SHRINKING PROJECTION METHOD FOR SOLVING ZERO POINT AND FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. In this article, we propose an algorithm for finding a common element of the set of fixed points of quasi-ϕ_p-nonexpansive mappings and zero points of nonnegative, l.s.c., convex mapping in a real p-uniformly convex and uniformly smooth Banach space by means of a modified shrinking projection method. Under some mild assumptions, strong convergence theorems are established. Preliminary numerical examples are performed to support our results.

Keywords. Fixed point; Quasi-ϕ_p-nonexpansive; Strong convergence; Shrinking projection; Zero point.

1. INTRODUCTION

Let E be a real Banach space with the topological dual space E^*. Let J_p ($p > 1$) denote the generalized duality mapping from E to 2^{E^*} defined by

$$J_p(x) := \{x^* \in E^* : \langle x^*, x \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}, \ \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. We call that J_2 is the normalized duality mapping. Because of the Hahn-Banach Theorem, we have that $J_p(x)$ is nonempty for any $x \in E$.

Let $f : E \to \mathbb{R}$ be a convex function. The set $\{x \in X : f(x) < +\infty\}$ is denoted by $dom f$. We assume that the interior of the domain of f, $int dom f$, is nonempty. For any $x \in int dom f$ and $y \in E$, we denote by $f^\circ(x, y)$ the right-hand derivative of f at x in the direction y, that is,

$$f^\circ(x, y) := \lim_{t \searrow 0} \frac{f(x + ty) - f(x)}{t}.$$

The function f is called Gâteaux differentiable at x if $\lim_{t \to 0}(f(x + ty) - f(x))/t$ exists for any y. In this case, $f^\circ(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x, see [17, 18]. Let $f : E \to \mathbb{R}$ be a convex function which is Gâteaux differentiable for any $x \in int dom f$. The function $D_f : dom f \times int dom f \to [0, +\infty)$, defined by

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle,$$

is called the Bregman distance with respect to f, see [3]. Let us recall a function $f_p : E \to \mathbb{R}$ by $f_p(x) = \frac{1}{p}\|x\|^p$, where $1 < p < \infty$. In this case, the Bregman distance associated with $f_p(\cdot)$ reduces to

$$\phi_p(y,x) = \frac{1}{q}\|y\|^p - \langle J_p y, x\rangle + \frac{1}{p}\|x\|^p,$$

where $x, y \in E$, $1 < p < \infty$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$ with $q > 1$. Under the assumption that E is a Hilbert space, we have that $f_2(x) = \frac{1}{2}\|x\|$, $\forall x \in E$ and J_2 is an identity mapping, then the Bregman distance associated with $f_2(\cdot)$ reduces to $\phi_2(x,y) = \frac{1}{2}\|x - y\|^2$, $\forall x, y \in E$.

We define a function $V_p : E^* \times E \to [0, +\infty)$ associated with $f_p(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ by $V_p(y,x) = \frac{1}{q}\|y\|^q - \langle y, x\rangle + \frac{1}{p}\|x\|^p$, $\forall x \in E, y \in E^*$, see [1]. Recall that V_p is nonnegative and

$$V_p(y,x) = \phi_p(J_p^{-1}(y),x), \ \forall x \in E, y \in E^*.$$

By making use of the subdifferential inequality, we obtain that

$$V_p(y,x) + \langle z, J_p^{-1}(y) - x\rangle \le V_p(y+z,x), \ \forall x \in E, y, z \in E^*,$$

see [9, 12]. Since V_p is convex in the first variable, hence we have that for all $z \in E$,

$$\sum_{i=1}^{N} t_i \phi_p(x_i, z) \ge \phi_p\left(J_p^{-1}(\sum_{i=1}^{N} t_i J_p(x_i)), z\right), \tag{1.1}$$

where $(x_i)_{i=1}^{N} \subset E$ and $(t_i)_{i=1}^{N} \subset (0,1)$ with $\sum_{i=1}^{N} t_i = 1$; see [10].

The modulus of convexity of E is the function $\tau : (0,2] \to [0,1]$ defined as follows

$$\tau(\omega) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1; \|x - y\| = \omega\right\}.$$

We call that E is uniformly convex if and only if $\tau(\omega) > 0, \forall \omega \in (0,2]$. Let $\lambda > 0$. We say that E is λ-uniformly convex if, there exists a positive constant δ such that $\tau(\omega) \ge \delta\omega^\lambda, \forall \omega \in (0,2]$. Let us define a unit sphere of E by $\mathscr{B}(E) := \{x \in E : \|x\| = 1\}$. We say that E is strictly convex if, $\frac{\|x+y\|}{2} < 1$, $\forall x, y \in \mathscr{B}(E)$, $x \ne y$. We say that if E is a strictly convex and reflexive Banach space with a strictly convex dual, one sees that the duality mapping from E^* into E satisfies that $(J_p)^{-1} J_p = I_E$ and $J_p(J_p)^{-1} = I_{E^*}$, see [8]. On the other hand, we say that the norm of E is Gâteaux differentiable iff the limit $\lim_{s\to 0} \frac{\|x+sy\| - \|x\|}{s}$ exists for any $x, y \in \mathscr{B}(E)$. In this case, E is said to be a smooth Banach space. Recall that if E is smooth, then J_p is single valued and $J_p(x) := \|x\|^{p-2}J(x)$, $\forall x \ne 0$, see [29]. The modulus of smoothness of E is a function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\iota) := \sup\left\{\frac{\|x+y\| - \|x-y\|}{2} - 1 : \|x\| \in \mathscr{B}(E), \|y\| = \iota\right\}.$$

We say that a Banach space E is uniformly smooth iff, $\lim_{\iota \to 0} \frac{\rho_E(\iota)}{\iota} = 0$. Given $r > 1$. We say that E is r-uniformly smooth iff, there exist a positive constant κ such that $\kappa\iota^r \ge \rho_E(\iota)$. If E is r-uniformly smooth, one sees that $r \le 2$ and E is uniformly smooth [28]. If E is uniformly smooth, we obtain that J_p is uniformly continuous on bounded subsets of E. If E is uniformly smooth, one obtains that ϕ_p is norm-to-norm uniformly continuous on bounded sets of E.

