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THE GENERIC STABILITY OF SOLUTIONS FOR VECTOR QUASI-EQUILIBRIUM PROBLEMS ON HADAMARD MANIFOLDS

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Abstract. In this paper, we first revisit two classes of set-valued vector quasi-equilibrium problems on Hadamard manifolds with established existence conditions of solutions. Then, we establish the generic stability of set-valued mappings whose set of essential points of a map is a dense residual subset of a (Hausdorff) metric space of the set-valued maps. As applications, we study generalized vector quasi-variational-like inequalities and vector quasi-optimization problems on Hadamard manifolds.

Keywords. Hadamard manifold; Quasi-variational inclusion; Quasi-equilibrium problem; Generic stability.

1. Introduction

The stability of solution maps is an important topic in optimization theory and applications. Up to now, there have been many results dealing with stability properties for related problems, such as, optimization problems (see [17, 23]), vector variational inequality problems (see [16]), vector equilibrium problems (see [1, 2, 6, 14]), traffic network problems (see [15]). On the other hand, in 1951, Fort [12] was the first one who introduced the notation of an essential solution for fixed points, which means that, for a fixed point x of a mapping h, if each mapping sufficiently near to h has a fixed point arbitrarily near to x, then x is said to be essential. However, it is not true that any continuous mapping has at least one essential fixed point, even though the space has the fixed point property. In 1952, Kinoshita [18] introduced the notion of essential components of the set of fixed points and showed that, for any continuous mapping of the Hilbert cube into itself, there exists at least one essential component of the set of its fixed points. Since then, there have been many authors studying the generic stability for various kinds of the optimization-related problems (see [8, 25, 26, 27] and the references therein).

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It is well-known that there are many important concepts and techniques of optimization and nonlinear analysis in Euclidean spaces, which have been extended to Riemannian manifolds (see, e.g., [21, 24] and the references therein). The development of optimization methods from Euclidean spaces to Riemannian manifolds has given rise to some advantages that some nonconvex and nonsmooth problems can be seen as convex and smooth from the Riemannian geometry point of view (see, e.g., [10, 11, 22]). In 2003, Németh [22] introduced and investigated the existence of solutions for the variational inequality problem on Hadamard manifolds under some suitable conditions. Recently, Colao et al. [9], and Batista, Bento and Ferreira [4] provided the existence of solutions for equilibrium problems and generalized vector equilibrium problems on Hadamard manifolds by using the KKM lemma in the setting of Hadamard manifolds. Very recently, Hung, Köbis and Tam [13] considered the existence of solutions for two classes of set-valued vector quasi-equilibrium problems on Hadamard manifolds by using the Kakutani-Fan-Glicksberg type fixed point theorem with the $C(x)$-strong quasiconcavity-like and $C(x)$-weak quasiconcavity-like in the setting of Hadamard manifolds. They applied to generalized vector quasi-variational-like inequalities and vector quasi-optimization problems. However, to the best of our knowledge, up to now, there is no paper on the generic stability for set-valued vector quasi-equilibrium problems, vector quasi-variational-like inequalities and vector quasi-optimization problems by using the Kakutani-Fan-Glicksberg type fixed point theorem with the $C(x)$-strong quasiconcavity-like and $C(x)$-weak quasiconcavity-like in the setting of Hadamard manifolds. Motivated and inspired by the above works, in this paper, we establish some results on the generic stability for set-valued vector quasi-equilibrium problems. We also apply them to generalized vector quasi-variational-like inequalities and vector quasi-optimization problems.

The rest of the paper is organized as follows. In Section 2, we revisit two classes of set-valued vector quasi-equilibrium problems on Hadamard manifolds which established existence conditions of solutions in [13] and recall definitions for later uses. In Section 3, we discuss the generic stability of solutions for vector quasi-equilibrium problems on Hadamard manifold. Some applications to generalized strong and weak vector quasi-variational-like inequalities and generalized vector optimization problems are given in Section 4 and Section 5, respectively.

2. PRELIMINARIES

First, we revisit our problems. Let $\mathbb{M}$ be a Hadamard manifold and $\mathbb{K}$ be a nonempty closed subset of $\mathbb{M}$. Note that some definitions and known results about Riemannian manifolds and Hadamard manifolds can be found in many introductory books on Riemannian geometry, such as, in [5, 19]. Let E be a nonempty subset of $T\mathbb{M}$ and $B : K \rightrightarrows E$ be a set-valued vector field. Let $\mathbb{Y}$ be a metric vector space, $D : K \rightrightarrows K$, $F : K \times E \times K \rightrightarrows \mathbb{Y}$ be two set-valued mappings and $C : K \rightrightarrows \mathbb{Y}$ be a set-valued mapping such that, for each $x \in K, C(x)$ is a proper, closed and convex cone in $\mathbb{Y}$ with $\text{int}C(x) \neq \emptyset$. We now consider the following set-valued strong vector quasi-equilibrium problem and set-valued weak vector quasi-equilibrium problem on Hadamard manifolds with variable domination structure:

(SSVQEPM): Find $x^* \in K$ and $z^* \in B(x^*)$ such that

$$x^* \in D(x^*) \text{ and } F(x^*, z^*, y) \subset C(x^*), \quad \forall y \in D(x^*).$$

(SWVQEPM): Find $x^* \in K$ and $z^* \in B(x^*)$ such that

$$x^* \in D(x^*) \text{ and } F(x^*, z^*, y) \not\subset -\text{int}C(x^*), \quad \forall y \in D(x^*).$$

For convenience, the solution set of (SSVQEPM) (resp. (SWVQEPM)) is denoted by $\mathscr{S}^s$ (resp. $\mathscr{S}^w$).

