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# EFFICIENCY CONDITIONS FOR MULTIOBJECTIVE BILEVEL PROGRAMMING PROBLEMS VIA CONVEXIFICATORS

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**Abstract.** Fritz John necessary conditions for solutions of nonsmooth multiobjective bilevel programming problems involving inequality constraints in terms of convexificators are established. Karush–Kuhn–Tucker necessary efficiency conditions are derived under the constraint qualification (CQ). With assumptions on generalized convexity, necessary efficiency conditions become sufficient ones.

**Keywords.** Multiobjective bilevel programming problem; Efficiency conditions; Constraint qualifications; Convexificators.

## 1. INTRODUCTION

Bilevel programming is an important part of optimization. This problem comprises two combined optimization problems, which are the upper and lower level problems, in which the feasible sets of the upper level problems are implicitly defined by the solution sets of the lower level problems. Bilevel programming problem has been studied by many authors in recent years; see, e.g., Bard [2, 3], Dempe [6, 8], Dempe, Dutta and Mordukhovich [7], Outrata [18], Wang, Wang and Rodriguez [21], Ye and Ye [22], etc. Bard [2] studied linear bilevel programming problems, while Dempe [7] and Outrata [18] derived necessary conditions for the optimistic bilevel programming problems, where the solution set of the lower level problem is a singleton. Ye and Zhu [23] derived optimality conditions for the general bilevel programming problems, where the solution sets of the lower level problems are not singleton. Note that the results mentioned above were established for scalar bilevel programming problems. Jeyakumar and Luc [10] introduced the notion of closed and nonconvex convexificators for extended-real-valued functions. The notion of convexificator yields good calculus rules for establishing optimality conditions in nonsmooth optimization. This notion is a generalization of some notions of known subdifferentials such as the subdifferentials of Clarke [4], Michel-Penot [16], Mordukhovich-Shao [17]. Optimality conditions for efficiency via convexificators have been developed by a lot of authors. By using this notion, Luu [13, 14] derived Lagrange multiplier rules for local efficient solutions of the multiobjective optimization problem. Luu [15] established Fritz John and Karush-Kuhn-Tucker necessary conditions for local efficient solutions of constrained

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vector equilibrium problems in Banach spaces via convexificators. Li and Zhang [12] derived necessary optimality conditions in Lipschitz optimization using convexificators. Babahadda and Gadhi [1] studied necessary optimality conditions for scalar bilevel programming problems via convexificators. Suneja and Kohli [20], and Kohli [11] derived optimality conditions and duality results for scalar bilevel programming problem using convexificators.

The purpose of this paper is to developing Fritz John necessary conditions for efficiency in multiobjective bilevel programming, Karush–Kuhn–Tucker necessary efficiency conditions under the constraint qualification (CQ), sufficient efficiency conditions under assumptions on generalized convexity. The paper is organized as follows. The mathematical preliminaries are presented in Section 2. After the preliminaries, Fritz John necessary conditions for solutions of multiobjective bilevel programming problems are derived via convexificators in Section 3. Theorem 3.1 obtained in this section is a significant extension of [1, Theorem 1] for scalar bilevel programming problem. In Section 4, Karush–Kuhn–Tucker necessary efficiency conditions are established under the constraint qualification (CQ). In Section 5, the last section, under assumptions on generalized convexity, sufficient optimality conditions for solutions of multiobjective bilevel programming problem are also given.

## 2. Preliminaries

We recall some notions on convexificators in [10]. The lower (upper) Dini directional derivatives of $f:\mathbb{R}^n \longrightarrow \overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$ at $\bar{x}\in\mathbb{R}^n$ in a direction $v\in\mathbb{R}^n$ is defined as

$$f^-(\bar{x};v) = \liminf_{t\downarrow 0} \frac{f(\bar{x}+tv)-f\bar{x})}{t}$$

$$\left(\text{resp. } f^+(\bar{x};v) = \limsup_{t\downarrow 0} \frac{f(\bar{x}+tv)-f(x)}{t}\right).$$

In case $f^-(\bar{x};v) = f^+(\bar{x};v)$, their common value is denoted by $f^{'}(\bar{x},v)$, which is called Dini derivative of $f$ in the direction $v$. The function $f$ is called Dini differentiable at $\bar{x}$ if its Dini derivative at $\bar{x}$ exists in all directions.

**Definition 2.1.** The function $f:\mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$ is said to have an upper (lower) convexificator $\partial^* f(\bar{x})$ (resp. $\partial_* f(\bar{x})$) at $\bar{x}$ if $\partial^* f(\bar{x})$ (resp. $\partial_* f(\bar{x})$) $\subseteq \mathbb{R}^n$ is closed, and for all $v\in\mathbb{R}^n$,

$$f^-(\bar{x},v) \le \sup_{x^*\in\partial^* f(\bar{x})} \langle x^*, v\rangle$$

$$\left(\text{resp. } f^+(\bar{x},v) \ge \inf_{x^*\in\partial_* f(\bar{x})} \langle x^*, v\rangle\right).$$

A closed set $\partial^* f(\bar{x}) \subseteq \mathbb{R}^n$ is said to be a convexificator of $f$ at $\bar{x}$ if it is both upper and lower convexificators of $f$ at $\bar{x}$. This means that, for each $v\in\mathbb{R}^n$,

$$f^-(\bar{x},v) \le \sup_{x^*\in\partial^* f(x)} \langle x^*, v\rangle, \quad f^+(\bar{x},v) \ge \inf_{x^*\in\partial_* f(\bar{x})} \langle x^*, v\rangle.$$

**Definition 2.2.** The function $f:\mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$ is said to have an upper (lower) semiregular convexificator $\partial^* f(\bar{x})$ (resp. $\partial_* f(\bar{x})$) at $\bar{x}$ if $\partial^* f(\bar{x})$ (resp. $\partial_* f(\bar{x})$) is closed and for all $v\in X$,

$$f^+(\bar{x},v) \leqslant \sup_{x^*\in\partial^* f(\bar{x})} \langle x^*, v\rangle.$$

$$\left(\text{resp. } f^-(\bar{x},v) \geqslant \inf_{x^* \in \partial_* f(\bar{x})} \langle x^*, v\rangle\right).$$

If "=" holds in these inequalities, then $\partial^* f(\bar{x})$ (resp. $\partial_* f(\bar{x})$) is called an upper (resp. lower) regular convexificator of $f$ at $\bar{x}$. Note that, if function $f$ admits an upper regular convexificator at $\bar{x}$, then it also admits an upper semi-regular convexificator at $\bar{x}$. Hence, it admits an upper convexificator at $\bar{x}$.

**Definition 2.3.** [4] The Clarke generalized directional derivative of $f$ at $\bar{x}$, with respect to a direction $\upsilon$, is defined as

$$f^0(\bar{x};\upsilon) = \limsup_{x\to\bar{x}, t\downarrow 0} \frac{f(x+t\upsilon)-f(x)}{t}.$$

The Clarke subdifferential of $f$ at $\bar{x}$ is

$$\partial f(\bar{x}) = \{\xi \in X^* : \langle \xi, v\rangle \leqslant f^0(\bar{x};v), \ \forall v \in X\}.$$

For a locally Lipschitz function $f$ at $\bar{x}$, $\partial f(\bar{x})$ is a convexificator of $f$ at $\bar{x}$ (see also [10]).

**Definition 2.4.** The Michel–Penot directional derivative of $f$ at $\bar{x}$ in a direction $v \in X$ is defined as

$$f^\diamondsuit(\bar{x};v) = \sup_{w\in X} \limsup_{x\downarrow 0} \frac{f(\bar{x}+t(v+w))-f(\bar{x}+tw)}{t}.$$

The Michel–Penot subdifferential of $f$ at $\bar{x}$ is

$$\partial^\diamondsuit f(\bar{x}) = \{\xi \in X^* : \langle \xi, v\rangle \leqslant f^\diamondsuit(\bar{x};v), \ \forall v \in X\}.$$

For a locally Lipschitz function $f$ at $\bar{x}$, $\partial^\diamondsuit f(\bar{x})$ is a convexificator of $f$ at $\bar{x}$. The convex hull of a convexificator of a locally Lipschitz function may be strictly contained in both the Clarke and Michel-Penot subdifferential.