Let C be a nonempty, closed, and convex subset of a real Banach space E. Let us give a mapping T from C into E. We denote by $Fix(T)$ the solution set of finding $u \in E$ such that $u = Tu$. According to [19], we say that $T : C \to E$ is

(i) ϕ_p-nonxpansive if

$$\phi_p(x,y) \geq \phi_p(Tx,Ty), \forall x,y \in C;$$

(ii) quasi-ϕ_p-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\phi_p(x^*,x) \geq \phi_p(Tx,x^*), \forall x \in C, x^* \in Fix(T).$$

We assume that C is a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. We recall that the generalized projection is a mapping $\Pi_C : E \to C$ that assigns to an arbitrary point $x \in E$ the minimum point of $\phi_p(x,y)$, that is $\bar{x} = \Pi_C x$, where $\bar{x}$ is the solution to the minimization problem

$$\Pi_C x = \arg\min_{y \in C} \phi_p(x,y),$$

see [2]. In the setting of Hilbert spaces, operator Π_C is the metric (nearest point) projection P_C. One knows ([29]) Π_C can be characterized by the following properties, for any $x \in E$,

$$\langle z - \Pi_C x, J_p(x) - J_p(\Pi_C x) \rangle \leq 0, \forall z \in C, \tag{1.2}$$

from which one sees that

$$\phi_p(\Pi_C x, z) \leq \phi_p(x,z) - \phi_p(x,\Pi_C x), \forall z \in C. \tag{1.3}$$

Fixed point problems find a number of applications in many practical fields, such as, medical imaging, traffic networks etc. Fixed point methods have been applied to monotone variational inequalities, convex optimization problems and convex feasibility problems [5, 6, 23, 24, 25, 26]. There is an extensive literature on numerical approaches to solving the fixed point problems, such as, the Halpern algorithm, the hybrid method and the Mann algorithm. Recently, the iterative constructions of fixed points of nonlinear mappings via the Mann-like method have been extensively investigated by many authors [7, 13, 16, 21]. However, it is worthwhile recalling that this method is weakly convergent in the framework of infinite dimensional Hilbert spaces. As we are working in infinite dimensional spaces, strong convergence, which is a convergence in norm topology, is much more essential and desirable than weak convergence, which is a convergence in weak topology.

Now we recall the shrinking projection method proposed by Takahasi, Takeuchi and Kubota in Hilbert spaces [22]. They obtained a strong convergence result of fixed points for nonexpansive mappings ($\|Tx - Ty\| \leq \|x - y\|, \forall x,y \in C$). To be more precisely, their result reads as follows.

Theorem 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Suppose that $T : C \to H$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$. The iterate sequence $\{x_n\}$ is defined by*

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \forall n \geq 0. \end{cases} \tag{1.4}$$

where $0 \leq \alpha_n < a < 1, \forall n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to the fixed point of T nearest to x_0, as $n \to \infty$.

As a generalization of the shrinking projection method, Qin, Huang and Wang [14] improved Theorem 3.1 from a Hilbert space to a Banach space. Their result reads as follows.

Theorem 1.2. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. Let C be a nonempty closed convex subset of E. Suppose that $T : C \to C$ is a closed and quasi-ϕ_2-nonexpansive mapping. Let $\{x_n\}$ be a sequence generated in the following manner*

$$
\begin{cases}
x_0 \in E \text{ chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \prod_{C_1} x_0, \\
y_n = J_2^{-1}[\alpha_n J_2 x_n + (1 - \alpha_n) J_2 T x_n], \\
C_{n+1} = \{z \in C_n : \phi_2(z, y_n) \le \phi_2(z, x_n)\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \ \forall n \ge 0,,
\end{cases}
\tag{1.5}
$$

where $\{\alpha_n\} \subset [0,1)$ and $\limsup_{n \to \infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to the fixed point of T nearest to x_0, as $n \to \infty$.

On the other hand, many physically important problems can be modeled by initial value problems expressed as $Sx(t) + x'(t) = 0, x(0) = x_0$, where S is a mapping defined in a Banach space. Some examples where such evolution equations occur can be found in the wave, heat and other fields. Under the assumption that $x(t)$ is dependent on t, the above problem can be reduced to $S(x) = 0$ whose solutions correspond to the equilibrium points of initial value problems. We denote the zero set of S by $S^{-1}(0)$, that is, $S^{-1}(0) = \{z \in \text{dom} S : 0 \in S(z)\}$. Recently, there are a number of numerical methods for solving zero point problems; see, e.g., [4, 7, 15, 20, 21].

Motivated by the works mentioned above and their real applications, we consider the following problem, which consists of

$$
\text{finding } x \in \bigcap_{i=1}^{n} Fix(T_i) \cap S^{-1}(0),
\tag{1.6}
$$

where $T_1, T_2, \ldots, T_n$ are quasi-ϕ_p-nonexpansive mappings, and S is a nonnegative, convex and lower semi-continuous function.

An outline of this paper is organized as follows. In Section 3, we present some lemmas, which are essential to obtain our main convergence theorem. In Section 3, the main convergence theorem is obtained in a real uniformly smooth and p-uniformly convex Banach space. In Section 4, numerical examples are presented to support our algorithm.

2. LEMMAS

In this section, we list some lemmas, which are essential to obtain our main convergence theorem.

Lemma 2.1. [27] *Let E be a r-uniformly smooth Banach space. Then there is a $\chi_r > 0$ such that $\|x - y\|^r \le \|x\|^r - r\langle y, J_r(x)\rangle + \chi_r \|y\|^r, \ \forall x, y \in E$.*

Lemma 2.2. [20] *Let E be a uniformly convex and smooth Banach space, and let C be a closed convex nonempty subset of E. Let T be a quasi-ϕ_p-nonexpansive and closed mapping from C into itself. Thus, $Fix(T)$ is a closed and convex subset of C.*

Lemma 2.3. [27] *Let $\varepsilon > 0$ and $r > 1$ be two real numbers. A Banach space E is uniformly convex if and only if there exists a strictly increasing, continuous and convex function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta(0) = 0$ and $\forall x, y \in \mathscr{B}_\varepsilon := \{x \in E : \|x\| \le \varepsilon\}$, $\alpha \in [0,1]$, $\|\alpha x + (1 - \alpha)y\|^r \le \alpha \|x\|^r + (1 - \alpha)\|y\|^r - \Upsilon_r(\alpha)\theta(\|x - y\|)$, where $\Upsilon_r(\alpha) := \alpha^r(1 - \alpha) + \alpha(1 - \alpha)^r$.*

Lemma 2.4. [11] *Let E be a real smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. Then $\lim_{n\to\infty}\phi_p(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