Definition 2.1. [24] A set $K \subset \mathbb{M}$ is said to be *geodesic convex* if, for any two distinct points x and y in K, the geodesic joining x to y is contained in K, that is, if $\gamma : [0,1] \to \mathbb{M}$ is a geodesic such that $x = \gamma(0)$ and $y = \gamma(1)$, then $\gamma(t) = \exp_x(t \exp_x^{-1} y) \in K$ for all $t \in [0,1]$.

Definition 2.2. [20] Let $\mathbb{M}$ be a Hadamard manifold, $G : \mathbb{M} \rightrightarrows \mathbb{M}$ be a set-valued mapping and $x_0 \in \mathbb{M}$. Then G is said to be

(i) *upper Kuratowski semicontinuous at* x_0 if, for any sequences $\{x_k\}, \{y_k\} \subset \mathbb{M}$ with each $y_k \in G(x_k)$, the relations $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} y_k = y_0$ imply $y_0 \in G(x_0)$;

(ii) *upper Kuratowski semicontinuous on* $K \subset \mathbb{M}$ if G is upper Kuratowski semicontinuous at every point $x \in K$.

Definition 2.3. [4] Let $\mathbb{M}$ be a Hadamard manifold, $K \subset \mathbb{M}$ be a nonempty set and $\mathbb{Y}$ be a topological vector space. A set-valued mapping $F : K \rightrightarrows \mathbb{Y}$ is said to be *upper semicontinuous on* K if, for each $u_0 \in K$ and any open set V in $\mathbb{Y}$ containing $F(u_0)$, there exists an open neighborhood U of u_0 in K such that $F(u) \subset V$ for all $u \in U$.

Now we recall some notions, which can be found in [3, 7]. Let (X,d) be a metric space. Denote $\mathscr{K}(X), BC(X)$ and $C\mathscr{K}(X)$ all nonempty compact subsets of X, all nonempty bounded closed subsets of X, and all nonempty convex compact subsets of X (if X is a linear metric space), respectively. Let $B_1, B_2 \subset X$ and define

$$H(B_1,B_2) := \max\{H^*(B_1,B_2), H^*(B_2,B_1)\},$$

where $H^*(B_1,B_2) := \sup_{b_1\in B_1} d(b_1,B_2)$ and $d(b_1,B_2) := \inf_{b_2\in B_2} ||b_1 - b_2||$. It is obvious that H is a Hausdorff metric in $\mathscr{K}(X), BC(X)$ and $C\mathscr{K}(X)$, respectively.

Lemma 2.1. *Let K be a nonempty compact subset of $(X, ||.||_X)$, and E be a nonempty compact subset of $(Y, ||.||_Y)$. Let $D : K \rightrightarrows K, B : K \rightrightarrows E$ be continuous set-valued mappings. Assume that, for each $x \in K$, $D(x), B(x)$ are nonempty compact subsets. Then, in K, we have, as $x \to x^*$,*

$$D(x) \xrightarrow{H} D(x^*), \text{ and } B(x) \xrightarrow{H'} B(x^*),$$

where H is a Hausdorff metric in $\mathscr{K}(K)$, and H' is a Hausdorff metric in $\mathscr{K}(E)$.

Lemma 2.2. *Let K be a nonempty compact subset of $(X, ||.||_X)$, and B be a nonempty compact subset of $(Y, ||.||_Y)$. Let $D : K \rightrightarrows K, B : K \rightrightarrows E$ be continuous set-valued mappings with nonempty compact valued. Then D is continuous if and if, for any $x^* \in K, x \to x^*$ implies $D(x) \xrightarrow{H} D(x^*)$, and so is it for B.*

Lemma 2.3. *Let (X,d) be a metric space and H be a Hausdorff metric in X. Then*

(i) $(BC(X),H)$ is complete if and if (X,d) is complete;

(ii) $(\mathscr{K}(X),H)$ is complete if and if (X,d) is complete;

(iii) if X is a linear metric space, then $(C\mathscr{K}(X),H)$ is complete if and if (X,d) is complete.

Lemma 2.4. ([28]) Let $\mathbb{Y}$ be a metric space and let $M, M_n (n = 1,2,...)$ be compact sets in $\mathbb{Y}$. Suppose that, for any open set $O \supset M$, there exists n_0 such that $M_n \subset O, \forall n \geq n_0$. Then, any sequence $\{x_n\}$ satisfying $x_n \in M_n$ has a convergent subsequence with limit in M.