**Example 2.1.** [10] Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = |x|-|y|$. Then, it can easily be verified that $\partial^* f(0) = \{(1,-1),(-1,1)\}$ is a convexificator of $f$ at 0, whereas $\partial^\diamondsuit f(0) = \partial f(0) = \text{conv}\{(1,1),(-1,1),(1,-1),(-1,-1)\}$. It is worth noting that $\text{conv}\partial^* f(0) \subset \partial^\diamondsuit f(0) = \partial f(0)$, where *conv* stands for the convex hull.

If $f$ is a convex function on $\mathbb{R}^n$, then the subdifferential of $f$ at $\bar{x}$ is defined as

$$\partial_C f(\bar{x}) := \left\{\xi \in X^* : \langle \xi, x-\bar{x}\rangle \leqslant f(x)-f(\bar{x}), \ \forall x \in \mathbb{R}^n\right\}.$$

[19, Proposition 7.3.9] pointed that if $f$ is convex on $\mathbb{R}^n$ and locally Lipschitz at $\bar{x} \in \mathbb{R}^n$, then

$$\partial_C f(\bar{x}) = \partial f(\bar{x}) = \partial^\diamondsuit f(\bar{x}).$$

Recall [4] that the Clarke tangent cone to a set $C \subseteq \mathbb{R}^n$ at $\bar{x} \in C$ is defined as

$$T(C;\bar{x}) := \{v \in \mathbb{R}^n : \forall x_n \in C, x_n \to \bar{x}, \forall t_n \downarrow 0, \exists v_n \to v \text{ such as } x_n + t_n v_n \in C, \forall n\}.$$

The Clarke normal cone to $C$ at $\bar{x}$ is

$$N(C;\bar{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, v\rangle \leqslant 0, \forall v \in T(C;\bar{x})\}.$$

For a set $D \subseteq \mathbb{R}^n$, the dual cone of cone $D$ is

$$D^* = \{\xi \in \mathbb{R}^n : \langle \xi, v\rangle \geqslant 0, \forall v \in D\}.$$

Note that the cones $T(C;\bar{x})$ and $N(C;\bar{x})$ are nonempty convex, and $T(C;\bar{x})$ and $N(C;\bar{x})$ are closed.

The following chain rule in [10] for composite functions is needed in the proof of Theorem 3.1 in Section 3.

**Proposition 2.1.** *Let $f=(f_1,...,f_p)$ be a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^p$, and $g$ be continuous function from $\mathbb{R}^p$ to $\mathbb{R}$. Suppose that, for each $i=1,...,p$, $f_i$ admits a bounded convexificator $\partial^* f_i(\bar{x})$ at $\bar{x}$, and $g$ admits a bounded convexificator $\partial^* g(f(\bar{x}))$ at $f(\bar{x})$. For each $i=1,2,...,p$, if $\partial^* f_i$ is upper semicontinuous at $\bar{x}$ and $\partial^* g$ is upper semicontinuous at $f(\bar{x})$, then the set*

$$\partial^*(g\circ f)(\bar{x}) = \partial^* g(f(\bar{x}))(\partial^* f_1(\bar{x}),...,\partial^* f_p(\bar{x}))$$

*is a convexificator of $g\circ f$ at $\bar{x}$.*

## 3. Necessary efficiency conditions of Fritz John type

Let $F:=(F_1,...,F_{r_1}):\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{r_1}$, $f:=(f_1,...,f_{r_2}):\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{r_2}$, $G:=(G_1,...,G_{m_1}):\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{m_1}$ and $g:=(g_1,...,g_{m_2}):\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{m_2}$, where $n_i$, $m_i$, $r_i, i=1,2$ are integers, with $n_i, r_i\geq 1$, $m_i\geq 0$. Let $Q_1$ be a pointed closed convex cone in $\mathbb{R}^{r_1}$ with $\operatorname{int} Q_1\neq\emptyset$. Denote by $J_i:=\{1,\dots,r_i\}\,(i=1,2)$.

Consider the following multiobjective bilevel programming problem:

$$(P)\qquad \begin{aligned} &\min_{x,y} F(x,y),\\ \text{s.t.}\quad &G_i(x,y)\leq 0\,(\forall i\in J_1),\\ &y\in S(x),\end{aligned}$$

where, for each $x\in\mathbb{R}^{n_1}, S(x)$ is the solution set of the following parametric multiobjective programming problem:

$$\min_y f(x,y)\quad \text{s.t.}\quad g_j(x,y)\leq 0\,(\forall j\in J_2).$$

Denote by $M$ the feasible set of Problem (P). Thus (P) is a sequence of two multiobjective optimization problems in which the feasible region of upper-level multiobjective optimization problem $(P_1)$ is defined implicitly by the solution set of lower-level multiobjective optimization problem $(P_2)$, where $(P_1)$ and $(P_2)$ are of the forms

$$(P_1)\qquad \begin{aligned} &\min_{x,y} F(x,y),\\ \text{s.t.}\quad &G_i(x,y)\leq 0\,(\forall i\in J_1),\\ &y\in S(x),\end{aligned}$$

$$(P_2)\qquad \begin{aligned} &\min_{y} f(x,y),\\ \text{s.t.}\quad &g_j(x,y)\leq 0\,(\forall j\in J_2).\end{aligned}$$

In this paper, we focus the notion of solutions for (P) as follows: weakly efficient solutions in multiobjective optimization problem $(P_1)$ are considered, which are defined with respect to cone $\operatorname{int} Q_1$, while efficient solutions of $(P_2)$ are considered with respect to cone $Q_2=\mathbb{R}^{r_2}_+$. Such efficient solutions of (P) are called solutions of (P).

We also focus on the optimistic approach, which means that the leader presuppose the cooperation of the follower in the sense that the follower will choose in each times an optimal solution, which is a best one from the leader's point of view. In this case, the optimal solution of lower-level problem $(P_2)$ is uniquely defined for all $x \in \mathbb{R}^{n_1}$.

In case $r_1 = r_2 = 1$, by Dempe, Dutta and Mordukhovich [7], Problem (P) can be reformulated as the following single level programming problem:

$$
(SP1) \qquad \begin{aligned} &\min_{x,y} F(x,y), \\ \text{s.t.} \quad & f(x,y) - V(x) \le 0, \\ & g_j(x,y) \le 0\,(\forall j \in J_2), \\ & G_i(x,y) \le 0\,(\forall i \in J_1), \end{aligned}
$$

where $V(x)$ is the optimal value function of (SP1)

$$
V(x) := \min_y \{ f(x,y) : g_j(x,y) \le 0, j \in J_2, y \in \mathbb{R}^{n_2} \}.
$$

In what follows, we say that $(\bar{x},\bar{y})$ is a solution of (P) in the sense that $(\bar{x},\bar{y})$ is a weakly efficient solution of $(P_1)$, while for any $x$, $\bar{y}(x)$ is an efficient solution of $(P_2)$ with $\bar{y}(\bar{x}) = \bar{y}$.

We are now in a position to formulate a necessary optimality condition for multiobjective bilevel programming problem (P).