3. Main results

Theorem 3.1. *Let E be a real uniformly smooth and p-uniformly convex Banach space with dual E^*, where $1 < p < \infty$. Let q be the conjugate of p, that is, $p + q = pq$. Let C be a nonempty, closed and convex subset of E. Let $T_i : C \to C$ $(i = 1, 2, \ldots, m)$ be a finite family of closed and quasi-ϕ_p-nonexpansive mappings. Let $S : E \to \mathbb{R}$ be a nonnegative, convex and lower semi-continuous (l.s.c) function. Let $\{x_n\}$ be a sequence generated by the following algorithm*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}x_0, \\ z_{n,m} = J_p^{-1}[a_nJ_px_n + b_nJ_pT_mx_n + c_n(J_px_n - t_nd_n)], \\ C_{n+1} = \{u \in C_n : \sup_{m\geq1}\phi_p(z_{n,m}, u) \leq \phi_p(x_n, u)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases} \quad (3.1)$$

where $\{a_n\},\{b_n\},\{c_n\} \subseteq (0,1)$ with $a_n + b_n + c_n = 1, \forall n \in \mathbb{N}$, d_n is a search direction with respect to S and

$$t_n = \begin{cases} \frac{\rho_nS(x_n)^{p-1}}{\|d_n\|^p}, & \text{if } d_n \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

for any $\rho_n \in \left(0, \left(\frac{q}{\chi_q}\right)^{\frac{1}{q-1}}\right)$, with respect of χ_q given in Lemma 2.1. We additionally assume that

(i) $0 \leq c \leq t_n \leq \bar{c}$ for some $c, \bar{c} \in \mathbb{R}$ and for any $n \in \Gamma := \{k \in \mathbb{N} : d_k \neq 0\}$;

(ii) $\inf_{n\in\Gamma}\left[\rho_n\left(1 - \frac{\rho_n^{r-1}\chi_q}{q}\right)\right] > 0$;

(iii) $S(x_n) \leq \langle d_n, x_n - z\rangle, \ \forall n \in \mathbb{N}$ and $\forall z \in S^{-1}(0)$.

If $\Lambda = \bigcap_{i=1}^m Fix(T_i) \cap S^{-1}(0) \neq \emptyset$, then $\{x_n\}$ converges strongly to $z = \Pi_\Lambda x_0$, as $n \to \infty$.

Proof. Since S is nonnegative l.s.c convex function, we find that $S^{-1}(0)$ is convex and closed. From this and Lemma 2.2, we obtain that Λ is closed and convex. It ensures that Π_Λ is well-defined. Observe that

$$\begin{aligned} C_{n+1} &= \Big\{u \in C_n : \sup_{m\geq1}\phi_p(z_{n,m}, u) \leq \phi_p(x_n, u)\Big\}, \\ &= \bigcap_{m\geq1}\big\{u \in E : \phi_p(z_{n,m}, u) \leq \phi_p(x_n, u)\big\} \cap C_n, \\ &= \bigcap_{m\geq1}\Big\{u \in E : \langle J_px_n - J_pz_{n,m}, u\rangle \leq \frac{1}{r}\|x_n\|^p - \frac{1}{r}\|z_{n,m}\|^p\Big\} \cap C_n. \end{aligned}$$

This implies that C_{n+1} is closed and convex. So, $\Pi_{C_{n+1}}$ is well-defined, for $n \in \mathbb{N}$. Let us now fix $\hat{x} \in \Omega$ and show $\Lambda \subset C_n$. Recalling the expression of $z_{n,m}$, it follows from (1.1) that

$$\begin{aligned} &\phi_p(z_{n,m}, \hat{x}) \\ &= \phi_p(J_p^{-1}[a_nJ_px_n + b_nJ_pT_mx_n + c_n(J_px_n - t_nd_n)], \hat{x}) \\ &= \phi_p\Big(J_p^{-1}\Big[(1 - c_n)\Big(\frac{a_n}{1 - c_n}J_px_n + \frac{b_n}{1 - c_n}J_pT_mx_n\Big) + c_n(J_px_n - t_nd_n)\Big], \hat{x}\Big). \end{aligned}$$

Setting briefly $y_{n,m} = J_p^{-1}\left(\frac{a_n}{1-c_n}J_p x_n + \frac{b_n}{1-c_n}J_p T_m x_n\right)$, we immediately find that

$$\phi_p(z_{n,m},\hat{x}) \leq (1-c_n)\phi_p(y_{n,m},\hat{x}) + c_n\phi_p\left(J_p^{-1}(J_p x_n - t_n d_n),\hat{x}\right). \tag{3.2}$$

By use of Lemma 2.1 and the definition of t_n, one has that

$$
\begin{aligned}
&\phi_p(J_p^{-1}(J_p x_n - t_n d_n),\hat{x}) \\
&= \frac{1}{q}\|J_p x_n - t_n d_n\|^q - \langle J_p x_n,\hat{x}\rangle + t_n\langle d_n,\hat{x}\rangle + \frac{1}{p}\|\hat{x}\|^p \\
&\leq \frac{1}{q}\left(\|J_p x_n\|^q - q t_n\langle d_n,x_n\rangle + t_n^q\chi_q\|d_n\|^q\right) - \langle J_p x_n,\hat{x}\rangle + t_n\langle d_n,\hat{x}\rangle + \frac{1}{p}\|\hat{x}\|^p \\
&= \frac{1}{r}\|x_n\|^p - \langle J_p x_n,\hat{x}\rangle + \frac{1}{p}\|\hat{x}\|^p - t_n\langle d_n,x_n-\hat{x}\rangle + \frac{t_n^q\chi_q}{q}\|d_n\|^q \\
&\leq \phi_p(x_n,\hat{x}) - \rho_n\left(1 - \frac{\rho_n^{q-1}\chi_q}{q}\right)\frac{S(x_n)^p}{\|d_n\|^p} \\
&\leq \phi_p(x_n,\hat{x}).
\end{aligned}
\tag{3.3}
$$