Definition 2.4. ([13, Definition 3.1]) Let $\mathbb{M}$ be a Hadamard manifold and $K \subset \mathbb{M}$ be a nonempty geodesic convex set. Let $\mathbb{Y}$ be a metric vector space and $C : K \rightrightarrows \mathbb{Y}$ be a set-valued mapping such that, for each $x \in K, C(x)$ is a proper, closed and convex cone in $\mathbb{Y}$ with $\operatorname{int}C(x) \neq \emptyset$. Then the set-valued mapping $F : K \rightrightarrows \mathbb{Y}$ is said to be

(i) $C(x)$-*strongly quasiconcave-like* if, for any $x_1, x_2 \in K$ and any $s \in [0,1]$ such that

$$\left.\begin{cases} F(x_1) \subset C(x_1) \\ F(x_2) \subset C(x_2) \end{cases}\right\} \Rightarrow F(\exp_{x_1} s \exp_{x_1}^{-1} x_2) \subset C(\exp_{x_1} s \exp_{x_1}^{-1} x_2);$$

(ii) $C(x)$-*weakly quasiconcave-like* if, for any $x_1, x_2 \in K$ and any $s \in [0,1]$ such that

$$\left.\begin{cases} F(x_1) \not\subset -\operatorname{int}C(x_1) \\ F(x_2) \not\subset -\operatorname{int}C(x_2) \end{cases}\right\} \Rightarrow F(\exp_{x_1} s \exp_{x_1}^{-1} x_2) \not\subset -\operatorname{int}C(\exp_{x_1} s \exp_{x_1}^{-1} x_2).$$

$\underline{\mathscr{H}_1}$: $K \subset \mathbb{M}$ is nonempty compact and geodesic convex.
$\underline{\mathscr{H}_2}$: $D : K \rightrightarrows K$ is a continuous set-valued mapping such that $D(x)$ is nonempty closed and geodesic convex for any $x \in K$.
$\underline{\mathscr{H}_3}$: $E \subset T\mathbb{M}$ is a compact and geodesic convex set, and $B : K \rightrightarrows E$ is an upper Kuratowski semicontinuous vector field such that $B(x)$ is compact and geodesic convex for any $x \in K$.

Theorem 2.1. ([13, Theorem 3.4]) *Let $\mathbb{M}$ be a Hadamard manifold. Suppose that the assumptions $\mathscr{H}_1$–$\mathscr{H}_3$ are satisfied and the following conditions hold:*

(i) *for all $(x,z) \in K \times E$, $F(x,z,x) \subset C(x)$*;
(ii) *for all $(x,z) \in K \times E$, the set $\{y \in K : F(x,z,y) \not\subset C(x)\}$ is geodesic convex*;
(iii) *for all $(z,y) \in E \times K$, $x \mapsto F(x,z,y)$ is strongly $C(x)$-quasiconcave-like*;
(iv) *the set $\{(x,z,y) \in K \times E \times K : F(x,z,y) \subset C(x)\}$ is closed.*

Then, the (SSVQEPM) *has a solution, i.e., there exist $x^* \in K$ and $z^* \in B(x^*)$ such that*

$$x^* \in D(x^*) \text{ and } F(x^*, z^*, y) \subset C(x^*), \quad \forall y \in D(x^*),$$

Moreover, the solution set of (SSVQEPM) *is a compact set.*

Theorem 2.2. ([13, Theorem 3.9]) *Let $\mathbb{M}$ be a Hadamard manifold. Suppose that the assumptions $\mathscr{H}_1$–$\mathscr{H}_3$ are satisfied and the following conditions hold:*

(i) *for all $(x,z) \in K \times E$, $F(x,z,x) \not\subset -\operatorname{int}C(x)$*;
(ii) *for all $(x,z) \in K \times E$, the set $\{y \in K : F(x,z,y) \subset -\operatorname{int}C(x)\}$ is geodesic convex*;
(iii) *for all $(z,y) \in E \times K$, $x \mapsto F(x,z,y)$ is weakly $C(x)$-quasiconcave-like*;
(iv) *the set $\{(x,z,y) \in K \times E \times K : F(x,z,y) \not\subset -\operatorname{int}C(x)\}$ is closed.*

Then, the (SWVQEPM) *has a solution, i.e., there exist $x^* \in K$ and $z^* \in B(x^*)$ such that*

$$x^* \in D(x^*) \text{ and } F(x^*, z^*, y) \not\subset -\operatorname{int}C(x^*), \quad \forall y \in D(x^*).$$

