case of (v).

$(vi) \Rightarrow (vii)$ Let $x \in (\operatorname{cl} C) \backslash C$ and let X the minimal exposed face (in $\operatorname{cl} C$) containing x. Since X is an exposed face of $\operatorname{cl} C$, there exists a hyperplane H such that $\operatorname{cl} C$ is contained in one of the closed halfspaces determined by H and $X = H \cap \operatorname{cl} C$, so that $x \in H$ and H supports $\operatorname{cl} C$ at x. Moreover, since $X \cap C = \emptyset$, we have

$$H \cap C = H \cap [(\operatorname{cl} C) \cap C] = X \cap C = \emptyset.$$

$(vii) \Rightarrow (i)$ Let $x \in \mathbb{R}^n \backslash C$. We obtain a hyperplane H such that $x \in H$ and $H \cap C = \emptyset$ as a consequence of (vii), if $x \in (\operatorname{cl} C) \backslash C$, and as a consequence of $\operatorname{cl} C$ being an evenly convex set, if $x \notin \operatorname{cl} C$ and, therefore, C satisfies condition (iii) in Theorem 2.1. □

According to [29, Theorem 11.2], given a nonempty relatively open convex set $C \subset \mathbb{R}^n$ and an affine manifold M such that $C \cap M = \emptyset$, there exists a hyperplane H such that $M \subset H$ and C is contained in one of the two open halfspaces determined by H. Applying this result to the zero dimensional affine manifolds, i.e., the singleton sets, it is easy to see that condition (iii) in Theorem 2.1 holds. Thus, any relatively open convex set is evenly convex. Analogously, any *strictly convex set* C (i.e., a convex set C whose boundary, $\operatorname{bd} C$, does not contain segments) is evenly convex since the exposed faces of C are the singleton sets determined by its boundary points. Observe that any convex set $C \neq \emptyset$ can be fitted from inside by its relative interior $\operatorname{rint} C$ and from outside by its closure $\operatorname{cl} C$, both approximating sets being evenly convex.

The next result allows to compare the cone of feasible directions at a point x, $D(C,x)$, the set $\operatorname{extr} C$ of extreme points, and the recession cone 0^+C of an evenly convex set C, with those of its closure $\operatorname{cl} C$. This comparison shows that evenly convex sets enjoy many of the well-known properties of closed convex sets (see, e.g., [29, Theorems 8.3 and 8.4, Corollary 8.4.1]).

Proposition 2.1 (Properties of evenly convex sets)**.** *If $C \subset \mathbb{R}^n$ is a nonempty evenly convex set, then the following statements hold:*

(i) *$D(C,x) = D(\operatorname{cl} C, x)$ for all $x \in C$.*
(ii) $\operatorname{extr} C = C \cap \operatorname{extr} \operatorname{cl} C$.
(iii) *$[x,y[\subset C$ for any $x \in C$ and $y \in \operatorname{cl} C$.*
(iv) $0^+C = 0^+(\operatorname{cl} C)$. *Consequently, C is bounded if and only if $0^+C = \{0_n\}$.*
(v) *If $y \neq 0_n$ and there exists $x \in C$ such that $\{x + \lambda y : \lambda \geq 0\} \subset C$, then $y \in 0^+C$.*
(vi) *If M is an affine manifold such that $C \cap M$ is a nonempty bounded set, then $M' \cap C$ is also bounded for each affine manifold M' which is parallel to M.*

Sources. Statement (iii) was already proved in [5, 3.5] while (vi) is [8, Corollary 3.2] and the remaining statements can be found in [8, Propositions 3.2 to 3.4].

The convex sets satisfying property (iii) are said to be *wholefaced* in the sense of Motzkin [23]. This property recalls the well-known accessibility lemma asserting that, for any convex set C, $[x,y[\subset C$ for any $x \in \operatorname{rint} C$ and $y \in \operatorname{cl} C$.

The class of evenly convex sets is closed for the same operations than the class of closed convex sets, except for the sum. Sufficient conditions for the sum of two evenly convex sets to be evenly convex will be given in Corollary 4.1 below.

Proposition 2.2 (Operations with evenly convex sets)**.** *The following statements hold:*

(i) *If $C \neq \emptyset$ is an evenly convex set, then αC is evenly convex for all $\alpha \in \mathbb{R}$.*
(ii) *If $\emptyset \neq C \subset \mathbb{R}^n$ is an evenly convex set and $A : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation such that $A^{-1}C \neq \emptyset$, then $A^{-1}C$ is evenly convex and $0^+(A^{-1}C) = A^{-1}(0^+C)$.*
(iii) *If C_1 and C_2 are nonempty sets, then $C_1 \times C_2$ is evenly convex if and only if C_1 and C_2 are evenly convex.*
(iv) *If C_1 and C_2 are nonempty evenly convex sets such that $(0^+C_1) \cap (-0^+C_2) = \{0_n\}$, then*

$$0^+(C_1 + C_2) = 0^+C_1 + 0^+C_2. \tag{2.6}$$

(v) *If $\{C_i \mid i \in I\}$ is a family of evenly convex sets such that $\bigcap_{i\in I} C_i \neq \emptyset$, then the recession cone of this evenly convex set is*

$$0^+\left(\bigcap_{i\in I} C_i\right) = \bigcap_{i\in I} 0^+C_i.$$

(vi) *Let C be a nonempty convex set with $\dim C = n$, $x \in \mathbb{R}^n$, and $k \in \mathbb{Z}$ such that $1 \leq k \leq n$. If $C \cap M$ is evenly convex for each $k-$dimensional affine manifold M containing x, with $k \geq 3$ or $x \in \operatorname{int} C$ and $k \geq 2$, then C is evenly convex.*

Sources. (i) is trivial; (ii) is [8, Proposition 3.5]; the 'if' part and the 'only if' part of (iii) are [8, Proposition 3.6] and [31, Proposition 1.2], respectively; (iv) is [8, Proposition 3.7]; (v) is [8, Proposition 3.8]; finally, (vi) is [15, Cororollary 2.3].