**Theorem 3.1.** *Let $(\bar{x},\bar{y})$ be a solution of (P). Assume that there exists a neighborhood $U$ of $(\bar{x},\bar{y})$ such that the functions $F_k, f_l, G_i, g_j$ are continuous on $U$ and admit bounded convexificators $\partial^* F_k(\bar{x},\bar{y}), \partial^* f_l(\bar{x},\bar{y})$, $\partial^* G_i(\bar{x},\bar{y}), \partial^* g_j(\bar{x},\bar{y})$ at $(\bar{x},\bar{y})$. Suppose also that $\partial^* F_1, \dots, \partial^* F_{r_1}$ are upper semicontinuous at $(\bar{x},\bar{y})$. Then, for every $s \in J_2$, there exist $\lambda := (\lambda_1, \dots, \lambda_{r_1}) \in Q_1^*$, $\theta \ge 0$, $\gamma_s \ge 0$, $\mu := (\mu_1, ..., \mu_{m_1}) \in \mathbb{R}_+^{m_1}$, $\nu := (\nu_1, ..., \nu_{m_2}) \in \mathbb{R}_+^{m_2}$, $\sigma := (\sigma_1, \dots, \sigma_{s-1}, \sigma_{s+1}, \dots, \sigma_{r_2}) \in \mathbb{R}_+^{r_2 - 1}$ such that $(\lambda, \theta) \ne 0$, $(\mu, \nu, \sigma, \gamma_s) \ne 0$, and*

$$
\begin{aligned}
(0,0) \in \sum_{k \in J_1} \lambda_k conv\, \partial^* F_k(\bar{x},\bar{y}) + \theta \Big[ & \gamma_s (conv\, \partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) \\
& + \sum_{i=1}^{m_1} \mu_i conv\, \partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j conv\, \partial^* g_j(\bar{x},\bar{y}) + \sum_{l \in J_2, l \ne s} \sigma_l conv\, \partial^* \widetilde{f_l}(\bar{x},\bar{y}) \Big], \qquad (3.1) \\
& \sum_{i=1}^{m_1} \mu_i G_i(\bar{x},\bar{y}) = 0 \quad \text{and} \quad \sum_{j=1}^{m_2} \nu_j g_j(\bar{x},\bar{y}) = 0,
\end{aligned}
$$

*where*

$$
\begin{aligned}
&\widetilde{f_l}(x,y) = f_l(x,y) - f_l(x,\bar{y})\,(l \in J_2, l \ne s), \\
&V_s(x) := \min_y \{ f_s(x,y) : g_j(x,y) \le 0 (j = 1, \dots, m_2), \widetilde{f_l}(x,y) \le 0 (l \in J_2, l \ne s), y \in \mathbb{R}^{n_2} \}, \\
&\partial^* V_s(\bar{x}) = conv\, \{ \partial^* f_s(.,y)(\bar{x}) : y \in J_s(\bar{x}) \}, \\
&J_s(\bar{x}) = \{ y \in \mathbb{R}^{n_2} : g_j(\bar{x},y) \le 0 (j = 1, \dots, m_2), \widetilde{f_l}(\bar{x},y) \le 0 (l \in J_2, l \ne s), f_s(\bar{x},y) = V_s(\bar{x}) \}.
\end{aligned}
$$

*Proof.* Since $\bar{y}(x)$ is an efficient solution of $(P_2)$, for any $s \in J_2$, it is a solution of the following problem:

$$
\begin{aligned}
(P_s) \qquad & \min_y f_s(x,y), \\
\text{s.t.} \quad & f_l(x,y) \leq f_l(x,\bar{y}) \ (\forall l \in J_2, l \neq s), \\
& g(x,y) \leq 0.
\end{aligned}
$$

We now invoke the scalarization theorem by Gong [9, Theorem 3.1] to Problem $(P)$ to deduce that there exists a continuous positively homogeneous subadditive function $\Lambda$ on $\mathbb{R}^{r_1}$ satisfying

$$z_2 - z_1 \in \operatorname{int} Q_1 \Longrightarrow \Lambda(z_1) < \Lambda(z_2)$$

and

$$(\Lambda \circ (F - F(\bar{x},\bar{y}))(x,y) \geq 0 \, (\forall (x,y) \in M).$$

Hence, $(\bar{x},\bar{y})$ is a solution of the following problem:

$$(SP_s) \qquad \min_{x,y} (\Lambda \circ F)(x,y) \quad \text{s.t.} \quad G(x,y) \leq 0, y \in S_s(x),$$

where $S_s(x)$ is the set of solutions of $P_s$. Taking account of [1, Theorem 1] to Problem $(SP_s)$, it follows that, for any $s \in J_2$, there exist $\theta \geq 0, \theta_1 \geq 0$, $\gamma_s \geq 0$, $\mu = (\mu_1,\dots,\mu_{m_1}) \in \mathbb{R}^{m_1}_+$, $\nu = (\nu_1,\dots,\nu_{m_2}) \in \mathbb{R}^{m_2}_+$, $\sigma := (\sigma_1,\dots,\sigma_{s-1},\sigma_{s+1},\dots,\sigma_{r_2}) \in \mathbb{R}^{r_2-1}_+$ such that $(\theta,\theta_1) \neq 0$, $(\mu,\nu,\sigma,\gamma_s) \neq 0$, and

$$
\begin{aligned}
(0,0) \in \theta_1 \operatorname{conv}\partial^*(\Lambda \circ F)(\bar{x},\bar{y}) + \theta\Big[&\gamma_s(\operatorname{conv}\partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) \\
&+ \sum_{i=1}^{m_1} \mu_i \operatorname{conv}\partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j \operatorname{conv}\partial^* g_j(\bar{x},\bar{y}) + \sum_{l \in J_2, l \neq s} \sigma_l \operatorname{conv}\partial^* \widetilde{f}_l(\bar{x},\bar{y})\Big], \qquad (3.2) \\
&\sum_{i=1}^{m_1} \mu_i G_i(\bar{x},\bar{y}) = 0 \quad \text{and} \quad \sum_{j=1}^{m_2} \nu_j g_j(\bar{x},\bar{y}) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
&\widetilde{f}_l(x,y) = f_l(x,y) - f_l(x,\bar{y}) \, (l \in J_2, l \neq s), \\
&V_s(x) := \min_y \{f_s(x,y) : g_j(x,y) \leq 0 (j = 1,\dots,m_2), \widetilde{f}_l(x,y) \leq 0 (l \in J_2, l \neq s), y \in \mathbb{R}^{n_2}\}, \\
&\partial^* V_s(\bar{x}) = \operatorname{conv}\{\partial^* f_s(.,y)(\bar{x}) : y \in J_s(\bar{x})\}, \\
&J_s(\bar{x}) = \{y \in \mathbb{R}^{n_2} : g_j(\bar{x},y) \leq 0 (j = 1,\dots,m_2), \widetilde{f}_l(\bar{x},y) \leq 0 \, (l \in J_2, l \neq s), f_s(\bar{x},y) = V_s(\bar{x})\}.
\end{aligned}
$$

We now check hypotheses of the chain rule by Jeyakumar-Luc (Proposition 2.1) to the composite function $(\Lambda \circ F)(x,y)$. Since the function $\Lambda$ is continuous convex, we can apply [4, Proposition 2.2.6] to deduce that it is locally Lipschitz. Hence, its subdifferential $\partial_C \Lambda(F(x,y))$ is a bounded convexificator of $\Lambda$ at $F(\bar{x},\bar{y})$. Since the function $\Lambda$ is convex and locally Lipschitz, [19, Proposition 7.3.9] pointed that

$$\partial_C \Lambda(F(\bar{x},\bar{y})) = \partial \Lambda(F(\bar{x},\bar{y})).$$

Moreover, due to [10, Proposition 5.1] on the chain rule, we claim that the set

$$\partial_C \Lambda(F(\bar{x},\bar{y}))(\partial^* F_1(\bar{x},\bar{y}),\dots,\partial^* F_{r_1}(\bar{x},\bar{y}))$$

is a convexificator of $\Lambda \circ F$ at $(\bar{x}, \bar{y})$. It follows from (3.2) that

$$
\begin{aligned}
(0,0) \in \theta_1\, \partial_C \Lambda(F(\bar{x},\bar{y}))(\partial^* F_1(\bar{x},\bar{y}), \ldots, \partial^* F_{r_1}(\bar{x},\bar{y})) + \theta \Big[ \gamma_s(\operatorname{conv} \partial^* f_s(\bar{x},\bar{y}) \\
- \partial^* V_s(\bar{x}) \times \{0\}) + \sum_{i=1}^{m_1} \mu_i \operatorname{conv} \partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j \operatorname{conv} \partial^* g_j(\bar{x},\bar{y}) \\
+ \sum_{l \in J_2, l \neq s} \sigma_l \operatorname{conv} \partial^* \widetilde{f}_l(\bar{x},\bar{y}) \Big].
\end{aligned} \tag{3.3}
$$