Moreover, the solution set of (SWVQEPM) *is a compact set.*

3. MAIN RESULTS

Throughout this section, let $\mathbb{M}$ be a Hadamard manifold and K be a nonempty closed and geodesic convex subset of $\mathbb{M}$. Let E be a nonempty subset of the tangent bundle $T\mathbb{M}$, and $\mathbb{Y}$ be a Banach space with norm denoted by $\|.\|$. Let $C : K \rightrightarrows \mathbb{Y}$ be a set-valued mapping such that, for each $x \in K, C(x)$ is a proper, closed and convex cone in $\mathbb{Y}$ with $\text{int}C(x) \neq \emptyset$. Let $\Psi_1 := \{(D,B,F) \mid D : K \rightrightarrows K$, and $B : K \rightrightarrows E$ be continuous in K with nonempty compact geodesic convex values, and let $F : K \times E \times K \rightrightarrows \mathbb{Y}$ be such that the set $\{(x,z,y) \in K \times E \times K : F(x,z,y) \subset C(x)\}$ is closed, and, for all $(z,y) \in E \times K$, $F(.,z,y)$ is strongly $C(x)$-quasiconcave-like$\}$. and $\Psi_2 := \{(D,B,F) \mid D : K \rightrightarrows K$, and $B : K \rightrightarrows E$ are continuous in K with nonempty compact geodesic convex values, and $F : K \times E \times K \rightrightarrows \mathbb{Y}$ is such that the set $\{(x,z,y) \in K \times E \times K : F(x,z,y) \not\subset -\text{int}C(x)\}$ is closed, and, for all $(z,y) \in E \times K$, $F(.,z,y)$ is weakly $C(x)$-quasiconcave-like$\}$.

For $u_1 = (D_1,B_1,F_1), u_2 = (D_2,B_2,F_2)$, $u_1,u_2 \in \Psi_1,\Psi_2$, we define

$$\xi(u_1,u_2) := \sup_{x\in K} H_D(D_1(x),D_2(x)) + \sup_{x\in K} H_B(B_1(x),B_2(x)) + \sup_{(x,z,y)\in K\times E\times K} H_F(F_1(x,z,y),F_2(x,z,y)),$$

where H_D, H_B are Hausdorff metrics in $C\mathscr{K}(K), C\mathscr{K}(E)$ and H_F is Hausdorff metric in $C(\mathbb{Y})$. Clearly, (Ψ_1,ξ) and (Ψ_2,ξ) are two metric spaces.

Theorem 3.1. *(Ψ_1,ξ) is a complete metric space.*

Proof. Let $\{u_n\}$ be any Cauchy sequence in Ψ_1 with $u_n = (D_n,B_n,F_n)$, $n = 1,2,....$ Then, for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\xi(u_n,u_m) < \frac{\varepsilon}{3}, \forall n,m \geq N. \tag{3.1}$$

It follows that, for any $(x,z,y) \in K \times E \times K$,

$$H_D(D_n(x),D_m(x)) < \frac{\varepsilon}{3}, \tag{3.2}$$

$$H_B(B_n(x),B_m(x)) < \frac{\varepsilon}{3}, \tag{3.3}$$

and

$$H_F(F_n(x,z,y),F_m(x,z,y)) < \frac{\varepsilon}{3}. \tag{3.4}$$

Then, for any fixed a point $(x,z,y) \in K \times E \times K, \{D_n(x)\}$ is a Cauchy sequence in $C\mathscr{K}(K)$, $\{B_n(x)\}$ is a Cauchy sequence in $C\mathscr{K}(E)$, and $\{F_n(x,z,y)\}$ are Cauchy sequences in $\mathscr{K}(\mathbb{Y})$. By Lemma 2.3 and the assumptions, we have that $(C\mathscr{K}(K),H_D),(C\mathscr{K}(E),H_B),(\mathscr{K}(\mathbb{Y}),H_F)$ are complete spaces. It follows that there exist $D(x) \in C\mathscr{K}(K), B(x) \in C\mathscr{K}(E)$ and $F(x,z,y) \in \mathscr{K}(\mathbb{Y})$ such that

$$D_n(x) \xrightarrow{H_D} D(x), \;\; B_n(x) \xrightarrow{H_B} B(x), \tag{3.5}$$

and

$$F_n(x,z,y) \xrightarrow{H_F} F(x,z,y). \tag{3.6}$$

Observe that $H_D(.,.), H_B(.,.)$ and $H_F(.,.)$ are continuous. Using (3.1), (3.2), (3.3) and (3.4), for any fixed $n \geq N$ and any $(x,z,y) \in K \times E \times K$, and leting $m \to +\infty$, we get

$$H_D(D_n(x), D(x)) < \frac{\varepsilon}{3}, \quad H_B(B_n(x), B(x)) < \frac{\varepsilon}{3}, \tag{3.7}$$

and

$$H_F(F_n(x,z,y), F(x,z,y)) < \frac{\varepsilon}{3}. \tag{3.8}$$

Now, we prove that D is continuous. From Lemma 2.2, we need to prove that, for any fixed a point $x_0 \in K$ and any $\varepsilon > 0$, there exists a neighborhood $N(x_0)$ of x_0 in K such that

$$H_D(D(x), D(x_0)) < \varepsilon, \forall x \in N(x_0) \cap K.$$

Since

$$H_D(D(x), D(x_0)) \leq H_D(D(x), D_n(x)) + H_D(D_n(x), D_n(x_0)) + H_D(D_n(x_0)), D(x_0)),$$

we find from (3.7) that there exists $N > 0$ such that, for any $n > N$,

$$H_D(D(x), D_n(x)) < \frac{\varepsilon}{3}, \forall x \in K.$$

Take a fixed $n > N$. From the continuity of D_n and Lemma 2.2, there exists a neighborhood $N(x_0)$ of x_0 in K such that

$$H_D(D_n(x), D_n(x_0)) < \frac{\varepsilon}{3}, \forall x \in N(x_0) \cap K.$$

We also have

$$\begin{aligned} H_D(D(x), D(x_0)) &\leq H_D(D(x), D_n(x)) + H_D(D_n(x), D_n(x_0)) + H_D(D_n(x_0)), D(x_0)) \\ &< \varepsilon, \forall x \in N(x_0) \cap K. \end{aligned}$$

Hence, D is continuous in K. Similarly, we can prove that B is continuous in K. It is easy see that the set $\{(x,z,y) \in K \times E \times K : F(x,z,y) \subset C(x)\}$ is closed.