Concerning the sum of closed convex sets, it is well-known that $(0^+C_1) \cap (-0^+C_2) = \{0_n\}$ guarantees that $C_1 + C_2$ is closed convex too (see, e.g., [29, Corollary 9.1.2]). This is not true for evenly convex sets (even though one of the two sets is bounded) as the pair of sets $C_1 = \{x \in C : x_1 + x_2 \leq 1\}$ and $C_2 = \left\{x \in \mathbb{R}^2 : x_1 \geq 0;\ x_2 \geq 0;\ x_1 + x_2 > 0\right\}$ show.

In statement (vi), conditions over k can be weakened when we replace 'evenly convex' by 'open' or 'closed'. So, C is open if $C \cap M$ is relatively open and $1 \leq k \leq n$ and C is closed if $C \cap M$ is closed and $k \geq 2$ or $x \in \operatorname{int} C$. However, with even convexity, statement (vi) fails when $k = 2$ and $x \notin \operatorname{int} C$.

Example 2.1 ([15, Example 2.5])**.** Consider in the plane of $\mathbb{R}^3$ given by $x_3 = 1$, a closed rectangle R and a closed circular halfdisk D that is disjoint of $\operatorname{int} R$ and whose diameter coincides with one of the sides of R, say the segment $[y,z]$, and let $G = (R \cup D) \setminus \{y,z\}$. The set $C = \operatorname{conv}(G \cup [0,y[\cup [0,z[)$ is not evenly convex. However, for each 2-dimensional affine manifold M containing $0 \notin \operatorname{int} C$, $C \cap M$ consists of a single point, a segment, a closed triangle, or a triangle with one or two vertices missing, and each of these sets is evenly convex.

3. Evenly convex hull

Since the intersection of evenly convex sets is evenly convex too (cf. Prop. 2.2 (v)), then the notion of evenly convex hull is well-defined (see [5, 4.2]).

Definition 3.1. The *evenly convex hull* of $X \subset \mathbb{R}^n$, denoted by $\operatorname{eco} X$, is the smallest evenly convex set which contains X.

Obviously, X is evenly convex if and only if $\operatorname{eco} X = X$. This happens, for instance, if X is either a closed or a relatively open convex set. Consequently, if X is a compact (open) set, then $\operatorname{conv} X$ is a compact (open) convex set and so $\operatorname{eco} X = \operatorname{conv} X$. This is the case, in particular, whenever X is finite. From the definition of evenly convex set, for any $\bar{x} \in \mathbb{R}^n$ one has

$$\bar{x} \notin \operatorname{eco} X \iff \exists z \in \mathbb{R}^n : \langle z, \bar{x} \rangle > \langle z, x \rangle\,,\ \forall x \in X. \tag{3.1}$$

For any $X \subset \mathbb{R}^n$, since $\operatorname{cl}\operatorname{conv} X$ is evenly convex and $\operatorname{eco} X$ is convex, we have

$$\operatorname{conv} X \subset \operatorname{eco} X \subset \operatorname{cl}\operatorname{conv} X. \tag{3.2}$$

From (3.2), if it does not exist a halfspace containing X, then $\mathbb{R}^n = \operatorname{conv} X \subset \operatorname{eco} X$ and $\operatorname{eco} X = \mathbb{R}^n$, too. For $\emptyset \neq X \subset \mathbb{R}^n$, since $\operatorname{aff}\operatorname{conv} X = \operatorname{aff}\operatorname{cl}\operatorname{conv} X$ [29, Theorem 6.2], we also have that $\operatorname{aff}\operatorname{eco} X = \operatorname{aff}\operatorname{conv} X$ and $\dim X = \dim\operatorname{conv} X$.

The next result establishes the relationship between the two latter sets in (3.2) (see [10, Proposition 2.1]).

Proposition 3.1 (Characterization of evenly convex hulls)**.** *For any $X \subset \mathbb{R}^n$, $\operatorname{eco} X$ is the result of eliminating from $\operatorname{cl}\operatorname{conv} X$ the union of all its exposed faces which do not intersect X.*

We now prove the Fenchel's extension of the bipolar theorem given in [5].

Definition 3.2. The *e-polar* of a nonempty set $X \subset \mathbb{R}^n$ is the evenly convex set

$$X^e := \{y \in \mathbb{R}^n : \langle y, x \rangle < 1, \forall x \in X\}$$

Corollary 3.1 (Involutory formula for e-polars)**.** *Let $\emptyset \neq X \subset \mathbb{R}^n$. The equation $X^{ee} = X$ characterizes those evenly convex sets containing 0_n.*

Proof. It is obvious that if $X^{ee}=X$, then X is an evenly convex set containing 0_n. Assume now that X is an evenly convex set containing 0_n. The inclusion $\operatorname{eco}X=X\subset X^{ee}$ holds as any $y\in X^e$ satisfies $\langle y,x\rangle<1$ for all $x\in X$. To prove the reverse inclusion, let $\overline{x}\notin \operatorname{eco}X$. Acording to (3.1), there exists $z\in\mathbb{R}^n$ such that $\langle z,x\rangle<\langle z,\overline{x}\rangle$ for all $x\in X$. Since $0_n\in X$, then $z\neq 0_n$ and $b:=\langle z,\overline{x}\rangle>0$. Letting $\widetilde{z}:=\frac{z}{b}$, one has $\langle \widetilde{z},x\rangle<1=\langle \widetilde{z},\overline{x}\rangle$ for all $x\in X$. Thus, $\widetilde{z}\in X^e$ and $\langle \widetilde{z},\overline{x}\rangle=1$, which shows that $\overline{x}\notin X^{ee}$. □

The next result describes how evenly convex hulls behave under different operators as closures, relative interiors and convex or conical hulls.