By using (3.3), we see that there exists $\chi \in \partial_C \Lambda(F(\bar{x},\bar{y})) \subset \mathbb{R}^{r_1}$ such that

$$
\begin{aligned}
(0,0) \in \theta_1 \chi(\partial^* F_1(\bar{x},\bar{y}), \ldots, \partial^* F_{r_1}(\bar{x},\bar{y})) + \theta \Big[ \gamma_s(\operatorname{conv} \partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) \\
+ \sum_{i=1}^{m_1} \mu_i \operatorname{conv} \partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j \operatorname{conv} \partial^* g_j(\bar{x},\bar{y}) + \sum_{l \in J_2, l \neq s} \sigma_l \operatorname{conv} \partial^* \widetilde{f}_l(\bar{x},\bar{y}) \Big].
\end{aligned} \tag{3.4}
$$

Observe that $\chi \in Q^* \setminus \{0\}$. Indeed, for any $z \in \operatorname{int} Q^*$, it can be written $0 - (-z) \in \operatorname{int} Q^*$. For $z_2 = F(\bar{x},\bar{y})$, and $z_1 = F(\bar{x},\bar{y}) - z$, one has $z_2 - z_1 = z \in \operatorname{int} Q$. Hence, $\Lambda(F(\bar{x},\bar{y}) - z) < \Lambda(F(\bar{x},\bar{y})$. Consequently,

$$
\begin{aligned}
\langle \chi, -z \rangle &= \langle \chi, F(\bar{x},\bar{y}) - z - F(\bar{x},\bar{y}) \rangle \\
&\le \Lambda(F(\bar{x},\bar{y}) - z) - \Lambda(F(\bar{x},\bar{y}) \\
&< 0.
\end{aligned}
$$

Hence,

$$
\langle \chi, z \rangle > 0 \ (\forall z \in \operatorname{int} Q),
$$

which implies $\chi \in Q^* \setminus \{0\}$. Putting $\lambda = \theta_1 \chi$, one has $\lambda = (\lambda_1, \ldots, \lambda_{r_1}) \in Q^* \subset \mathbb{R}^{r_1}$ with $(\lambda, \theta) \neq 0$. It follows from (3.4) that

$$
\begin{aligned}
(0,0) \in \lambda(\partial^* F_1(\bar{x},\bar{y}), \ldots, \partial^* F_{r_1}(\bar{x},\bar{y})) \\
+ \theta \Big[ \gamma_s(\operatorname{conv} \partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) + \sum_{i=1}^{m_1} \mu_i \operatorname{conv} \partial^* G_i(\bar{x},\bar{y}) \\
+ \sum_{j=1}^{m_2} \nu_j \operatorname{conv} \partial^* g_j(\bar{x},\bar{y}) + \sum_{l \in J_2, l \neq s} \sigma_l \operatorname{conv} \partial^* \widetilde{f}_l(\bar{x},\bar{y}) \Big].
\end{aligned} \tag{3.5}
$$

By virtue of (3.5), it holds that

$$
\begin{aligned}
(0,0) \in \sum_{k=1}^{r_1} \lambda_k \operatorname{conv} \partial^* F_k(\bar{x},\bar{y}) + \theta \Big[ \gamma_s(\operatorname{conv} \partial^* f_s(\bar{x},\bar{y}) \\
- \partial^* V_s(\bar{x}) \times \{0\}) + \sum_{i=1}^{m_1} \mu_i \operatorname{conv} \partial^* G_i(\bar{x},\bar{y}) \\
+ \sum_{j=1}^{m_2} \nu_j \operatorname{conv} \partial^* g_j(\bar{x},\bar{y}) + \sum_{l \in J_2, l \neq s} \sigma_l \operatorname{conv} \partial^* \widetilde{f}_l(\bar{x},\bar{y}) \Big].
\end{aligned}
$$

This completes the proof. $\square$

We illustrated Theorem 3.1 by the following example.

**Example 3.1.** Let $F := (F_1, F_2), f := (f_1, f_2) : \mathbb{R}^2_+ \to \mathbb{R}^2$ be defined as

$$F(x,y) = \begin{cases} (x^2 - x^2 \sin\frac{1}{x} + |\sin y|, -e^x + 1), & \text{if} \quad x \neq 0, \\ (|\sin y|, 0), & \text{if} \quad x = 0, \end{cases}$$

and

$$f(x,y) = (e^x + 2y^2 + \frac{1}{2}y - 1, 2^x - 1 + |y|).$$

Let $G, g : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$G(x) = \begin{cases} -\frac{x}{1+e^{\frac{1}{x}}} - \frac{1}{3}x, & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0, \end{cases}$$

and

$$g(x,y) = y^2 + y.$$

Let $Q_1 = \mathbb{R}^2_+$. Then, for any $x \in \mathbb{R}_+, y = 0$ is an efficient solution of the following convex mutiobjective optimization problem:

$$\min_y \{f(x,y) : g(x,y) \leq 0\},$$

and $(\bar{x}, \bar{y}) = (0,0)$ is a weakly solution of the following problem:

$$\min_{(x,y)} \{F(x,y) : G(x,y) \leq 0, y \in S(x)\}.$$

For $s = 1$, $\bar{y} = 0$ is a solution of the following problem (I):

$$\min_y \{f_1(x,y) : g(x,y) \leq 0, f_2(x,y) \leq f_2(x,\bar{y})\}.$$

For $s = 2$, $y = 0$ is a solution of the following problem (II):

$$\min_y \{f_2(x,y) : g(x,y) \leq 0, f_1(x,y) \leq f_1(x,\bar{y})\}.$$

The value functions for Problem (I) is $V_1(x) = e^x - 1$, and for Problem (II) is $V_2(x) = 2^x - 1$ and $S_1(x) = S_2(x) = \{0\}$ for any $x \in \mathbb{R}_+$. We have $M = \{(x,y) \in \mathbb{R}^2 : G(x,y) \leq 0, y \in S(x)\} = \{(x,0) : x \geq 0\}$. It can be seen that

$$\begin{aligned}
&\partial^* F_1(0,0) = \{(1,1), (-1,1), (1,-1), (-1,-1)\}, \\
&\partial^* F_2(0,0) = \{(-1,0)\}, \\
&\partial^* f_1(0,0) = \{(1,1/2)\}, \\
&\partial^* f_2(0,0) = \{(\ln 2, -1), (\ln 2, 1)\}, \\
&\partial^* G(0,0) = \{(-4/3,0)\}, (-1/3,0)\}, \\
&\partial^* g(0,0) = \{(0,1)\}, \\
&\partial^* V_1(0) = \{1\}, \\
&\partial^* V_2(0) = \{\ln 2\}.
\end{aligned}$$

Necessary condition (3.1) holds with $\lambda_1 = \lambda_2 = 1$, $\theta = \mu = \nu = 0$, $\gamma_1 = 1/2$, and $\gamma_2 = 1/2$. Note that necessary condition (3.1) involves two inclusions with $\gamma_1 = 1/2$ and $\gamma_2 = 1/2$.

We remak here that Theorem 3.1 is a significant extension of Theorem 1 in [1].

## 4. Necessary optimality conditions of Karush–Kuhn–Tucker type

To derive Karush–Kuhn–Tucker necessary conditions, we introduce some constraint qualifications in this section.

**Definition 4.1.** The problem (P) is said to satisfy the constraint qualification (CQ) at $(\bar{x},\bar{y}) \in M$ if, for each $s \in J_2$, $\gamma_s \geq 0$, $\mu = (\mu_1,...,\mu_{m_1}) \in \mathbb{R}^{m_1}_+, \nu = (\nu_1,...,\nu_{m_2}) \in \mathbb{R}^{m_2}_+$ $\sigma_l (l \in J_2, l \neq s)$ not all zero,

$$(0,0) \notin \gamma_s(\operatorname{conv}\partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) \\ + \sum_{i=1}^{m_1} \mu_i \operatorname{conv}\partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j \operatorname{conv}\partial^* g_j(\bar{x},\bar{y}) + \sum_{l\in J_2, l\neq s} \sigma_l \operatorname{conv}\partial^* \widetilde{f_l}(\bar{x},\bar{y}),$$

where $\widetilde{f_l}(x,y) = f_l(x,y) - f_l(x,\bar{y})\,(l \in J_2, l \neq s)$.