Now, we show that, for all $(z,y) \in E \times K$, $F(.,z,y)$ is strongly $C(x)$-quasiconcave-like. Indeed, $\forall x_1, x_2 \in K$, $\forall s \in [0,1]$ and $\forall n$, we have $F_n(x_1,y,z) \subset C(x_1)$ and $F_n(x_2,y,z) \subset C(x_2)$. By the notion of the strong $C(x)$-quasiconcave-likeness, we have

$$F_n(\exp_{x_1}(s \exp_{x_1}^{-1} x_2), y, z) \subset C(\exp_{x_1}(s \exp_{x_1}^{-1} x_2)).$$

Since

$$F_n(x_1,y,z) \xrightarrow{H_F} F(x_1,y,z), \quad F_n(x_2,y,z) \xrightarrow{H_F} F(x_2,y,z),$$

and

$$F_n(\exp_{x_1}(s \exp_{x_1}^{-1} x_2), z, y) \xrightarrow{H_F} F(\exp_{x_1}(s \exp_{x_1}^{-1} x_2), z, y),$$

it follows that

$$F(\exp_{x_1}(s \exp_{x_1}^{-1} x_2), z, y) \subset C(\exp_{x_1}(s \exp_{x_1}^{-1} x_2)).$$

So, $F(.,z,y)$ is strongly $C(x)$-quasiconcave-like. By (3.7) and (3.8), for any fixed $n \geq N$ and any $x \in K$, we have

$$H_D(D_n(x), D(x)) < \frac{\varepsilon}{3}, \quad H_B(B_n(x), B(x)) < \frac{\varepsilon}{3},$$

and

$$H_F(F_n(x,z,y),F(x,z,y)) < \frac{\varepsilon}{3}.$$

Hence,

$$\sup_{x\in K} H_D(D_n(x),D(x)) < \frac{\varepsilon}{3}, \ \sup_{x\in K} H_B(B_n(x),B(x)) < \frac{\varepsilon}{3},$$

and

$$\sup_{(x,z,y)\in K\times E\times K} H_F(F_n(x,z,y),F(x,z,y)) < \frac{\varepsilon}{3}.$$

Setting $u=(D,B,F)$, we know that $u\in\Psi_1$ and $\xi(u_n,u)\leq\varepsilon, \forall n\geq N$, i.e., $u_n \xrightarrow{\xi} u$. Thus, (Ψ_1,ξ) is a complete metric space. □

Theorem 3.2. (Ψ_2,ξ) *is a complete metric space.*

We omit the proof since the technique is similar as that in Theorem 3.1 with suitable modifications.

Assume that all the conditions of Theorem 2.1 and Theorem 2.2 are satisfied. Then, for each $u=(D,B,F)\in\Psi_1,\Psi_2$, (SSVQEPM) and (SWVQEPM) have solutions.

For $(D,B,F)\in\Psi_1,\Psi_2$, let

$$\Phi_1(D,B,F) := \{\bar{x}\in D(\bar{x}) \text{ and } \bar{z}\in B(\bar{x}) \text{ such that } F(\bar{x},\bar{z},y)\subset C(\bar{x}), \forall y\in D(\bar{x})\}.$$

and

$$\Phi_2(D,B,F) := \{\bar{x}\in D(\bar{x}) \text{ and } \bar{z}\in B(\bar{x}) \text{ such that } F(\bar{x},\bar{z},y)\not\subset -\mathrm{int}(\bar{x}), \forall y\in D(\bar{x})\}.$$

Then $\Phi_1(D,B,F)\neq\emptyset$, and $\Phi_2(D,B,F)\neq\emptyset$. It follows that $\Phi_1(D,B,F)$ and $\Phi_2(D,B,F)$ define set-valued mappings from Ψ_1 into K and from Ψ_2 into K, respectively.