Invoking that T_i ($i = 1,2,\cdots,m$) is quasi-ϕ_p-nonexpansive, one finds from Lemma 2.3 that

$$
\begin{aligned}
\phi_p(y_{n,m},\hat{x}) &= \frac{1}{q}\left\|\frac{a_n}{1-c_n}J_p x_n + \frac{b_n}{1-c_n}J_p T_m x_n\right\|^q - \left\langle\frac{a_n}{1-c_n}J_p x_n + \frac{b_n}{1-c_n}J_p T_m x_n,\hat{x}\right\rangle + \frac{1}{p}\|\hat{x}\|^p \\
&\leq \frac{1}{q}\left(\frac{a_n}{1-c_n}\|J_p x_n\|^q + \frac{b_n}{1-c_n}\|J_p T_m x_n\|^q - \Upsilon_q\left(\frac{a_n}{1-c_n}\right)\theta(\|J_p x_n - J_p T_m x_n\|)\right) \\
&\quad - \left\langle\frac{a_n}{1-c_n}J_p x_n + \frac{b_n}{1-c_n}J_p T_m x_n,\hat{x}\right\rangle + \frac{1}{p}\|\hat{x}\|^p \\
&\leq \frac{a_n}{1-c_n}\left(\frac{1}{q}\|J_p x_n\|^q - \langle J_p x_n,\hat{x}\rangle + \frac{1}{p}\|\hat{x}\|^p\right) \\
&\quad + \frac{b_n}{1-c_n}\left(\frac{1}{q}\|J_p T_m x_n\|^q - \langle J_p T_m x_n,\hat{x}\rangle + \frac{1}{p}\|\hat{x}\|^p\right) \\
&\quad - \frac{1}{q}\Upsilon_q\left(\frac{a_n}{1-c_n}\right)\theta(\|J_p x_n - J_p T_m x_n\|) \\
&\leq \frac{a_n}{1-c_n}\phi_p(J_p^{-1}(J_p x_n),\hat{x}) + \frac{b_n}{1-c_n}\phi_p(J_p^{-1}(J_p T_m x_n),\hat{x}) \\
&\quad - \frac{1}{q}\Upsilon_q\left(\frac{a_n}{1-c_n}\right)\theta(\|J_p x_n - J_p T_m x_n\|) \\
&\leq \phi_p(x_n,\hat{x}) - \frac{1}{q}\Upsilon_q\left(\frac{a_n}{1-c_n}\right)\theta(\|J_p x_n - J_p T_m x_n\|).
\end{aligned}
\tag{3.4}
$$

By collecting the above results (3.2), (3.3) and (3.4), one finds that

$$
\begin{aligned}
\phi_p(z_{n,m},\hat{x}) \leq &(1-c_n)\left(\phi_p(x_n,\hat{x}) - \frac{1}{q}\Upsilon_q\left(\frac{a_n}{1-c_n}\right)\theta(\|J_p x_n - J_p T_m x_n\|)\right) \\
&+ c_n\left(\phi_p(x_n,\hat{x}) - \rho_n\left(1 - \frac{\rho_n^{q-1}\chi_q}{q}\right)\frac{S(x_n)^p}{\|d_n\|^p}\right) \\
\leq &\phi_p(x_n,\hat{x}) - \frac{1-c_n}{q}\Upsilon_r\left(\frac{a_n}{1-c_n}\right)\theta(\|J_p x_n - J_p T_m x_n\|) \\
&- c_n\rho_n\left(1 - \frac{\rho_n^{q-1}\chi_q}{q}\right)\frac{S(x_n)^p}{\|d_n\|^p}\,.
\end{aligned}
\tag{3.5}
$$

From this, one concludes that

$$
\phi_p(z_{n,m},\hat{x}) \leq \phi_p(x_n,\hat{x})\,. \tag{3.6}
$$

As a consequence, we obtain that $\Lambda \subset C_n$. In light of $x_n = \Pi_{C_n}x_0$, $\forall n \geq 1$, one finds from (1.3) that

$$
\phi_p(x_n,x_0) = \phi_p(\Pi_{C_n}x_0,x_0) \leq \phi_p(\hat{x},x_0) - \phi_p(\hat{x},\Pi_{C_{n+1}}x_0) \leq \phi_p(\hat{x},x_0),\ \forall n \geq 1\,.
$$

This implies that $\{\phi_p(x_n,x_0)\}$ is bounded. According to $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we refer that

$$
\phi_p(x_n,x_0) \leq \phi_p(x_{n+1},x_0),\ \forall n \geq 1\,.
$$

As a result, we immediately have that $\{\phi_p(x_n,x_0)\}$ is nondecreasing and bounded. Then the limit $\lim_{n\to\infty}\phi_p(x_n,x_0)$ exists. Again, by making use of (1.3), we have that

$$
\begin{aligned}
\phi_p(x_{n+1},x_n) = \phi_p(x_{n+1},\Pi_{C_n}x_0) &\leq \phi_p(x_{n+1},x_0) - \phi_p(\Pi_{C_n}x_0,x_0) \\
&= \phi_p(x_{n+1},x_0) - \phi_p(x_n,x_0)\,,
\end{aligned}
$$

which implies that

$$
\lim_{n\to\infty}\phi_p(x_{n+1},x_n) = 0\,. \tag{3.7}
$$

It follows from (3.7) and Lemma 2.4 that

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0\,. \tag{3.8}
$$

On the other hand, taking account of (3.6) and (3.8), with the help of Lemma 2.4, we obtain that

$$
\lim_{n\to\infty}\|z_{n,m} - x_{n+1}\| = 0\,. \tag{3.9}
$$

Let us give some positive integers k,l with $k \leq l$. By combining $x_l = \Pi_{C_l}x_0 \subseteq C_k$ with (1.3), we infer that

$$
\begin{aligned}
\phi_p(x_k,x_l) = \phi_p(x_k,\Pi_{C_l}x_0) &\leq \phi_p(x_k,x_0) - \phi_p(\Pi_{C_l}x_0,x_0) \\
&= \phi_p(x_k,x_0) - \phi_p(x_l,x_0)\,.
\end{aligned}
\tag{3.10}
$$

Since $\lim_{n\to\infty}\phi_p(x_n,x_0)$ exists, we have that (3.10) yields that $\lim_{l\to\infty}\phi_p(x_k,x_l) = 0$. From Lemma 2.4, we get that $\lim_{l\to\infty}\|x_k - x_l\| = 0$. Hence, $\{x_n\}$ is a Cauchy sequence. On the other

hand, with the help of $\phi_p(\cdot,\cdot)$, we derive that

$$
\begin{aligned}
&\phi_p(x_n,\hat{x}) - \phi_p(z_{n,m},\hat{x}) \\
&= \left(\frac{1}{q}\|x_n\|^p - \langle J_p x_n,\hat{x}\rangle + \frac{1}{p}\|\hat{x}\|^p\right) - \left(\frac{1}{q}\|z_{n,m}\|^p - \langle J_p z_{n,m},\hat{x}\rangle + \frac{1}{p}\|\hat{x}\|^p\right) \\
&= \frac{1}{q}(\|x_n\|^p - \|z_{n,m}\|^p) + \langle J_p z_{n,m} - J_p x_n,\hat{x}\rangle \\
&= \phi_p(x_n,z_{n,m}) + \langle J_p z_{n,m} - J_p x_n,\hat{x} - z_{n,m}\rangle.
\end{aligned}
\tag{3.11}
$$