A Karush-Kuhn-Tucker necessary condition for solutions of (P) can be stated as follows.

**Theorem 4.1.** *Let $(\bar{x},\bar{y})$ be a solution of (P). Assume all the hypotheses of Theorem 3.1 are fulfilled and (CQ) holds. Then, for each $s \in J_2$, there exist $\lambda := (\lambda_1,\dots,\lambda_{r_1}) \in Q_1^* \setminus \{0\}$, $\theta \geq 0$, $\gamma_s \geq 0$, $\mu := (\mu_1,...,\mu_{m_1}) \in \mathbb{R}^{m_1}_+, \nu := (\nu_1,...,\nu_{m_2}) \in \mathbb{R}^{m_2}_+$, $\sigma := (\sigma_1,\dots,\sigma_{s-1},\sigma_{s+1},\dots,\sigma_{r_2}) \in \mathbb{R}^{r_2-1}_+$ such that $||(\mu,\nu,\sigma,\gamma_s)|| = 1$, and*

$$(0,0) \in \sum_{k\in J_1} \lambda_k \mathit{conv}\,\partial^* F_k(\bar{x},\bar{y}) + \theta\Big[\gamma_s(\mathit{conv}\,\partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) \\ + \sum_{i=1}^{m_1} \mu_i \mathit{conv}\,\partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j \mathit{conv}\,\partial^* g_j(\bar{x},\bar{y}) + \sum_{l\in J_2, l\neq s} \sigma_l \mathit{conv}\,\partial^* \widetilde{f_l}(\bar{x},\bar{y})\Big], \tag{4.1}$$
$$\sum_{i=1}^{m_1} \mu_i G_i(\bar{x},\bar{y}) = 0 \quad \mathit{and} \quad \sum_{j=1}^{m_2} \nu_j g_j(\bar{x},\bar{y}) = 0.$$

*Proof.* We invoke Theorem 3.1 to deduce that, for each $s \in J_2$, there exist $\lambda := (\lambda_1,\dots,\lambda_{r_1}) \in Q_1^*$, $\theta \geq 0$, $\gamma_s \geq 0$, $\mu := (\mu_1,..., \mu_{m_1}) \in \mathbb{R}^{m_1}_+, \nu := (\nu_1,...,\nu_{m_2}) \in \mathbb{R}^{m_2}_+$, $\sigma := (\sigma_1,\dots,\sigma_{s-1},\sigma_{s+1}, \dots,\sigma_{r_2}) \in \mathbb{R}^{r_2-1}_+$ such that $(\lambda,\theta) \neq (0,0)$, $||(\mu,\nu,\sigma,\gamma_s)|| = 1$, and

$$(0,0) \in \sum_{k\in J_1} \lambda_k \operatorname{conv}\partial^* F_k(\bar{x},\bar{y}) + \theta\Big[\gamma_s(\operatorname{conv}\partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) \\ + \sum_{i=1}^{m_1} \mu_i \operatorname{conv}\partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j \operatorname{conv}\partial^* g_j(\bar{x},\bar{y}) + \sum_{l\in J_2, l\neq s} \sigma_l \operatorname{conv}\partial^* \widetilde{f_l}(\bar{x},\bar{y})\Big], \tag{4.2}$$
$$\sum_{i=1}^{m_1} \mu_i G_i(\bar{x},\bar{y}) = 0 \quad \text{and} \quad \sum_{j=1}^{m_2} \nu_j g_j(\bar{x},\bar{y}) = 0.$$

If $\lambda = 0$, then $\theta > 0$. It follows from (4.2) that, for each $s \in J_2$,

$$(0,0) \in \gamma_s(\operatorname{conv}\partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) \\ + \sum_{i=1}^{m_1} \mu_i \operatorname{conv}\partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j \operatorname{conv}\partial^* g_j(\bar{x},\bar{y}) + \sum_{l\in J_2, l\neq s} \sigma_l \operatorname{conv}\partial^* \widetilde{f_l}(\bar{x},\bar{y}),$$

which contradicts (CQ). Hence, $\lambda \neq 0$. This completes the proof. $\square$

## 5. SUFFICIENT OPTIMALITY CONDITIONS

Adapting the definitions in Chuong and Kim [5], we give some definitions on $\partial^*$-generalized convexity. Suppose that a function $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ admits a convexificator $\partial^* f(\bar{x},\bar{y})$ at $(\bar{x},\bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

**Definition 5.1.** (i) $f$ is said to be $\partial^*$-invex-infine on $C \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ at $(\bar{x},\bar{y})$ if, for any $(x,y) \in C$, $\xi \in \partial^* f(\bar{x},\bar{y})$, there exists $(u,v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

$$f(x,y) \geq f(\bar{x},\bar{y}) + \langle \xi, (u,v) \rangle.$$

If "$>$" holds in above definition for $(x,y) \neq (\bar{x},\bar{y})$, then $f$ is called strictly $\partial^*$-invex-infine on $C$ at $(\bar{x},\bar{y})$.
(ii) Given $f_1,\dots,f_{r_1} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, the vector-valued function $(f_1,\dots,f_{r_1})$ is said to be $\partial^*$-invex-infine on $C$ at $(\bar{x},\bar{y})$ if, for any $(x,y) \in C$, $\xi_i \in \partial^* f_i(\bar{x},\bar{y})$ $(i = 1,\dots,r_1)$, there exists $(u,v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

$$f_i(x,y) \geq f_i(\bar{x},\bar{y}) + \langle \xi_i, (u,v) \rangle \; (i = 1,\dots,r_1).$$

(iii) $f$ is said to be $\partial^*$-pseudoinvex-infine on $C$ at $(\bar{x},\bar{y})$, for any $(x,y) \in C$, $\xi \in \partial^* f(\bar{x},\bar{y})$, there exists $(u,v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

$$f(x,y) < f(\bar{x},\bar{y}) \Longrightarrow \langle \xi, (u,v) \rangle < 0.$$

(iv) $f$ is said to be $\partial^*$-quasiinvex-infine on $C$ at $(\bar{x},\bar{y})$ if, for any $(x,y) \in C$, $\xi \in \partial^* f(\bar{x},\bar{y})$, there exists $(u,v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

$$f(x,y) \leq f(\bar{x},\bar{y}) \Rightarrow \langle \xi, (u,v) \rangle \leq 0.$$

(v) The function $(f_1,\dots,f_m,f_{m+1},\dots,f_{m+r})$ with $m \geq 1$ is said to be $m-\partial^*$-semiinvex-infine on $C$ at $(\bar{x},\bar{y})$ if the functions $f_1,\dots,f_m$ are $\partial^*$-pseudoinvex-infine on $C$ at $(\bar{x},\bar{y})$, and $f_{m+1},\dots,f_{m+r}$ are $\partial^*$-quasiinvex-infine on $C$ at $(\bar{x},\bar{y})$.

**Remark 5.1.** (a) If $f$ is a convex function, then it is $\partial^*$-invex-infine on $C$ at $(\bar{x},\bar{y})$, as it can be taken $(u,v) := (x,y) - (\bar{x},\bar{y})$, and $\partial^*$ as subdifferential of convex function.
(b) If $f$ is $\partial^*$-pseudoconvex on $C$ at $(\bar{x},\bar{y})$ (see [20]), then it is $\partial^*$-pseudoinvex-infine on $C$ at $(\bar{x},\bar{y})$.
(c) If $f$ is $\partial^*$-quasiconvex on $C$ at $(\bar{x},\bar{y})$ (see [20]), then it is $\partial^*$-quasiinvex-infine on $C$ at $(\bar{x},\bar{y})$.

In what follows, we give a sufficient condition for solutions of Problem (P).