Theorem 3.3. $\Phi_1:\Psi_1\rightrightarrows K$ *is upper semicontinuous with compact values.*

Proof. Since K is compact, we only need to show that Φ_1 is a closed mapping. Let $\{(u_n,x_n)\}\subset$ graph(Φ_1) be given a sequence such that $(u_n,x_n)\to(u,x_0)\in\Phi_1\times K$, where

$$u_n=(D_n,B_n,F_n), u=(D,B,F).$$

We now show $(u,x_0)\subset$ graph(Φ_1). For any n, since $x_n\in\Phi_1(u_n)$, we have $x_n\in D_n(x_n)$ and $z_n\in B_n(x_n)$ such that

$$F_n(x_n,z_n,y_n)\subset C(x_n), \forall y_n\in D_n(x_n). \tag{3.9}$$

For any open set $O\supset D(x_0)$, since $D(x_0)$ is a compact set, there exists $\varepsilon>0$ such that

$$\{x\in K : d(x,D(x_0))<\varepsilon\}\subset O, \tag{3.10}$$

where $d(x,D(x_0))=\inf_{x'\in D(x_0)}||x-x'||$. Since $\xi(u_n,u)\to 0, x_n\to x_0$ and D is upper semicontinuous at x_0, we have that there exists n_0 such that

$$\sup_{x\in K} H_D(D_n(x),D(x)) < \frac{\varepsilon}{2}, \tag{3.11}$$

and

$$D(x_n)\subset\{x\in K : d(x,D(x_0))<\frac{\varepsilon}{2}\}, \forall n\geq n_0. \tag{3.12}$$

From (3.10), (3.11) and (3.12), we have

$$D(x_n) \subset \{x \in K : d(x, D(x_0)) < \frac{\varepsilon}{2}\} \subset \{x \in K : d(x, D(x_0)) < \varepsilon\} \subset O, \forall n \geq n_0. \tag{3.13}$$

Observe $D(x_0) \subset O$ and $x_n \in D_n(x_n)$. From Lemma 2.4, we have that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ convergent to x_0. It follows that $x_0 \in D(x_0)$. By using the same argument as above, we can show that $z_0 \in B(x_0)$.

Next, we only need to show that

$$F(x_0, y_0, z_0) \subset C(x_0), \forall y_0 \in D(x_0). \tag{3.14}$$

Since $x_n \to x_0$ and D is lower semicontinuous at x_0, for any $y_0 \in D(x_0)$, we see that there exists $y_n \in D(x_n)$ such that $y_n \to y_0$. Since $\xi(u_n, u) \to 0$, we can choose a subsequence $\{D_{n_k}\}$ of $\{D_n\}$ such that

$$\sup_{x \in K} H_D(D_{n_k}(x), D(x)) < \frac{1}{k}. \tag{3.15}$$

Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$H_D(D_{n_k}(x_{n_k}), D(x_{n_k})) < \frac{1}{k}.$$

This implies that there exists $\hat{y}_{n_k} \in D_{n_k}(x_{n_k})$ such that

$$||\hat{y}_{n_k} - y_{n_k}|| < \frac{1}{k}.$$

As

$$||\hat{y}_{n_k} - y_0|| \leq ||\hat{y}_{n_k} - y_{n_k}|| + ||y_{n_k} - y_0|| < \frac{1}{k} + ||y_{n_k} - y_0|| \to 0,$$

we have $\hat{x}^*_{n_k} \to x^*$. Since $\hat{y}_{n_k} \in D_{n_k}(x_{n_k}), x_{n_k} \in D_{n_k}(x_{n_k})$, $z_{n_k} \in B_{n_k}(x_{n_k})$, we obtain from (3.9) that

$$F_{n_k}(x_{n_k}, z_{n_k}, \hat{y}_{n_k}) \subset C(x_{n_k}). \tag{3.16}$$

By use of the assumption (iv) in Theorem 2.1, we get that

$$F(x_0, y_0, z_0) \subset C(x_0), \forall y_0 \in D(x_0). \tag{3.17}$$

From $x_0 \in D(x_0)$ and (3.17), we have that graph(Φ_1) is closed, so is Φ_1. Since K is a compact set and $\Phi_1(u) \subset K$, we have that Φ_1 is a upper semicontinuous with compact values. □

From the same result obtained in Theorem 3.3, we can obtain the following result for problem (SWVQEPM).

Theorem 3.4. *$\Phi_2 : \Psi_2 \rightrightarrows K$ is upper semicontinuous with compact values.*

We will need the following well-known result, which can be found in [12]:

Lemma 3.1. *Let X be a complete metric space, Y be a space possessing the Baire property (e.g., a Banach space enjoys this property) and $F : X \rightrightarrows Y$ be upper semicontinuous with compact values. Then the set of points, where F is lower semicontinuous, is a dense residual set of X.*

Theorem 3.5. *The set Γ_1 of the essential points of Φ_1 is a dense residual set of Ψ_1.*

Proof. From Lemma 3.1, Theorem 3.1 and Theorem 3.3, we obtain the desired conclusion immediately. □

From the same result obtained in Theorem 3.4, we can also obtain the following result for problem (SWVQEPM).

Theorem 3.6. *The set Γ_2 of the essential points of Φ_2 is a dense residual set of Ψ_2.*

4. Applications to vector quasi-variational-like inequalities

Let $\mathbb{M}, \mathbb{Y}, K, E, D, B, C$ be as in Section 3 and $\eta : K\times K \to T\mathbb{M}$ be a vector field. Then we consider the generalized strong and weak vector quasi-variational-like inequalities on Hadamard manifolds with variable domination structure:

(GSQVIM): Find $x^* \in K$ and $z^* \in B(x^*)$ such that

$$x^* \in D(x^*) \text{ and } \langle z^*, \eta(y,x^*)\rangle \in C(x^*), \quad \forall y \in D(x^*).$$

(GWQVIM): Find $x^* \in K$ and $z^* \in B(x^*)$ such that

$$x^* \in D(x^*) \text{ and } \langle z^*, \eta(y,x^*)\rangle \notin -\text{int}C(x^*), \quad \forall y \in D(x^*).$$

First, we recall the following two lemmas, which were established in [13].