From the same argument above, we have that

$$
\phi_p(z_{n,m},\hat{x}) - \phi_p(x_n,\hat{x}) = \phi_p(z_{n,m},x_n) + \langle J_p x_n - J_p z_{n,m},\hat{x} - x_n\rangle.
\tag{3.12}
$$

Combining (3.11) with (3.12), one obtains that

$$
\phi_p(x_n,z_{n,m}) + \phi_p(z_{n,m},x_n) = \langle J_p z_{n,m} - J_p x_n, z_{n,m} - x_n\rangle.
$$

As a classical result, one deduces that

$$
\phi_p(x_n,z_{n,m}) \leq \langle J_p z_{n,m} - J_p x_n, z_{n,m} - x_n\rangle.
\tag{3.13}
$$

By making use of (3.11) and (3.13), one immediately finds that

$$
\phi_p(x_n,\hat{x}) - \phi_p(z_{n,m},\hat{x}) \leq \langle J_p z_{n,m} - J_p x_n,\hat{x} - x_n\rangle.
\tag{3.14}
$$

Keeping in mind that E is a p-uniformly convex and uniformly smooth real Banach space, we have that J_p is uniformly norm to norm continuous. Coming back to (3.8) and (3.9), we have that

$$
\lim_{n\to\infty} \|J_p z_{n,m} - J_p x_n\| = 0.
\tag{3.15}
$$

Returning to (3.5), together with (3.14), (3.15), we deduce that

$$
\begin{aligned}
&\frac{1-c_n}{q}\Upsilon_r\left(\frac{a_n}{1-c_n}\right)\theta(\|J_p x_n - J_p T_m x_n\|) + c_n\rho_n\left(1 - \frac{\rho_n^{q-1}\chi_q}{q}\right)\frac{S(x_n)^p}{\|d_n\|^p} \\
&\leq \phi_p(x_n,\hat{x}) - \phi_p(z_{n,m},\hat{x}) \to 0, \quad \text{as } n\to\infty.
\end{aligned}
\tag{3.16}
$$

By using successively (3.16) and $\Upsilon_r\left(\frac{a_n}{1-c_n}\right) \neq 0$, we check that

$$
\lim_{n\to\infty} \theta(\|J_p x_n - J_p T_m x_n\|) = 0 \Rightarrow \lim_{n\to\infty} \|J_p x_n - J_p T_m x_n\| = 0.
\tag{3.17}
$$

On the other hand, invoking $\rho_n \in \left(0,\left(\frac{q}{\chi_q}\right)^{\frac{1}{q-1}}\right)$, we obtain that

$$
\lim_{n\to\infty} \frac{S(x_n)^p}{\|d_n\|^p} = 0 \Rightarrow \lim_{n\to\infty} S(x_n) = 0.
\tag{3.18}
$$

Recalling that J_p^{-1} is also uniformly norm-to-norm continuous on bounded sets, (3.17) yields that

$$
\lim_{n\to\infty} \|x_n - T_m x_n\| = 0.
\tag{3.19}
$$

Observing that $\{x_n\}$ is a Cauchy sequence, we find that there exists $\bar{x} \in E$ such that $\{x_n\}$ converges strongly to $\bar{x}$ as $n\to\infty$. Owning to (3.19), we have that $\{T_m x_n\}$ also converges strongly to $\bar{x}$ as $n\to\infty$. Recalling that T_i $(i=1,2,\cdots,m)$ is closed, we immediately find that $\bar{x} \in \bigcap_{i=1}^m Fix(T_i)$. Returning to (3.18), one sees that $\bar{x} \in S^{-1}(0)$. In conclusion, we obtain that $\bar{x} \in \Lambda$.

Now we are in a position to prove that $\{x_n\}$ converges strongly to an element $z = \Pi_\Omega x_0$, as $n \to \infty$. Invoking $x_n = \Pi_{C_n} x_0$, it follows from (1.3) that

$$\langle x_n - x, J_p x_0 - J_p x_n \rangle \geq 0, \ \forall x \in \Lambda. \tag{3.20}$$

Taking the limit as $n \to \infty$ in (3.20), we obtain that

$$\langle \bar{x} - x, J_p x_0 - J_p \bar{x} \rangle \geq 0, \ \forall x \in \Lambda.$$

As a classical result, we conclude that $\bar{x} = \Pi_\Omega x_0$. Thus, $z = \bar{x}$. It follows that $\{x_n\}$ converges strongly to z, as $n \to \infty$. The proof is completed. $\square$

In Theorem 3.1, if direction vector d_n coincides with gradient $\nabla S(x_n)$, then we obtain the following corollary.

Corollary 3.1. *Let E be a real uniformly smooth and p-uniformly convex Banach space with dual E^*, where $1 < p < \infty$. Let C be a nonempty, closed and convex subset of E. Let $T_i : C \to C (i = 1, 2, \ldots, m)$ be a finite family of closed and quasi-ϕ_p-nonexpansive mappings. Let $S : E \to \mathbb{R}$ be a nonnegative, convex lower semi-continuous (l.s.c) and Gâteaux differentiable function with the gradient $\nabla S : E \to E$. Let $\{x_n\}$ be a sequence generated by the following algorithm*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_{n,m} = J_p^{-1} [a_n J_p x_n + b_n J_p T_m x_n + c_n (J_p x_n - t_n \nabla S(x_n))], \\ C_{n+1} = \{u \in C_n : \sup_{m \geq 1} \phi_p(z_{n,m}, u) \leq \phi_p(x_n, u)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.21}$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subseteq (0,1)$ and $a_n + b_n + c_n = 1, \forall n \in \mathbb{N}$. If $\Lambda = \bigcap_{i=1}^m F(T_i) \cap S^{-1}(0) \neq \emptyset$, then $\{x_n\}$ converges strongly to $z = \Pi_\Lambda x_0$, as $n \to \infty$.

4. NUMERICAL EXPERIMENTS

In this section, we use two numerical examples to illustrate the computational performance of our proposed algorithm. All the programs are performed in MATLAB2018a on a PC Desktop Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz 1.800 GHz, RAM 8.00 GB.