**Theorem 5.1.** *Let $(\bar{x},\bar{y})$ be a feasible point of (P), and $Q_1 = \mathbb{R}^{r_1}_+$. Assume that*
*(i) For each $s \in J_2$, there exist numbers $\lambda_k \geq 0, k = 1,2,..,r_1$ with $\lambda := (\lambda_1,\dots,\lambda_{r_1}) \neq 0, \theta \geq 0, \gamma_s \geq 0, \mu_i \geq 0$, $i = 1,...,m_1, \nu_j \geq 0$, $j = 1,...,m_2$, $\sigma_l \geq 0\,(l \in J_2, l \neq s)$ such that, for all $s \in J_2$,*

$$(a) \qquad (0,0) \in \sum_{k\in J_1} \lambda_k conv\, \partial^* F_k(\bar{x},\bar{y}) + \theta \Big[\gamma_s(conv\, \partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\}) + \sum_{i=1}^{m_1} \mu_i conv\, \partial^* G_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \nu_j conv\, \partial^* g_j(\bar{x},\bar{y}) + \sum_{l\in J_2, l\neq s} \sigma_l conv\, \partial^* \widetilde{f}_l(\bar{x},\bar{y})\Big], \tag{5.1}$$

$(b) \qquad \sum_{i=1}^{m_1} \mu_i G_i(\bar{x},\bar{y}) = 0 \qquad$ *and* $\qquad \sum_{j=1}^{m_2} \nu_j g_j(\bar{x},\bar{y}) = 0.$

*(ii) The function $(F_1,\dots,F_{r_1}, f_s - V_s, \mu_1 G_1,\dots,\mu_{m_1} G_{m_1}, \nu_1 g_1,\dots,\nu_{m_2} g_{m_2}, \sigma_1 \widetilde{f}_1,\dots,\ \sigma_{s-1}\widetilde{f}_{s-1}, \sigma_{s+1}\widetilde{f}_{s+1},\dots,\sigma_{r_2}\widetilde{f}_{r_2})$ is $r_1$-$\partial^*$-semiinvex-infine on $M$ at $(\bar{x},\bar{y})$. Then $(\bar{x},\bar{y})$ is a solution of (P).*

*Proof.* Denote by $\arg(P_2)$ and $\arg(P_s)$ the sets of solutions of Problems $(P_2)$ and $(P_s)$, respectively. We first note that $(x,y)\in\arg(P_2)\Longrightarrow(x,y)\in\arg(P_s)\,(\forall s\in J_2)$. Since $(\bar{x},\bar{y})$ is a feasible point of (P), for every $s\in J_2$, it is also a feasible point of the following problem

$$(SP^1_\lambda)\qquad \min_{x,y}\lambda F(x,y)\quad \text{s.t.}\quad G(x,y)\le 0,\, y\in S_s(x),$$

where $S_s(x)$ is the set of solutions of the following problem

$$(P_s)\qquad \begin{aligned}&\min_y f_s(x,y),\\ &\text{s.t.}\quad f_l(x,y)\le f_l(x,\bar{y})\ (\forall l\in J_2, l\neq s),\\ &\qquad g(x,y)\le 0.\end{aligned}$$

From [7], we have that problem $(SP^1_\lambda)$ is equivalent to the following optimization problem

$$(SP^2_\lambda)\qquad \begin{aligned}&\min_{x,y}(\lambda F)(x,y),\\ &\text{s.t.}\quad G(x,y)\le 0,\, g(x,y)\le 0,\\ &\qquad f_l(x,y)\le f_l(x,\bar{y})\ (\forall l\in J_2, l\neq s),\\ &\qquad f_s(x,y)-V_s(x)\le 0,\\ &\qquad (x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},\end{aligned}$$

where $V_s(x)$ is the optimal value function of problem $(P_s)$. By virtue of (5.1), there exists $\chi_k\in\operatorname{conv}\partial^*F_k(\bar{x},\bar{y}), k=1,...,r_1$, $\chi_s\in\operatorname{conv}\partial^*f_s(\bar{x},\bar{y})-\partial^*V(\bar{x})\times\{0\}$, $\zeta_i\in\operatorname{conv}\partial^*G_i(\bar{x},\bar{y}), i=1,...,m_1$, $\eta_j\in\operatorname{conv}\partial^*g_j(\bar{x},\bar{y})$, $j=1,...,m_2$, and $\xi_l\in\operatorname{conv}\partial^*\widetilde{f_l}(\bar{x},\bar{y}), l\in J_2, l\neq s$ such that

$$0=\sum_{k=1}^{r_1}\lambda_k\chi_k+\theta\gamma_s\chi_s+\theta\sum_{i=1}^{m_1}\mu_i\zeta_i+\theta\sum_{j=1}^{m_2}\nu_j\eta_j+\theta\sum_{l\in J_2,l\neq s}\sigma_l\xi_l. \tag{5.2}$$

Since the function $(F_1,\dots,F_{r_1},f_s-V_s,\mu_1G_1,\dots,\mu_{m_1}G_{m_1},\nu_1g_1,\dots,\nu_{m_2}g_{m_2},\sigma_1\widetilde{f}_1,\dots,\sigma_{s-1}\widetilde{f}_{s-1}$, $\sigma_{s+1}\widetilde{f}_{s+1},\dots,\sigma_{r_2}\widetilde{f}_{r_2})$ is $r_1$-$\partial^*$-invex-infine on $M$ at $(\bar{x},\bar{y})$ for $(x,y)\in M$, $\chi_k\in\operatorname{conv}\partial^*F_k(\bar{x},\bar{y})$, $k=1,2,...,r_1$, $\chi_s\in\operatorname{conv}\partial^*f_s(\bar{x},\bar{y})-\partial^*V_s(\bar{x})\times\{0\}$, $\zeta_i\in\operatorname{conv}\partial^*G_i(\bar{x},\bar{y}), i=1,...,m_1$, $\eta_j\in\operatorname{conv}\partial^*g_j(\bar{x},\bar{y}), j=1,...,m_2$, $\xi_l\in\operatorname{conv}\widetilde{\partial}^*f_l(\bar{x},\bar{y}), l\in J_2, l\neq s$, there exists $(u,v)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ such that

$$\sum_{k=1}^{r_1}\lambda_k(F_k(x,y)-F_k(\bar{x},\bar{y}))\ge\Big\langle\sum_{k=1}^{r_1}\lambda_k\chi_k,(u,v)\Big\rangle, \tag{5.3}$$

$$\sum_{i=1}^{m_1}\mu_i(G_i(x,y)-G_i(\bar{x},\bar{y}))\ge\Big\langle\sum_{i=1}^{m_1}\mu_i\zeta_i,(u,v)\Big\rangle, \tag{5.4}$$

$$\sum_{j=1}^{m_2}\nu_j(g_j(x,y)-g_j(\bar{x},\bar{y}))\ge\Big\langle\sum_{j=1}^{m_2}\nu_j\eta_j,(u,v)\Big\rangle, \tag{5.5}$$

$$\gamma_s\{f_s(x,y)-V_s(x)-(f_s(\bar{x},\bar{y})-V_s(\bar{x}))\}\ge\langle\gamma_s\chi_s,(u,v)\rangle, \tag{5.6}$$

and

$$\sum_{l\in J_2,l\neq s}\sigma_l(\widetilde{f_l}(x,y)-\widetilde{f_l}(\bar{x},\bar{y}))\ge\Big\langle\sum_{l\in J_2,l\neq s}\sigma_l\xi_l,(u,v)\Big\rangle. \tag{5.7}$$

On the other hand, it follows from (5.2) that

$$\begin{aligned}\Big\langle \sum_{k=1}^{r_1} \lambda_k \chi_k, (u,v)\Big\rangle = -\big\langle \theta\gamma_s\chi_s, (u,v)\big\rangle - \Big\langle \theta \sum_{i=1}^{m_1} \mu_i\zeta_i, (u,v)\Big\rangle \\ - \Big\langle \theta \sum_{j=1}^{m_2} \nu_j\eta_j, (u,v)\Big\rangle - \Big\langle \theta \sum_{l\in J_2, l\neq s} \sigma_l\xi_l, (u,v)\Big\rangle.\end{aligned} \tag{5.8}$$