Lemma 4.1. ([13, Theorem 5.1]) *Let $\mathbb{M}$ be a Hadamard manifold. Suppose that the assumptions $\mathscr{H}_1$–$\mathscr{H}_3$ are satisfied and the following conditions hold:*

(i) *for all $(x,z) \in K\times E$, $\langle z, \eta(x,x)\rangle \in C(x)$;*
(ii) *for all $(x,z) \in K\times E$, the set $\{y\in K : \langle z,\eta(y,x)\rangle \notin C(x)\}$ is geodesic convex;*
(iii) *for all $(z,y) \in E\times K$, $x \mapsto \langle z,\eta(y,x)\rangle$ is strongly $C(x)$-quasiconcave-like;*
(iv) *the set $\{(x,z,y) \in K\times E\times K : \langle z,\eta(y,x)\rangle \in C(x)\}$ is closed.*

Then, there exist $x^* \in K$ and $z^* \in T(x^*)$ such that

$$x^* \in D(x^*) \text{ and } \langle z^*, \eta(y,x^*)\rangle \in C(x^*), \quad \forall y \in D(x^*).$$

Lemma 4.2. ([13, Theorem 6.1]) *Let $\mathbb{M}$ be a Hadamard manifold. Suppose that the assumptions $\mathscr{H}_1$–$\mathscr{H}_3$ are satisfied and the following conditions hold:*

(i) *for all $(x,z) \in K\times E$, $\langle z, \eta(x,x)\rangle \notin -\text{int}C(x)$;*
(ii) *for all $(x,z) \in K\times E$, the set $\{y\in K : \langle z,\eta(y,x)\rangle \in -\text{int}C(x)\}$ is geodesic convex;*
(iii) *for all $(z,y) \in E\times K$, $x \mapsto \langle z,\eta(y,x)\rangle$ is weakly $C(x)$-quasiconcave-like;*
(iv) *the set $\{(x,z,y) \in K\times E\times K : \langle z,\eta(y,x)\rangle \notin -\text{int}C(x)\}$ is closed.*

Then, there exist $x^* \in K$, $z^* \in T(x^*)$ such that

$$x^* \in D(x^*) \text{ and } \langle z^*, \eta(y,x^*)\rangle \notin -\text{int}C(x^*), \quad \forall y \in D(x^*).$$

Next, we let $\Psi_3 := \{(D,B) \mid D : K \rightrightarrows K$, and $B : K \rightrightarrows E$ be continuous in K with nonempty compact geodesic convex values such that the set $\{(x,z,y) \in K\times E\times K : \langle z,\eta(y,x)\rangle \in C(x)\}$ is closed, and, for all $(z,y)\in E\times K$, $x\mapsto \langle z,\eta(y,x)\rangle$ is strongly $C(x)$-quasiconcave-like$\}$, and $\Psi_4 := \{(D,B) \mid D : K \rightrightarrows K$, and $B : K \rightrightarrows E$ be continuous in K with nonempty compact geodesic convex values such that the set $\{(x,z,y) \in K\times E\times K : \langle z,\eta(y,x)\rangle \notin -\text{int}C(x)\}$ is closed, and, for all $(z,y)\in E\times K$, $x\mapsto \langle z,\eta(y,x)\rangle$ is weakly $C(x)$-quasiconcave-like$\}$.

Theorem 4.1. (Ψ_3, ξ) *is a complete metric space.*

Proof. Set $F(x,z,y) := \{\langle z,\eta(y,x)\rangle\}$ for all $(x,z,y)\in K\times E\times K$. It is easy to check that all the assumptions in Theorem 3.1 are satisfied. This completes the proof. □

From Theorem 3.2, we have the following result.

Theorem 4.2. (Ψ_4, ξ) *is a complete metric space.*

Proof. Setting $F(x,z,y) := \{\langle z, \eta(y,x)\rangle\}$ for all $(x,z,y) \in K \times E \times K$, we have that all the assumptions in Theorem 3.2 are satisfied. This completes the proof. □

Assume that all the conditions of Lemma 4.1 and Lemma 4.2 are satisfied. Then, for each $u = (D,B) \in \Psi_3, \Psi_4$, (GSQVIM) and (GWQVIM) have solutions.