Example 4.1. In the first experiment, we assume that $E = \mathbb{R}^N$ and $C = E$. Let us define $T_i : \mathbb{R}^N \to \mathbb{R}^N$ $(i = 1, 2, \ldots, m)$ by $T_i x := \frac{1}{i+1} x (\forall x \in E)$. Meanwhile, we define $S : \mathbb{R}^N \to \mathbb{R}^N$ by $S(x) := x^2 (\forall x \in E)$. Then we check that T_i $(i = 1, 2, \ldots, m)$ is quasi-ϕ_p-nonexpansive. In this case, we observe that $\Lambda = \bigcap_{i=1}^m Fix(T_i) \cap S^{-1}(0) = \mathbf{0} \neq \emptyset$.

We use our Algorithm (3.1) to solve Example 4.1 with $N = 3$, $m = 3$ and $C_0 = \mathbb{R}^N$. Our parameters are given as follows. We set $a_n = b_n = c_n = \frac{1}{3}$ and $\rho_n = 0.5$ $(\forall n \in \mathbb{N})$. Our initial value is randomly generated in the range of $(0,1)^N$. In this test, we consider four cases (Case I: $N = 1$; Case II: $N = 10$; Case III: $N = 50$; Case IV: $N = 100$). Let us define a sequence $\{E_n\}$ by $E_n = \|x_n - z\|^2$ with $z = \mathbf{0}$. In this experiment, the sequence $\{E_n\}$ can be regarded as the iteration error of Algorithm (3.21), due to the fact that the unique solution of Λ is $z = \mathbf{0}$. We take $E_n < 10^{-4}$ as the stopping criteria. Our numerical results are reported in Figure 1 and Figure 2.

In addition, we also tested the numerical behaviors of our algorithm (3.1) with different values of N and m. Our numerical results are represented in Figure 3 and Figure 4.
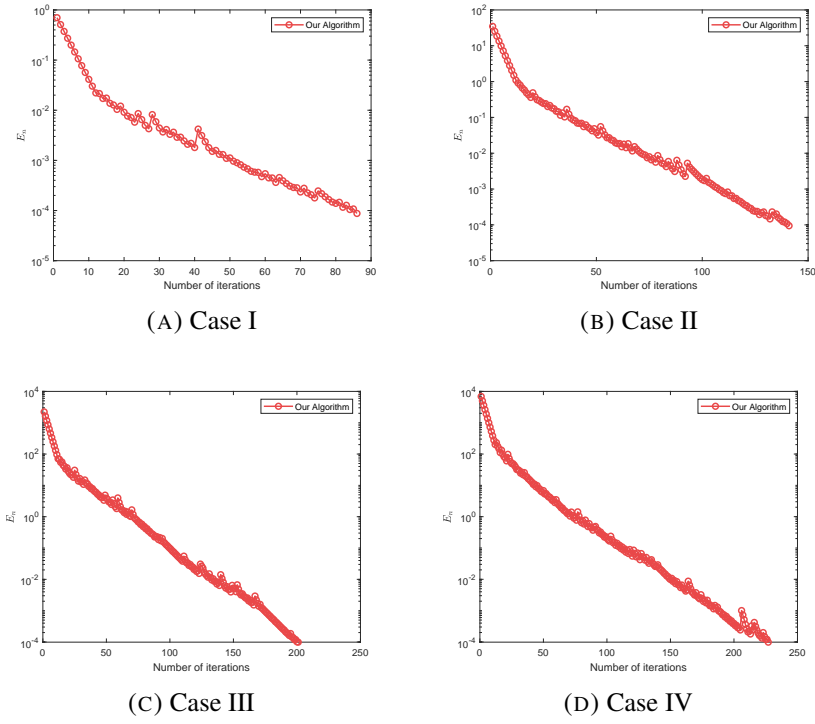
FIGURE 1. Numerical behaviors of error sequence $\{E_n\}$ for Example 4.1.
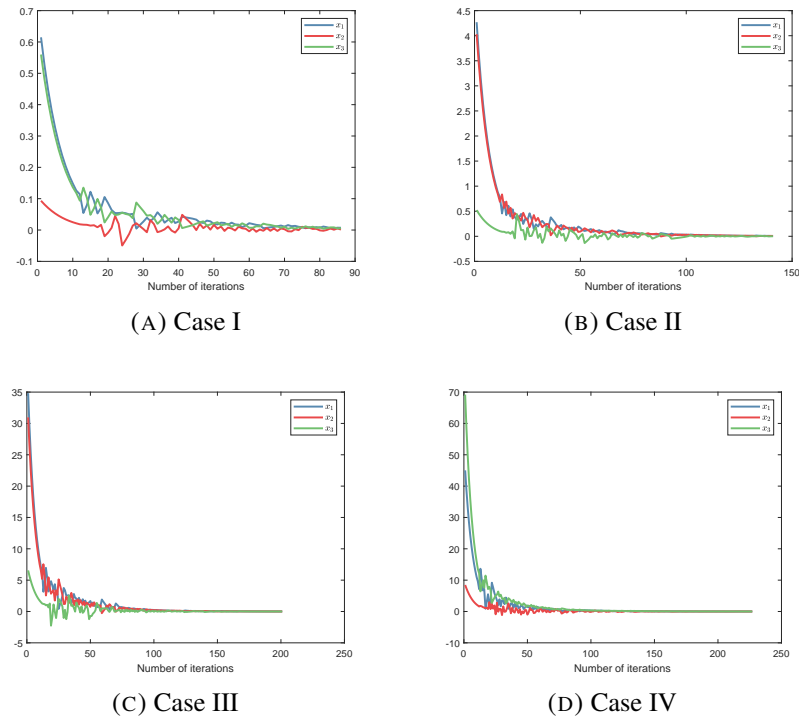
FIGURE 2. Numerical behaviors of each component of x_n for Example 4.1.