Proposition 3.2 (Relationships between the eco hull and other hulls)**.** *Given $X\subset\mathbb{R}^n$, the following statements hold:*

(i) $\operatorname{cl}\operatorname{eco}X=\operatorname{cl}\operatorname{conv}X$.
(ii) $\operatorname{rint}\operatorname{eco}X=\operatorname{rint}\operatorname{conv}X$.
(iii) $\operatorname{eco}\operatorname{conv}X=\operatorname{eco}X=\operatorname{conv}\operatorname{eco}X$.
(iv) $\operatorname{cone}\operatorname{eco}X\subset\operatorname{eco}\operatorname{cone}X=\operatorname{cl}\operatorname{cone}X$.
(v) *If X is a nonempty bounded set , then* $\operatorname{cl}\operatorname{eco}X=\operatorname{eco}\operatorname{cl}X=\operatorname{conv}\operatorname{cl}X$.

Sources. Statements (i) and (ii) are easily obtained by taking closures and relative interiors, respectively, in (3.2). The proof of (iii) and (iv) can be found in [10, (2.2) and (2.4)] while (v) is [10, Proposition 2.7].

An immediate consequence of the equality in statement (iv) is that a convex cone containing its apex is evenly convex if and only if it is closed [15, Proposition 3.3].

The set $X=\{x\in\mathbb{R}^2: x_2(1+x_1^2)=1\}$ shows that the inclusions in (3.2) and Proposition 3.2 (iv) may be strict as $\operatorname{conv}X=(\mathbb{R}\times]0,1[)\cup\{(0,1)\}$, $\operatorname{eco}X=\mathbb{R}\times]0,1]$, $\operatorname{cl}\operatorname{conv}X=\mathbb{R}\times[0,1]$, $\operatorname{cone}\operatorname{eco}X=(\mathbb{R}\times]0,+\infty[)\cup\{0_2\}$, and $\operatorname{eco}\operatorname{cone}X=\mathbb{R}\times[0,+\infty[$. Observe also that $\operatorname{eco}X$ is obtained by eliminating from $\operatorname{cl}\operatorname{conv}X$ its unique exposed face which does not intersect X (that is, the line $\mathbb{R}\times\{0\}$).

Proposition 3.3 (Operations with evenly convex hulls)**.** *The following statements hold:*

(i) *If $X,Y\subset\mathbb{R}^n$ and $X\subset Y$, then* $\operatorname{eco}X\subset\operatorname{eco}Y$.
(ii) *If $X\subset\mathbb{R}^n$ and $Y\subset\mathbb{R}^m$, then* $\operatorname{eco}(X\times Y)=(\operatorname{eco}X)\times(\operatorname{eco}Y)$.
(iii) *If X is a nonempty set in $\mathbb{R}^m$ and $A:\mathbb{R}^m\to\mathbb{R}^n$ is a linear transformation, then* $A(\operatorname{eco}X)\subset\operatorname{eco}AX$.
(iv) *If $X,Y\subset\mathbb{R}^n$, then* $\operatorname{eco}X+\operatorname{eco}Y\subset\operatorname{eco}(X+Y)$.
(v) *If X is a nonempty set in $\mathbb{R}^n$ and $A:\mathbb{R}^m\to\mathbb{R}^n$ is a linear transformation such that $A^{-1}X\neq\emptyset$, then* $\operatorname{eco}(A^{-1}X)\subset A^{-1}(\operatorname{eco}X)$.
(vi) *If $\{X_i\mid i\in I\}$ is a family of nonempty sets in $\mathbb{R}^n$, then* $\operatorname{eco}\left(\bigcap_{i\in I}X_i\right)\subset\bigcap_{i\in I}(\operatorname{eco}X_i)$.

Sources. Statement (i) easily follows from the definition of evenly convex hull. Proofs of statements (ii), (iii), (v), (vi) and (iv) can be found in [10, Propositions 2.3 to 2.6 and Corollary 2.1], respectively.

4. SEPARATION OF EVENLY CONVEX SETS

In 1968, Klee [14] gave separation theorems for pairs of evenly convex sets that are useful in the study of optimization problems with strict inequality constraints. These results involve two types of desirable separation properties.

Definition 4.1. Given two nonempty disjoint sets $X,Y \subset \mathbb{R}^n$, we say that a hyperplane H *separates openly* (respectively, *separates nicely*) X from Y if H is disjoint from X (respectively, H is disjoint from X or from Y, without specifying which).

Obviously, open separation implies nice separation. From $(i) \Leftrightarrow (iii)$ in Theorem 2.1, one has that $C \subset \mathbb{R}^n$ is evenly convex if and only if it is openly separated from any singleton set contained in $\mathbb{R}^n \backslash C$.

Given a hyperplane H determining two open halfspaces, H_+ and H_-, and two different points, $x,y \in H$, defining the disjoint convex sets $X := H_+ \cup \{x\}$ and $Y := H_- \cup \{y\}$, H separates weakly X from Y (in the sense that any of these sets lies in one of the two closed halfspaces determined by H), but not nicely. If one aggregates the condition that X and Y should be closed, the counterexample must be built in dimension at least 3, as the following one shows (see [13]).

Example 4.1. Consider the line $X = \{(0,x_2,1) : x_2 \in \mathbb{R}\}$ and the closed convex cone

$$Y = \left\{ y \in \mathbb{R}^3_+ : y_3^2 \leq y_1 y_2 \right\}.$$

The hyperplane $H = \left\{ x \in \mathbb{R}^3 : x_1 = 0 \right\}$ contains X and Y lies in the halfspace $\left\{ x \in \mathbb{R}^3 : x_1 \geq 0 \right\}$. In fact, H is the unique hyperplane separating weakly X from Y, but the separation is not nice.

The next two results collect six open and six nice separation theorems, respectively. Their statements involve several concepts we introduce next.