For $(D,B) \in \Psi_3, \Psi_4$, let

$$\Phi_3(D,B) := \{\bar{x} \in D(\bar{x}) \text{ and } \bar{z} \in B(\bar{x}) \text{ such that } \langle \bar{z}, \eta(y,\bar{x})\rangle \in C(\bar{x}), \forall y \in D(\bar{x})\},$$

and

$$\Phi_4(D,B) := \{\bar{x} \in D(\bar{x}) \text{ and } \bar{z} \in B(\bar{x}) \text{ such that } \langle \bar{z}, \eta(y,\bar{x})\rangle \notin -\text{int}C(\bar{x}), \forall y \in D(\bar{x})\}.$$

Then $\Phi_3(D,B) \neq \emptyset$, $\Phi_4(D,B) \neq \emptyset$. So, $\Phi_3(D,B)$ and $\Phi_4(D,B)$ define set-valued mappings from Ψ_3 into K and from Ψ_4 into K, respectively.

From Lemma 4.1 and Theorem 3.3, we obtain the following results.

Theorem 4.3. $\Phi_3 : \Psi_3 \rightrightarrows K$ *is upper semicontinuous with compact values.*

From Lemma 4.2 and Theorem 3.4, the following result can be derived easily.

Theorem 4.4. $\Phi_4 : \Psi_4 \rightrightarrows K$ *is upper semicontinuous with compact values.*

Applying the proof of Theorem 3.5 and Theorem 3.6, we can get the following results.

Theorem 4.5. *The set* Γ_3 *of the essential points of* Φ_3 *is a dense residual set of* Ψ_3.

Theorem 4.6. *The set* Γ_4 *of the essential points of* Φ_4 *is a dense residual set of* Ψ_4.

5. Applications to vector quasi-optimization problems

Let $\mathbb{M}, \mathbb{Y}, K, E, D, C$ be as in Section 3 and $\eta : K \times K \to T\mathbb{M}$ be a vector field. Then we consider the generalized strong and weak vector quasi-variational-like inequalities on Hadamard manifolds with variable domination structure:

(GVOP): Find $x^* \in K$ such that $x^* \in D(x^*)$ and

$$f(y) - f(x^*) \notin -\text{int}C(x^*), \text{ for all } y \in D(x^*)$$

where $f : K \to \mathbb{Y}$ is a vector-valued function.

Lemma 5.1. *([13, Theorem 6.1]) Let* $\mathbb{M}$ *be a Hadamard manifold. Suppose that the assumptions* $\mathscr{H}_1$, $\mathscr{H}_2$ *are satisfied and the following conditions hold:*

(i) *for all* $x \in K$*, the set* $\{y \in K : f(y) - f(x) \in -\text{int}C(x)\}$ *is geodesic convex*;

(ii) *the mapping* $-f$ *is weakly* $C(x)$*-quasiconcave-like*;

(iii) *the set* $\{(x,y) \in K \times K : f(y) - f(x) \notin -\text{int}C(x)\}$ *is closed.*

Then, the solution set of (GVOP) *is nonempty.*

Now, we let $\Psi_5 := \{(D, f) \mid D : K \rightrightarrows K$ be continuous in K with nonempty compact geodesic convex values, and $f : K \to \mathbb{Y}$ be such that the set $\{(x,y) \in K \times K : f(y) - f(x) \notin -\text{int}C(x)\}$ is closed, and the mapping $-f$ is weakly $C(x)$-quasiconcave-like$\}$.

Theorem 5.1. (Ψ_5, ξ) *is a complete metric space.*

Proof. Setting $F(x,z,y) := \{f(y) - f(x)\}$ for all $(x,y) \in K \times K$, we have that all the assumptions in Theorem 3.2 are satisfied. This completes the proof. □

Assume that all the conditions of Lemma 5.1 are satisfied. Then, for each $u = (D, f) \in \Psi_5$, (GVOP) has a solution. For $(D, f) \in \Psi_5$, let

$$\Phi_5(D, f) := \{\bar{x} \in D(\bar{x}) \text{ such that } f(y) - f(\bar{x}) \notin -\mathrm{int}C(\bar{x}), \forall y \in D(\bar{x})\}.$$

Then $\Phi_5(D, f) \neq \emptyset$. So, $\Phi_5(D, f)$ defines set-valued mapping from Ψ_5 into K.

From Lemma 5.1 and Theorem 3.3, we obtain the following results.

Theorem 5.2. $\Phi_5 : \Psi_5 \rightrightarrows K$ *is upper semicontinuous with compact values.*

Applying the proof of Theorem 3.6, we obtain the following result:

Theorem 5.3. *The set* Γ_5 *of the essential points of* Φ_5 *is a dense residual set of* Ψ_5.

6. Conclusions

In this paper, we first recalled from [13] the existence conditions of solutions of two classes of set-valued vector quasi-equilibrium problems on Hadamard manifolds. Then, we established the generic stability of set-valued mappings, where the set of essential points is a dense residual subset of a (Hausdorff) metric space. As applications of these problems, we studied generalized vector quasi-variational-like inequalities and vector quasi-optimization problems on Hadamard manifolds. In particular, we studied the generic stability for set-valued vector quasi-equilibrium problems, vector quasi-variational-like inequalities and vector quasi-optimization problems by using the Kakutani-Fan-Glicksberg type fixed point theorem with the $C(x)$-strong quasiconcavity-like and $C(x)$-weak quasiconcavity-like conditions in the setting of Hadamard manifolds.

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