(A) $N = 3, m = 10$

(B) $N = 10, m = 3$

(C) $N = 10, m = 10$

(D) $N = 20, m = 20$
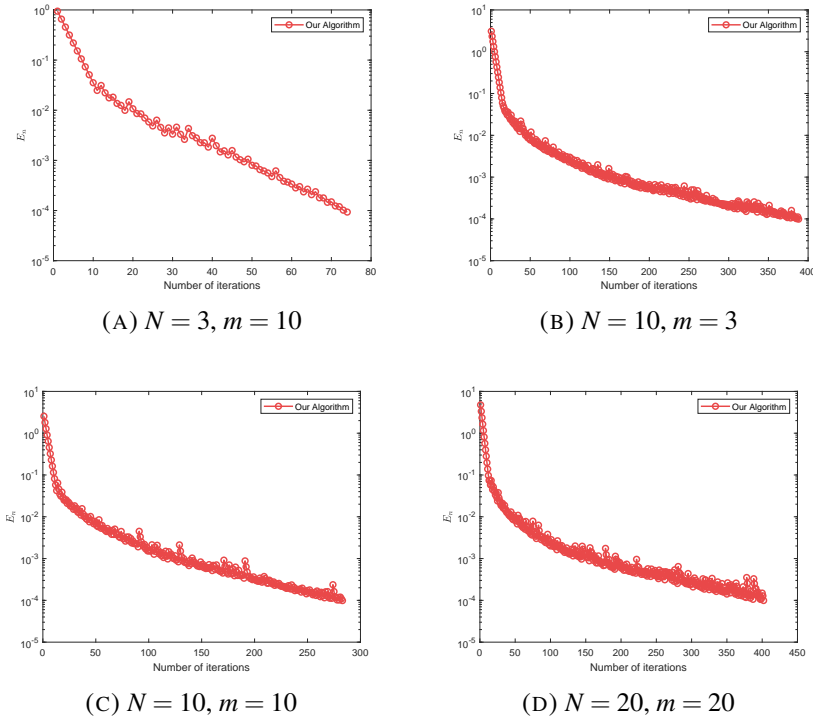
FIGURE 3. Numerical behaviors of error sequence $\{E_n\}$ for Example 4.1.

(A) $N = 3, m = 10$

(B) $N = 10, m = 3$

(C) $N = 10, m = 10$

(D) $N = 20, m = 20$
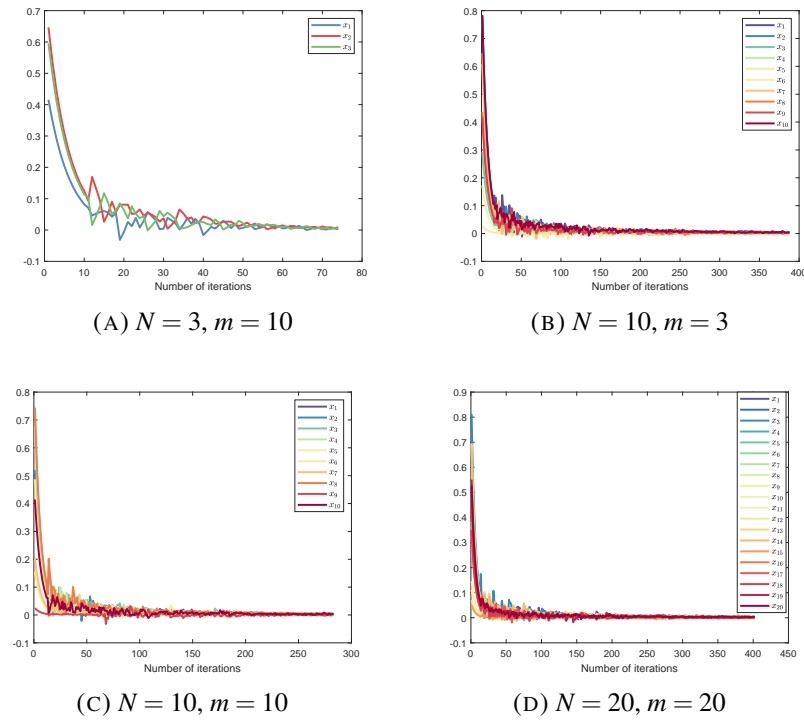
FIGURE 4. Numerical behaviors of each component of x_n for Example 4.1.
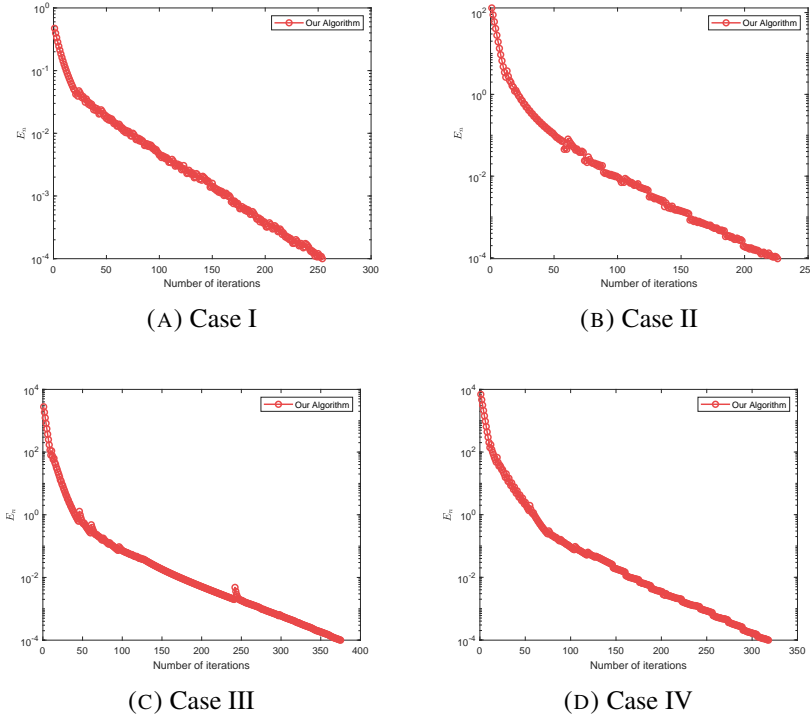
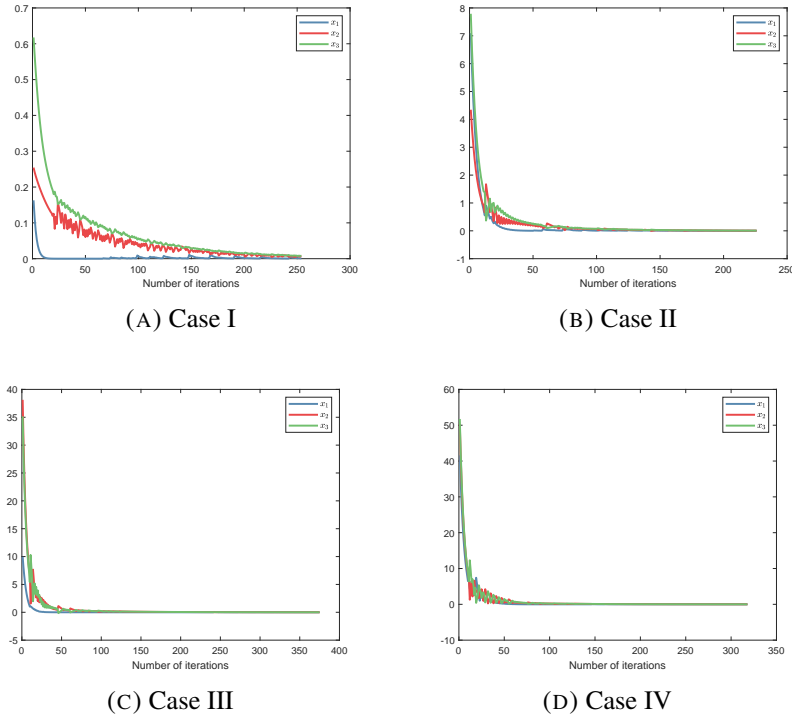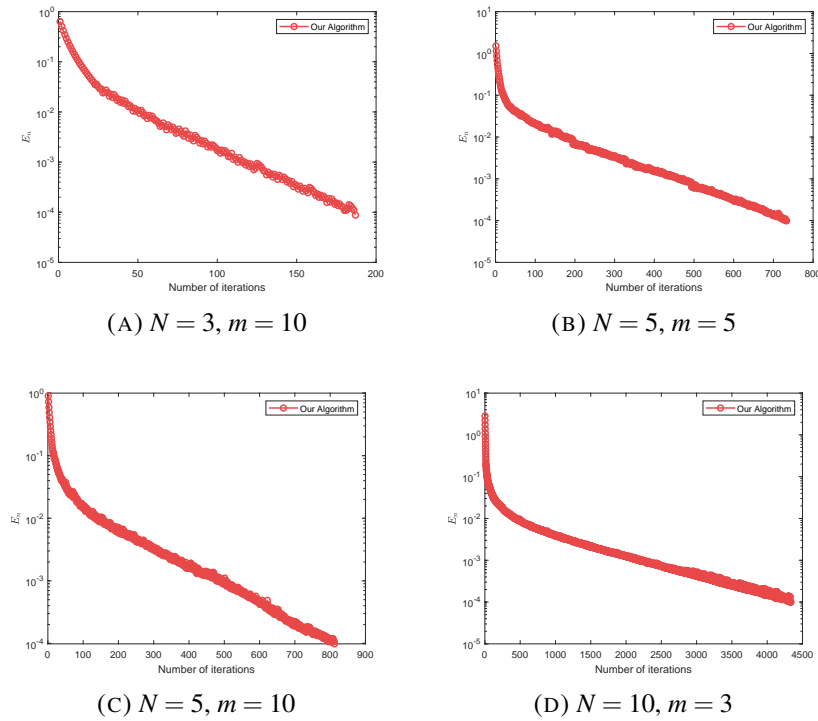FIGURE 5. Numerical behaviors of error sequence $\{E_n\}$ for Example 4.2.