Given a set X such that $\emptyset \neq X \subsetneq \mathbb{R}^n$, if $Y \subset \mathbb{R}^n \backslash X$ is a j-dimensional affine manifold such that $d(X,Y) := \inf\{d(x,y) : x \in X, y \in Y\} = 0$, then Y is called a *j-asymptote* of X. A convex set X is called *continuous* provided that X is closed and its support function $\delta^*_X(\cdot) := \sup\{\langle \cdot, x\rangle : x \in X\}$ is continuous (this is equivalent to saying that there is no halfline contained in $\operatorname{bd} X$ and no 1-asymptote). Given a supporting hyperplane H of X, we say that X is *continuous relative to* H if $H \cap X$ is closed and convex but it has neither ray contained in its relative boundary nor 1-asymptote relative to H. A convex set $\emptyset \neq X \subsetneq \mathbb{R}^n$ is called a *strip* provided that it is a union of translates of a hyperplane. Equivalently, a strip is a hyperplane, an open or closed halfspace, or a set of the form S or $H_1 \cup S$, or $H_1 \cup S \cup H_2$, where H_1 and H_2 are parallel hyperplanes and S is the set of all points of $\mathbb{R}^n$ lying between H_1 and H_2. All strips are evenly convex. A set $X \subset \mathbb{R}^n$ is said to be *quasi-polyhedral* (or *boundedly polyhedral*) provided that its intersection with any polytope is a polytope and to be *polyhedral* at $x \in X$ provided that X contains a polytope which is a neighborhood of x relative to X. A set is quasi-polyhedral if and only if it is closed, convex, and polyhedral at each of its points.

Theorem 4.1 (Open separation theorems)**.** *For $X,Y \subset \mathbb{R}^n$ disjoint nonempty convex sets, each of the following conditions implies X is openly separated from Y.*

- (i) *X is open; Y is arbitrary.*
- (ii) *X is evenly convex and its intersection with any supporting hyperplane is compact; Y is closed.*

(iii) *X admits no asymptote in any supporting hyperplane intersecting X; Y admits no asymptote.*

(iv) *X is evenly convex and its intersection with any supporting hyperplane is closed; Y is evenly convex, Y admits no hyperplane asymptote and Y is continuous relative to every supporting hyperplane.*

(v) *X's projections are all evenly convex; Y admits no asymptote and is quasi-polyhedral.*

(vi) *X is evenly convex; Y is singleton or a closed strip.*

Concerning Theorem 4.1, each statement "(i) (respectively, (ii), ..., (vi)) implies X is openly separated from Y" is an open separation theorem, and all of them are maximal in Klee's sense [14], except that (vi) does not when $n = 2$.

Theorem 4.2 (Nice separation theorems)**.** *For $X,Y \subset \mathbb{R}^n$ disjoint nonempty convex sets, each of the following conditions implies X is nicely separated from Y.*

(I) *X is open or a strip; Y is arbitrary.*

(II) *X is evenly convex and is continuous relative to any supporting hyperplane; Y is evenly convex and its intersection with any supporting hyperplane is closed.*

(III) *X admits no asymptote in any supporting hyperplane; Y admits no asymptote in any supporting hyperplane.*

(IV) *X's projections are all evenly convex and X is polyhedral at each of its points; Y's projections are all evenly convex and Y is polyhedral at each of its points.*

(V) *X's projections are all evenly convex; Y admits no asymptote in any supporting hyperplane and Y is polyhedral at each of its points.*

(VI) *X is evenly convex; Y is singleton or open or a strip.*

Regarding Theorem 4.2, each statement "(I) (respectively, (II), ..., (VI)) implies X is nicely separated from Y" is a nice separation theorem, and all of them are maximal in Klee's sense [14], except that (VI) does not when $n = 2$. The common keys for the proofs of the above results are Proposition 2.1 (iii) and the following lemma.

Lemma 4.1. *Let $X,Y \subset \mathbb{R}^n$ be disjoint nonempty convex sets. Then, X is openly separated from Y if and only if there is no point $p \in X$ which lies in every hyperplane separating X from Y. Any such point p satisfies at least one of the following conditions:*

(a) $p \in \operatorname{cl} Y$.

(b) *There is $w \in \operatorname{cl} Y$ such that $[p,w] \subset (\operatorname{cl} X) \cap H$ for every hyperplane H separating X from Y.*

(c) *There are sequences $\{p^k\} \subset \mathbb{R}^n$, $\{x^k\} \subset X$, and $\{y^k\} \subset Y$ such that $y^k \in [p^k, x^k]$ for all k, $\lim p^k = p$, $\lim x^k = x$, and $[p,x]$ is contained in some ray which lies in $(\operatorname{cl} X) \cap H$ for every hyperplane H separating X from Y.*

If X and Y are evenly convex, then condition (c) is satisfied for each point $p \in X$ which lies in every hyperplane separating X from Y and each separating hyperplane H such that $X \cap H$ and $Y \cap H$ are both closed and nonempty.

Sketch of the proof for Theorems 4.1 and 4.2. By the standard separation theorem, there is a hyperplane H separating X from Y.

If X is open, then $X \cap H = \emptyset$. If X is a strip, it can happen that H supports X and, therefore, $H \subset X$ and $Y \cap H = \emptyset$, or that $H \cap X = \emptyset$. So, statements (i) and (I) imply that X is openly and nicely separated from Y, respectively.

The proofs for statements (vi) and (VI) are trivial.

The separation theorems corresponding to statements (ii), (iv) and (II) are proved by contradiction. Supposing that X is not openly separated from Y, by Lemma 4.1, there is a point $p \in X$ which lies in every hyperplane H separating X from Y. Therefore, $p \in X \cap H$ and H is a supporting hyperplane of X, which, under statements (ii), (iv) or (II), implies that X is evenly convex and $X \cap H$ is nonempty and closed.

Regarding the set Y, if $Y \cap H \neq \emptyset$, any of the three conditions implies that Y is evenly convex and $Y \cap H$ is nonempty and closed, and then, by the last assertion in Lemma 4.1, condition (c) is satisfied for p, X, Y and H.

If $Y \cap H = \emptyset$, under (ii), conditions (a) and (b) in Lemma 4.1 are excluded by the fact that Y is closed, so (c) is satisfied; under (iv), the fact that Y admits no hyperplane asymptote yields to a contradiction; and finally, under (II), $Y \cap H = \emptyset$ implies that X is nicely separated from Y and there is nothing to prove.