Combining (5.3) and (5.8) yields that

$$\begin{aligned}\sum_{k=1}^{r_1} \lambda_k(F_k(x,y)-F_k(\bar{x},\bar{y})) \geq -\big\langle \theta\gamma_s\chi_s, (u,v)\big\rangle - \Big\langle \theta \sum_{i=1}^{m_1} \mu_i\zeta_i, (u,v)\Big\rangle \\ - \Big\langle \theta \sum_{j=1}^{m_2} \nu_j\eta_j, (u,v)\Big\rangle - \Big\langle \theta \sum_{l\in J_2, l\neq s} \sigma_l\xi_l, (u,v)\Big\rangle.\end{aligned} \tag{5.9}$$

Combining (5.4)-(5.7) and (5.9) gives that

$$\begin{aligned}\sum_{k=1}^{r_1} \lambda_k(F_k(x,y)-F_k(\bar{x},\bar{y})) \geq \theta\gamma_s(f_s(\bar{x},\bar{y})-V_s(\bar{x}))-(f_s(x,y)-V_s(x)) \\ +\theta\sum_{i=1}^{m_1}\mu_i(G_i(\bar{x},\bar{y})-G_i(x,y))+\theta\sum_{j=1}^{m_2}\nu_j(g_j(\bar{x},\bar{y})-g_j(x,y)) \\ +\theta\sum_{l\in J_2, l\neq s}\sigma_l(\widetilde{f}_l(\bar{x},\bar{y})-\widetilde{f}_l(x,y)).\end{aligned}$$

Moreover, by condition $(b)$, one gets

$$\mu_i G_i(\bar{x},\bar{y}) = 0\,(i=1,\ldots,m_1) \quad \text{and} \quad \nu_j g_j(\bar{x},\bar{y}) = 0\,(j=1,\ldots,m_2). \tag{5.10}$$

Due to the fact that $(x,y)$ is feasible of $(SP^2_\lambda)$ and (5.10), we obtain

$$\sum_{k=1}^{r_1} \lambda_k(F_k(x,y)-F_k(\bar{x},\bar{y})) \geq 0,$$

which is equivalent to the following

$$\lambda F(x,y) \geq \lambda F(\bar{x},\bar{y}).$$

Hence, $(\bar{x},\bar{y})$ is a solution of Problem $(SP^2_\lambda)$. So, it is a solution of $(SP^1_\lambda)$ for any $s\in J_2$. Consequently, $(\bar{x},\bar{y})$ is a solution of $(P)$. □

Set

$$I_1 := \{i\in\{1,...,m_1\} : G_i(\bar{x},\bar{y})=0\},$$

$$I_2 := \{j\in\{1,...,m_2\} : g_j(\bar{x},\bar{y})=0\},$$

and $|I_i|$ is the capacity of $I_i$, $i=1,2$.

In case that $Q_1 \neq \mathbb{R}^{r_1}_+$, we obtain the following sufficient condition for solutions of (P).

**Theorem 5.2.** *Let $(\bar{x},\bar{y})$ be a feasible point of (P). Assume that*
*(i) For every $s \in J_2$, there exist $\lambda := (\lambda_1,\dots,\lambda_{r_1}) \in Q_1^* \setminus \{0\}, \gamma_s \geq 0, \mu_i \geq 0$ $i = 1,...,m_1, \nu_j \geq 0, j = 1,...,m_2$, $\sigma_l \geq 0\,(l \in J_2, l \neq s)$, $\theta \geq 0$ such that*

$$\begin{aligned}&(0,0) \in conv\,\partial^*\lambda F(\bar{x},\bar{y}) + \theta\Big[\gamma_s(conv\,\partial^* f_s(\bar{x},\bar{y}) - \partial^* V_s(\bar{x}) \times \{0\})\\&+\sum_{i\in I_1}\mu_i conv\,\partial^* G_i(\bar{x},\bar{y}) + \sum_{j\in I_2}\nu_j conv\,\partial^* g_j(\bar{x},\bar{y}) + \sum_{l\in J_2, l\neq s}\sigma_l conv\,\partial^*\widetilde{f_l}(\bar{x},\bar{y})\Big].\end{aligned} \tag{5.11}$$

*(ii) The function $(\lambda F, f_s - V_s, \widetilde{f}_1,\dots,\widetilde{f}_{s-1},\widetilde{f}_{s+1},\dots,\widetilde{f}_{r_2}, G_1,\dots,G_{|I_1|}, g_1,\dots,g_{|I_2|})$ is 1-$\partial^*$-semiinvex-infine on M at $(\bar{x},\bar{y})$. Then $(\bar{x},\bar{y})$ is a solution of (P).*

*Proof.* Suppose that $(\bar{x},\bar{y})$ is not an efficient solution of (P). Hence, there exists $(\hat{x},\hat{y}) \in M$ such that

$$F(\hat{x},\hat{y}) - F(\bar{x},\bar{y}) \in -\mathrm{int}Q_1.$$

Since $\lambda \in Q_1^*$, it follows from this that

$$\lambda F(\hat{x},\hat{y}) - \lambda F(\bar{x},\bar{y}) < 0.$$

Observe that $(\hat{x},\hat{y})$ is an efficient solution of the following problem

$$(P_2) \qquad \min_y f(x,y), \quad \text{s.t.} \quad g_j(x,y) \leq 0\,(\forall j \in J_2).$$

Hence, for any $s \in J_2$, it is also an solution of the following problem

$$(P_s) \qquad \begin{aligned}&\min_y f_s(x,y),\\ \text{s.t.}\quad & f_l(x,y) \leq f_l(x,\bar{y})\ (\forall l \in J_2, l \neq s),\\ & g(x,y) \leq 0.\end{aligned}$$

Consequently, $(\bar{x},\bar{y})$ is not a solution of the following problem

$$(SP_\lambda^1) \qquad \min_{x,y} \lambda F(x,y) \quad \text{s.t.} \quad G(x,y) \leq 0, y \in S_s(x),$$

where $S_s(x)$ is the set of solutions of the following problem

$$(P_s) \qquad \begin{aligned}&\min_y f_s(x,y),\\ \text{s.t.}\quad & f_l(x,y) \leq f_l(x,\bar{y})\ (\forall l \in J_2, l \neq s),\\ & g(x,y) \leq 0.\end{aligned}$$

From [7], problem $(SP_\lambda^1)$ is equivalent to problem $(SP_\lambda^2)$. Hence, $(\bar{x},\bar{y})$ is not a solution of problem $(SP_\lambda^2)$

$$\begin{aligned}&\min_{x,y} (\lambda F)(x,y),\\ \text{s.t.}\quad & G(x,y) \leq 0, g(x,y) \leq 0,\\ & f_l(x,y) \leq f_l(x,\bar{y})\ (\forall l \in J_2, l \neq s),\\ & f_s(x,y) - V_s(x) \leq 0,\\ & (x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},\end{aligned}$$

Therefore, there exists $(x_o, y_o) \in M$ such that

$$\lambda F(x_o, y_o) < \lambda F(\bar{x}, \bar{y}).$$

On the other hand, we have that there exists $\chi \in \operatorname{conv} \partial^* \lambda F(\bar{x}, \bar{y})$, $\chi_s \in \operatorname{conv} \partial^* f_s(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}$, $\zeta_i \in \operatorname{conv} \partial^* G_i(\bar{x}, \bar{y}), i = 1, ..., m_1$, $\eta_j \in \operatorname{conv} \partial^* g_j(\bar{x}, \bar{y}), j = 1, ..., m_2$, and $\chi_l \in \operatorname{conv} \partial^* f_l(\bar{x}, \bar{y}) (l \in J_2, l \neq s)$ such that

$$0 = \chi + \theta \gamma \chi_s + \theta \sum_{i \in I_1} \mu_i \zeta_i + \theta \sum_{j \in I_2} \nu_j \eta_j + \theta \sum_{l \in J_2, l \neq s} \sigma_l \chi_l. \tag{5.12}$$