Condition (c) in Lemma 4.1 claims the existence of a ray

$$r := \{p + \lambda u : \lambda \geq 0\} \subset (\operatorname{cl} X) \cap H$$

and, since $p \in X$ and X is evenly convex, by Proposition 2.1 (iii), $r \subset X \cap H$, which is a contradiction under statements (ii) and (II). Finally, under (iv), [7, 1.1 and 1.2] assert the existence of a parallel ray to r which is a boundary ray or an asymptote of $Y \cap H$ and we obtain a contradiction again.

The remaining statements are proved by induction on n. □

Corollary 4.1 (Even convexity of the sum of convex sets)**.** *If X and Y are two proper convex sets in $\mathbb{R}^n$, not necessarily disjoint, and they satisfy any of the conditions of Theorems 4.1 and 4.2, then the set $X+Y$ is evenly convex.*

Proof. The conditions on Y in Theorems 4.1 and 4.2 are symmetric in the sense that they hold for Y if and only if they hold for $-Y$ and they are also preserved under translations. If we take $z \notin X+Y$, then the sets X and $-Y+z$ are disjoint (otherwise, there exist $x \in X$ and $y \in Y$ such that $x = -y+z$ and $z = x+y \in X+Y$). Then, since X and $-Y+z$ are disjoint, any of the conditions of Theorems 4.1 and 4.2 implies the existence of $a \in \mathbb{R}^n \setminus \{0_n\}$ such that $\langle a, x\rangle < \langle a, -y+z\rangle$ for all $x \in X$ and $y \in Y$, whence, $\langle a, x+y\rangle < \langle a, z\rangle$ and we have that $H = \{x \in \mathbb{R}^n : \langle a, x\rangle = \langle a, z\rangle\}$ is a hyperplane which contains z and misses $X+Y$ and, since $X+Y$ is convex, by Proposition 2.1 (iii), $X+Y$ is evenly convex. □

Observe that X and Y are simultaneously evenly convex under conditions (ii), (iv), (vi), (II) and (IV), so that Corollary 4.1 can be interpreted, in those cases, as providing sufficient conditions for the sum of two evenly convex sets to be evenly convex.

5. Evenly quasiconvex functions

In this section, we briefly review the relationship between evenly quasiconvex functions and other close families of functions. Firstly, we state some notation related to functions. For an extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we denote by $\operatorname{epi} f$ its *epigraph*, while its *sublevel set* and its *strict sublevel set* for $r \in \mathbb{R}$ are defined by $L_f(r) := \{x \in \mathbb{R}^n : f(x) \leq r\}$ and $L_f^s(r) := \{x \in \mathbb{R}^n : f(x) < r\}$, respectively. The function f is said to be *lower semicontinuous*, lsc in brief, (*upper semicontinuous*, usc in brief) at $\overline{x} \in \mathbb{R}^n$ if for any $\lambda \in \mathbb{R}$, $\lambda < f(\bar{x})$ (resp. $\lambda > f(\bar{x})$), there exists a neigborhood of $\bar{x}$, $V_{\bar{x}}$, such that $\lambda < f(x)$ (resp. $\lambda > f(x)$) for all $x \in V_{\bar{x}}$. It is well-known that a function f is lsc at any point of $\mathbb{R}^n$ if and only if $\operatorname{epi} f$ is closed, or equivalently, if $L_f(r)$ is closed for every $r \in \mathbb{R}$. The function f is said to be *quasiconvex* if $L_f(r)$ is convex for every $r \in \mathbb{R}$. Although lsc quasiconvex functions (those whose sublevel sets are closed and convex) play an important role in optimization, they do not constitute the class of regular functions in quasiconvex programming. In this field, the larger class of *evenly quasiconvex* functions arises in a natural way.

Definition 5.1. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *evenly quasiconvex* (respectively, *strictly evenly quasiconvex*) if the sublevel set $L_f(r)$ (respectively, the strict sublevel set $L_f^s(r)$) is evenly convex for every $r \in \mathbb{R}$.

It is obvious that every lsc quasiconvex function is evenly quasiconvex and, as a function f is usc if and only if $-f$ is lsc, every usc quasiconvex function is strictly evenly quasiconvex. Moreover, since $L_f(r) = \cap_{r<q} L_f^s(q)$ and the intersection of evenly convex sets is evenly convex, every strictly evenly quasiconvex function is evenly quasiconvex. The converse is not true, as the following example shows.

Example 5.1 ([4, p. 64])**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x_1,x_2) = \begin{cases} 0, & \text{if } x_1 \geq x_2 \text{ and } x_2 \leq 0, \\ x_2/x_1, & \text{if } x_1 > x_2 > 0, \\ 1, & \text{elsewhere.} \end{cases}$$

All the sublevel sets of f are closed and convex, and so evenly convex, showing that f is evenly quasiconvex. However, $L_f^s(1) = \{x \in \mathbb{R}^2 : x_1 > x_2 > 0\} \cup \{x \in \mathbb{R}^2 : x_1 \geq x_2, x_2 \leq 0\}$ is not evenly convex.

As the sublevel sets of the pointwise supremum of a family of functions are intersections of sublevel sets of the members of the family, and even convexity is preserved under intersections, then evenly quasiconvex functions are closed under pointwise suprema. Thus, every function f has a largest evenly quasiconvex minorant, which is called its *evenly quasiconvex hull* and denoted by $\operatorname{eqco} f$.

Definition 5.2. A function f is said to be *evenly quasiconvex at* $x_0 \in \mathbb{R}^n$ if $f(x_0) = (\operatorname{eqco} f)(x_0)$.

Clearly, as stated in [4, Proposition 11], f is evenly quasiconvex if and only if it is evenly quasiconvex at every $x_0 \in \mathbb{R}^n$. Next result gathers together four characterizations of even quasiconvexity at a point given in the literature.

Theorem 5.1 (Characterizations of the even quasiconvexity at a point)**.** *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $x_0 \in \mathbb{R}^n$. *The following statements are equivalent:*

(i) f *is evenly quasiconvex at* x_0.
(ii) $f(x_0) = \inf\{r \in \mathbb{R} : x_0 \in \operatorname{eco} L_f(r)\}$.
(iii) $f(x_0) = \sup_{x^* \in \mathbb{R}^n} \inf\{f(x) : \langle x, x^*\rangle \geq \langle x_0, x^*\rangle\}$.
(iv) $x_0 \notin \operatorname{eco} L_f(r)$ *for all* $r < f(x_0)$.

(*v*) *f is quasiconvex and for every $y_0 \in \mathbb{R}^n$ such that $f(y_0) < f(x_0)$, every $\{y_n\}_{n\geq 1} \to y_0$ and every $\{\mu_n\}_{n\geq 1} \subset (0,+\infty)$, one has $f(x_0) \leq \liminf_{n\to+\infty} f(x_0 + \mu_n(x_0 - y_n))$.*

Sources. See, e.g., [21, Corollary 6.4] for $(i) \Leftrightarrow (iii)$. For the rest of equivalences, see [4, Propositions 8, 10, 12].

From $(i) \Leftrightarrow (ii)$, one obtains the following representation for the evenly quasiconvex hull of a given function f:

$$(\text{eqco}\, f)(x_0) = \inf\{r \in \mathbb{R} : x_0 \in \text{eco}\, L_f(r)\}. \tag{5.1}$$

This identity shows that, for any $r \in \mathbb{R}$,

$$L^s_{\text{eqco}\, f}(r) \subset \text{eco}\, L_f(r) \subset L_{\text{eqco}\, f}(r).$$

Regarding the statement $(i) \Leftrightarrow (iii)$, it means that every evenly quasiconvex function is the pointwise supremum of a collection of evenly quasiaffine functions (recall that a function φ is *evenly quasiaffine* if there is $x^* \in \mathbb{R}^n$ and a nondecreasing univariate function h such that $\varphi = h \circ \langle \cdot, x^* \rangle$), that is,

$$(\text{eqco}\, f)(x_0) = \sup_{x^* \in \mathbb{R}^n} \varphi_{x^*}(x_0)$$

where $\varphi_{x^*}(x_0) := \inf\{f(x) : \langle x, x^* \rangle \geq \langle x_0, x^* \rangle\}$. This characterization was obtained by Martínez-Legaz by employing the generalized convex conjugation theory developed in [21] as the biconjugate of f for an appropriate coupling functional.

6. Even quasiconvexity in the graph space and subdifferentiability

The objective of this section is to present a further analysis on evenly quasiconvex functions extending the one given in [4]. To this end, we shall employ the methodology developed in [34] and [36] for the class of evenly convex functions.

Definition 6.1. We say that the family $\mathscr{K} := \{K_t\}_{t\in\mathbb{R}}$ of (possibly empty) subsets of $\mathbb{R}^n$ is an *ascending family* if $K_t \subset K_{t'}$ for all $t, t' \in \mathbb{R}$ with $t \leq t'$. We also associate to the ascending family $\mathscr{K} := \{K_t\}_{t\in\mathbb{R}}$ the function $\psi_{\mathscr{K}} : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\psi_{\mathscr{K}}(x) := \inf\{t \in \mathbb{R} : x \in K_t\}. \tag{6.1}$$

These notions were previously considered in [3, p. 126] indeed. Observe that, if $K_t = \emptyset$ for all $t \in \mathbb{R}$, then $\psi_{\mathscr{K}}(x) = +\infty$ for all $x \in \mathbb{R}^n$. We recall from [36] that a set $K \subset \mathbb{R}^n \times \mathbb{R}$ is *ascending* if either $K = \emptyset$ or there exists $(x_0, t_0) \in K$ such that $(x_0, t) \in K$ for all $t \geq t_0$. According to [36, Corollary 2.2], if $K \subset \mathbb{R}^n \times \mathbb{R}$ is a nonempty evenly convex set, then it is ascending if and only if $(0_n, 1) \in 0^+K$.

Proposition 6.1 (Ascending set from an ascending family)**.** *If $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}}$ is an ascending family of sets of $\mathbb{R}^n$, then $K := \bigcup_{t\in\mathbb{R}} (K_t \times \{t\})$ is ascending.*

Proof. On the one hand, if $K_t = \emptyset$ for all $t \in \mathbb{R}$, then $K := \bigcup_{t\in\mathbb{R}} (K_t \times \{t\}) = \emptyset$ which is ascending. On the other hand, if there exists $t_0 \in \mathbb{R}$ such that $K_{t_0} \neq \emptyset$, since $K_{t_0} \subset K_t$ for all $t \in \mathbb{R}$ with $t_0 \leq t$, by taking any $x_0 \in K_{t_0}$ one has $(x_0, t_0) \in K := \bigcup_{t\in\mathbb{R}} (K_t \times \{t\})$ and $(x_0, t) \in K$ for all $t \geq t_0$. Thus, K is ascending. □

We denote by $\text{proj}_{\mathbb{R}^n}$ the projection (mapping) from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n$ such that $\text{proj}_{\mathbb{R}^n}(x, t) = x$.

Proposition 6.2 (Ascending family from an ascending set)**.** *If $K \subset \mathbb{R}^{n+1}$ is ascending and evenly convex, then $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}}$ with*

$$K_t := \operatorname{proj}_{\mathbb{R}^n}\left(K \cap (\mathbb{R}^n \times \{t\})\right) \tag{6.2}$$

is an ascending family.

Proof. Assume the non-trivial case in which K is nonempty. If K is ascending and evenly convex, then $(0_n, 1) \in 0^+K$ according to [36, Corollary 2.2], which means that, for every $(x_0,t_0) \in K$, $(x_0,t) \in K$ for all $t \geq t_0$. This implies, by defining $K_t := \operatorname{proj}_{\mathbb{R}^n}\left(K \cap (\mathbb{R}^n \times \{t\})\right)$, that $K_{t_0} \subset K_t$ for all $t \geq t_0$, i.e., $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}}$ is an ascending family. □

The above proposition does not hold in general if the even convexity assumption is removed, even though we keep just convexity. We illustrate this fact in the following example.