From the assumption, the function $(\lambda F, f_s - V_s, \widetilde{f}_1, \ldots, \widetilde{f}_{s-1}, \widetilde{f}_{s+1}, \ldots, \widetilde{f}_{r_2}, G_1, \ldots, G_{|I_1|}, g_1, \ldots, g_{|I_2|})$ is 1-$\partial^*$-semiinvex-infine on $M$ at $(\bar{x}, \bar{y})$. Hence, the function $\lambda F$ is $\partial^*$-pseudoinvex-infine on $M$ at $(\bar{x}, \bar{y})$ and the functions $f_s - V_s, \widetilde{f}_1, \ldots, \widetilde{f}_{s-1}, \widetilde{f}_{s+1}, \ldots, \widetilde{f}_{r_2}, G_1, \ldots, G_{|I_1|}, g_1, \ldots, g_{|I_2|}$ are $\partial^*$-quasiinvex-infine on $M$ at $(\bar{x}, \bar{y})$. For $(x_o, y_o) \in M$, by the $\partial^*$-pseudoinvexity-infine of $\lambda F$, it follows from (5.12) that there exists $(u, v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

$$\langle \chi, (u, v) \rangle < 0. \tag{5.13}$$

Since $(x_o, y_o)$ is feasible for $(SP_\lambda^2)$, we have $f_s(x_o, y_o) - V_s(x_o) \leqslant 0 = f_s(\bar{x}, \bar{y}) - V_s(\bar{x})$. Hence, $\langle \chi_s, (u, v) \rangle \leq 0$, and then

$$\langle \theta \gamma_s \chi_s, (u, v) \rangle \leq 0. \tag{5.14}$$

For each $i \in I_1$, one has $G_i(x_o, y_o) \leq 0 = G_i(\bar{x}, \bar{y})$. By virtue of the $\partial^*$-quasiinvexity-infine of $G_i\,(i \in I_1)$, we deduce that $\langle \zeta_i, (u, v) \rangle \leq 0\ (\forall i \in I_1)$. Therefore,

$$\langle \theta \sum_{i \in I_1} \mu_i \zeta_i, (u, v) \rangle \leq 0, \tag{5.15}$$

as $\mu_i \geq 0\,(\forall i \in I_1)$. For each $j \in I_2$, we have $g_j(x_o, y_o) \leq 0 = g_j(\bar{x}, \bar{y})$. By the $\partial^*$-quasiinvexity-infine of $g_j\,(j \in I_2)$, we obtain that $\langle \eta_j, (u, v) \rangle \leq 0\ (\forall j \in I_2)$. Since $\theta \geq 0$, $\nu_j \geq 0\,(\forall j \in I_2)$, the latter implies that

$$\langle \theta \sum_{j \in I_2} \nu_j \eta_j, (u, v) \rangle \leq 0. \tag{5.16}$$

For each $l \in J_2, l \neq s$, we have $f_l(x_o, y_o) \leq f_l(x, \bar{y})$. By the $\partial^*$-quasiinvexity-infine of $f_l\,(l \in J_2, l \neq s)$, we obtain that $\langle \chi_l, (u, v) \rangle \leq 0\ (\forall l \in J_2, l \neq s)$. Since $\theta \geq 0$, $\sigma_l \geq 0\,(\forall l \in J_2, l \neq s)$, the latter implies that

$$\langle \theta \sum_{l \in J_2, l \neq s} \sigma_l \chi_l, (u, v) \rangle \leq 0. \tag{5.17}$$

Combining (5.13)-(5.17) yields that

$$\langle \chi + \theta \gamma \chi_s + \theta \sum_{i \in I_1} \mu_i \zeta_i + \theta \sum_{j \in I_2} \nu_j \eta_j, (u, v) + \theta \sum_{l \in J_2, l \neq s} \sigma_l \chi_l, (u, v) \rangle < 0,$$

which contradicts (5.12). Hence $(\bar{x}, \bar{y})$ is a solution of (P). □

**Remark 5.2.** In this paper, we studied a nonsmooth optimistic multiobjective bilevel programming problem, where the upper and lower level multiobjective problems involve inequality constraints. Weakly efficient solutions in multiobjective optimization problem $(P_1)$ were considered with respect to cone $\operatorname{int} Q_1$, while efficient solutions of $(P_2)$ with respect to cone $Q_2 = \mathbb{R}_+^{r_2}$ were considered. Necessary conditions for solutions of bilevel programming problems were established via convexificators. Theorem 3.1 obtained in this paper can be viewed as an extension of

[1, Theorem 1] for scalar bilevel programming problems. Under assumptions on the generalized convexity, Karush–Kuhn–Tucker necessary conditions for weakly efficient solutions of the problem become sufficient ones. It should be noted that necessary optimality conditions that are expressed in terms of convexificators can be sharper than those expressed via the Clarke, Michel–Penot and Mordukhovich subdifferentials even for locally Lipschitz funtions (see, e.g., [10, Example 2.1]).

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## REFERENCES

[1] H. Babahadda, N. Gadhi, Necessary optimality conditions for bilevel optimization problems using convexificators, J. Glob. Optim. 34 (2006), 535-549.

[2] J.F. Bard, Optimality conditions for the bilevel programming problem, Nav. Res. Logist. Q. 31 (1984), 13-26.

[3] J.F. Bard, Some properties of the bilevel programming problem, J. Optim. Theory Appl. 68 (1991), 371-378.

[4] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.

[5] T.D. Chuong, D.S. Kim, Optimality conditions and duality in nonsmooth multiobjective optimization problems, Ann. Oper. Res. 217 (2014), 117-136.

[6] S. Dempe, First-order necessary optimality conditions for general bilevel programming problems, J. Optim. Theory Appl. 95 (1997), 735-739.

[7] S. Dempe, J. Dutta, B.S. Mordukhovich, New necessary optimality conditions in optimistic bilevel programming, Optimization 56 (2007), 577-604.

[8] S. Dempe, A necessary and sufficient optimality condition for bilevel programming problem, Optimization 25 (1992), 314-354.

[9] X.H. Gong, Scalarization and optimality conditions of vector equilibrium problems, Nonlinear Anal. 73 (2010), 3598-3612.

[10] V. Jeyakumar, D.T. Luc, Nonsmooth calculus, minimality and monotonicity of convexificators, J. Optim. Theory Appl. 101 (1999), 599-612.

[11] B. Kohli, Optimality conditions for optimistic bilevel programming problem using convexifactors, J. Optim. Theory Appl. 152 (2012), 632-651.

[12] X.F. Li, J.Z. Zhang, Necessary optimality conditions in terms of convexificators in Lipschitz optimization, J. Optim. Theory Appl. 131 (2006), 429-452.

[13] D.V. Luu, Convexficator and necessary conditions for efficient, Optimization 63 (2014), 321-335.

[14] D.V. Luu, Necessary and sufficient conditions for efficiency vis convexificators, J. Optim. Theory Appl. 160 (2014), 510-526.

[15] D.V. Luu, Optimality condition for local efficient solutions of vector equilibrium problems via convexificators and applications, J. Optim. Theory Appl. 171 (2016), 643-665.

[16] P. Michel, J.P. Penot, Calcul sous-differentiel pour des fonctions lipschitziennes et nonlipschitziennes, C. R. Math. Acad. Sci. 12 (1984), 269-272.

[17] B.S. Mordukhovich, Y. Shao, On nonconvex subdifferential calculus in Banach spaces, J. Convex Anal. 2 (1995), 211-228.

[18] J.V. Outrata, On necessary optimality conditions for Stackelberg problems, J. Optim. Theory Appl. 76 (1993), 306-320.

[19] W. Schirotzek, Nonsmooth Analysis, Springer, Berlin Heidelberg New York, 2007.

[20] S.K. Suneja, B. Kohli, Optimality and duality results for bilevel programming problem using convexificators, J. Optim. Theory Appl. 150 (2011), 1-19.

[21] S. Wang, Q. Wang, S. Romano-Rodriguez, Optimality condition and algorithm for linear-quadratic bilevel programs, Optimization 4 (1993), 521-536.

[22] J.J. Ye, X.Y. Ye, Necessary optimality conditions for optimization problems with variational inequality constraint, Math. Oper. Res. 22 (1997), 977-997.
[23] J.J. Ye, D.L. Zhu, Optimality conditions for bilevel programming problems, Optimization 33 (1995), 9-27.