Example 6.1. Consider the set $K = ([-1,1] \times \{0\}) \cup (\{0\} \times [0,+\infty[) \subset \mathbb{R}^2$. Clearly, K is ascending but it is neither convex nor evenly convex. It is easy to see that the family $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}}$ defined as in (6.2) is not an ascending one. Observe that the convex and the evenly convex hulls, $\operatorname{conv} K = ([-1,1] \times \{0\}) \cup (]-1,1[\times]0,+\infty[)$ and $\operatorname{eco} K = [-1,1] \times \mathbb{R}_+$ (recall Proposition 3.1), are obviously ascending, but the family obtained by replacing K by $\operatorname{eco} K$ in (6.2) is ascending while the result of replacing K by $\operatorname{conv} K$ does not.

Proposition 6.3 (Ascending family and even quasiconvexity)**.** *Let $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}}$ be an ascending family of evenly convex sets of $\mathbb{R}^n$. Then, the function $\psi_{\mathscr{K}}$ in (6.1) is evenly quasiconvex.*

Proof. We just need to show that $L_{\psi_{\mathscr{K}}}(a)$ is evenly convex for every $a \in \mathbb{R}$. Consider the non-trivial case in which there is $t_0 \in \mathbb{R}$ such that $K_{t_0} \neq \emptyset$, otherwise $\psi_{\mathscr{K}} \equiv +\infty$ which is obviously evenly quasiconvex. Since $L_{\psi_{\mathscr{K}}}(a) \subset \operatorname{eco} L_{\psi_{\mathscr{K}}}(a)$ always holds, we next prove the reverse set containment.

Let $\bar{x} \notin L_{\psi_{\mathscr{K}}}(a)$, that is, $\psi_{\mathscr{K}}(\bar{x}) = \inf\{t \in \mathbb{R} : \bar{x} \in K_t\} > a$. Then, $\bar{x} \notin K_a$. As K_a is evenly convex by hypothesis, then $K_a = \operatorname{eco} K_a$ and so $\bar{x} \notin \operatorname{eco} K_a$. By (3.1), there exists $z \in \mathbb{R}^n$ such that

$$\langle z, \bar{x} \rangle > \langle z, x \rangle \tag{6.3}$$

for all $x \in K_a$. Since $\mathscr{K}$ is an ascending family, that is, $K_t \subset K_a$ for all $t \leq a$, then $\inf\{t \in \mathbb{R} : x \in K_t\} \leq a$ implies that $x \in K_a$, or equivalently, $L_{\psi_{\mathscr{K}}}(a) \subset K_a$. Thus, (6.3) holds in particular for all $x \in L_{\psi_{\mathscr{K}}}(a)$, which again in virtue of (3.1) implies that $\bar{x} \notin \operatorname{eco} L_{\psi_{\mathscr{K}}}(a)$, and this completes the proof. □

Corollary 6.1. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. If there is an ascending family $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}}$ of evenly convex sets in $\mathbb{R}^n$ such that $L_f^s(t) \subset K_t \subset L_f(t)$ for all $t \in \mathbb{R}$, then f is evenly quasiconvex. In particular, any function whose strict sublevel sets are evenly convex, is evenly quasiconvex as well.*

Proof. As $f = \psi_{\mathscr{K}}$ and $\mathscr{K}$ is an ascending family of evenly convex sets, due to Proposition 6.3, one has that f is evenly quasiconvex. □

Next, for a family of sets $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}}$, we denote $\operatorname{eco} \mathscr{K} := \{\operatorname{eco} K_t\}_{t\in\mathbb{R}}$.

Proposition 6.4. *Let $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}}$ be a family of sets in $\mathbb{R}^n$ such that $\operatorname{eco} \mathscr{K}$ is ascending. Then,*

$$\operatorname{eqco} \psi_{\mathscr{K}} = \psi_{\operatorname{eco} \mathscr{K}}.$$

Proof. Since $K_t \subset \operatorname{eco} K_t$ for every $t \in \mathbb{R}$, one has $\psi_{\operatorname{eco}\mathscr{K}} \leq \psi_{\mathscr{K}}$. It follows from Proposition 6.3 that

$$\psi_{\operatorname{eco}\mathscr{K}} = \operatorname{eqco}\psi_{\operatorname{eco}\mathscr{K}} \leq \operatorname{eqco}\psi_{\mathscr{K}}.$$

On the other hand, since $K_t \subset L_{\psi_{\mathscr{K}}}(t)$ for every $t \in \mathbb{R}$, then

$$\operatorname{eco} K_t \subset \operatorname{eco} L_{\psi_{\mathscr{K}}}(t) \subset \operatorname{eco} L_{\operatorname{eqco}\psi_{\mathscr{K}}}(t) = L_{\operatorname{eqco}\psi_{\mathscr{K}}}(t).$$

So, $\psi_{\operatorname{eco}\mathscr{K}} \geq \operatorname{eqco}\psi_{\mathscr{K}}$. □

Corollary 6.2 (Characterization of the evenly quasiconvex hull)**.** *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\mathscr{K} = \{K_t\}_{t\in\mathbb{R}} \subset \mathbb{R}^n$ *be such that* $L^s_f(t) \subset K_t \subset L_f(t)$ *for every* $t \in \mathbb{R}$*. Then,*

$$\operatorname{eqco} f = \psi_{\operatorname{eco}\mathscr{K}}.$$

Proof. It easily follows from Proposition 6.4 since $f = \psi_{\mathscr{K}}$ and $\operatorname{eco}\mathscr{K}$ is ascending. □

This result generalizes the well-known characterization given in (5.1). As a consequence, we observe that a strictly evenly quasiconvex function is always evenly quasiconvex. Next we recover the equivalence $(i) \Leftrightarrow (iv)$ in Theorem 5.1.

Corollary 6.3 (Characterization of the even quasiconvexity at a point)**.** *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $x_0 \in \mathbb{R}^n$*. The following statements are equivalent:*

(i) $f(x_0) = (\operatorname{eqco} f)(x_0)$.
(ii) $x_0 \in \operatorname{eco} L_f(t)$ *if and only if* $f(x_0) \leq t$.

Proof. Consider the ascending family $\mathscr{K} := \{\operatorname{eco} L_f(t)\}_{t\in\mathbb{R}}$.

$(i) \Rightarrow (ii)$ On the one hand, if $f(x_0) \leq t$, then $x_0 \in L_f(t)$ and so $x_0 \in \operatorname{eco} L_f(t)$. On the other hand, let $t \in \mathbb{R}$ be such that $x_0 \in \operatorname{eco} L_f(t)$. By applying (i) and Corollary 6.2, one has

$$f(x_0) = (\operatorname{eqco} f)(x_0) = \psi_{\operatorname{eco} L_f}(x_0) \leq t.$$

$(ii) \Rightarrow (i)$ By applying (ii) and Corollary 6.2, we get

$$(\operatorname{eqco} h)(x_0) = \psi_{\operatorname{eco} L_f}(x_0) = \psi_{L_f}(x_0) = f(x_0).$$

□

Even convexity and (Fenchel) subdifferentiability via the strict epigraph were recently linked in [36]. When dealing with quasiconvex functions, Fenchel subdifferential is not appropriate and, because of that, several subdifferential notions have been proposed in the literature, being the Greenberg-Pierskalla one [11] the most remarkable by its properties (see, e.g., [35]). Next, we aim to link even quasiconvexity of a function and subdifferentiability (in the sense of Greenberg-Pierskalla) via its strict sublevel sets.

Definition 6.2. Given $\varepsilon \geq 0$, a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be ε-*subdifferentiable* (in the sense of Greenberg-Pierskalla) at a point $\bar{x} \in f^{-1}(\mathbb{R})$ if there exists $u \in \mathbb{R}^n$ such that

$$\langle u, x - \bar{x}\rangle \geq 0 \;\Rightarrow\; f(x) \geq f(\bar{x}) - \varepsilon. \tag{6.4}$$

The set of those points u satisfying (6.4) is the ε-*subdifferential* of f at $\bar{x}$, denoted by $(\partial^{GP}_\varepsilon f)(\bar{x})$. When $\varepsilon = 0$ we just write $(\partial^{GP} f)(\bar{x})$ and it is called the *subdifferential* (in the sense of Greenberg-Pierskalla) of f at $\bar{x}$. The function f is said to be ε-subdifferentiable on $A \subset \mathbb{R}^n$ if it is ε-subdifferentiable at each point of A.

Next result characterizes the ε-subdifferentiability of a function at a given point in terms of the even convexity of a given strict sublevel set.

Proposition 6.5. *Let $\varepsilon \geq 0$, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in f^{-1}(\mathbb{R})$. Then, the following statements are equivalent:*

(i) $(\partial_{\varepsilon}^{GP} f)(\overline{x}) \neq \emptyset$.
(ii) $\overline{x} \notin \operatorname{eco} L_f^s(f(\overline{x}) - \varepsilon)$.

Proof. It follows from the definition of ε-subdifferential since (6.4) is equivalent to

$$\langle u, x \rangle < \langle u, \overline{x} \rangle, \quad \forall x : f(x) < f(\overline{x}) - \varepsilon,$$

and this is equivalent to (ii) according to (3.1). □

Corollary 6.4. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in f^{-1}(\mathbb{R})$. Then, $(\partial^{GP} f)(\overline{x}) \neq \emptyset$ if and only if $\overline{x} \notin \operatorname{eco} L_f^s(f(\overline{x}))$.*

Corollary 6.5. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in f^{-1}(\mathbb{R})$. If $L_f^s(f(\overline{x}))$ is evenly convex, then f is subdifferentiable at $\overline{x}$.*

We have obtained in the above corollary a sufficient condition for subdifferentiability based on the even convexity of an strict sublevel set. The last result of this paper shows that, under certain even convexity assumptions on either the function or its domain, the even convexity of all strict sublevel sets is a necessary condition for the subdifferentiability.

Proposition 6.6. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Assume that f is subdifferentiable on $f^{-1}(\mathbb{R})$ and either f is evenly quasiconvex or $\operatorname{dom} f$ is evenly convex. Then, $L_f^s(r)$ is evenly convex for every $r \in \mathbb{R}$.*

Proof. Let $r \in \mathbb{R}$ and $\overline{x} \notin L_f^s(r)$, that is, $f(\overline{x}) \geq r$.

Firstly, assume that $f(\overline{x}) < +\infty$ and so, $f(\overline{x}) \in \mathbb{R}$. As f is subdifferentiable on $f^{-1}(\mathbb{R})$, there exists $u \in (\partial^{GP} f)(\overline{x})$ such that if $\langle u, x - \overline{x} \rangle \geq 0$, then $f(x) \geq f(\overline{x}) \geq r$. Hence, $\overline{x} \notin \operatorname{eco} L_f^s(r)$ and so, $L_f^s(r)$ is evenly convex.

Now, if $f(\overline{x}) = +\infty$ and $\operatorname{dom} f$ is evenly convex, there exists $u \in \mathbb{R}^n$ such that $\langle u, \overline{x} \rangle > \langle u, x \rangle$ for all $x \in \operatorname{dom} f$. Since $L_f^s(r) \subset \operatorname{dom} f$, then $\overline{x} \notin \operatorname{eco} L_f^s(r)$ and so, $L_f^s(r)$ is evenly convex.

Finally, assume that $f(\overline{x}) = +\infty$ and f is evenly quasiconvex. If $\overline{x} \in \operatorname{eco} L_f^s(r)$, then $\overline{x} \in L_f(r) = \operatorname{eco} L_f(r)$, but this is impossible as $f(\overline{x}) = +\infty$. Consequently, $\overline{x} \notin \operatorname{eco} L_f^s(r)$ and the conclusion follows. □

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