Example 4.2. Let us consider a convex feasibility problem with $E = \mathbb{R}^N$ and $C = E$. Given any nonempty closed convex set $Q_i \subset \mathbb{R}^N$ $(i = 1, \ldots, m)$ with $Q := \bigcap_{i=1}^m Q_i \neq \emptyset$. We attempt to find the feasible solution $z \in Q := \bigcap_{i=1}^m Q_i$. We define a finite family of mappings $T_i : \mathbb{R}^N \to \mathbb{R}^N$ $(i = 1, 2, \cdots, m)$ by $T_i x := P_i(x)$ $(x \in \mathbb{R}^N)$, where $P_i(x) = P_{Q_i}(x)$ $(x \in \mathbb{R}^N)$ stands for the metric projection from $\mathbb{R}^N$ onto Q_i. We define a mapping $S : \mathbb{R}^N \to \mathbb{R}^N$ by $S(x) := x^2$ $(x \in \mathbb{R}^N)$. We observe that T_i $(i = 1, 2, \cdots, m)$ is quasi-ϕ_p-nonexpansive with respect of $Fix(T_i) \neq \emptyset$. In this experiment, we set Q_i $(i = 1, 2, \cdots, m)$ as a closed ball with center $c_i \in \mathbb{R}^N$ and radius $r_i > 0$. Thus P_i $(i = 1, 2, \cdots, m)$ can be computed with

$$P_i(x) := \begin{cases} c_i + \frac{r_i}{\|c_i - x\|}(x - c_i), & \text{if } \|c_i - x\| > r_i, \\ x, & \text{if } \|c_i - x\| \leq r_i. \end{cases}$$

We choose $r_i = 1$ $(i = 1, \ldots, m)$, $c_1 = (1, 0, \ldots, 0)^T$ and $c_2 = (-1, 0, \ldots, 0)^T$. Meanwhile, c_i is randomly generated in the range of $(-1/\sqrt{N}, 1/\sqrt{N})^N (i = 3, \ldots, m)$. Give the values of c_1, c_2, r_1 and r_2 as above. In this case, one sees that $\Lambda = \bigcap_{i=1}^m Fix(T_i) \cap S^{-1}(0) = \bigcap_{i=1}^m Q_i \cap S^{-1}(0) = \mathbf{0} \neq \emptyset$.

We use our Algorithm (3.21) to solve Example 4.2, our parameter settings are given the same as in Example 4.1. Our initial value is randomly generated by the MATLAB function $k \times$ rand (Case I: $k = 1$; Case II: $k = 10$; Case III: $k = 50$; Case IV: $k = 100$). Let us denote $E_n = \|x_n - z\|^2$ by the iteration error of Algorithm (3.21) with $z = \mathbf{0}$. Let us take $E_n < 10^{-4}$ as the stopping criteria. In addition, we also test the numerical behavior of our algorithm (3.21) with different values of N and m. Our numerical results are reported in Figure 5–Figure 8.

Remark 4.1.　　(i) Figure 1, Figure 3, Figure 5 and Figure 7 show the computational performance of our proposed Algorithms (3.1) and (3.21) with different initial values and

(A) Case I

(B) Case II

(C) Case III

(D) Case IV

FIGURE 6. Numerical behaviors of each component of x_n for Example 4.2.

(A) $N = 3, m = 10$

(B) $N = 5, m = 5$

(C) $N = 5, m = 10$

(D) $N = 10, m = 3$

FIGURE 7. Numerical behaviors of error sequence $\{E_n\}$ for Example 4.2.

(A) $N = 3$, $m = 10$

(B) $N = 5$, $m = 5$

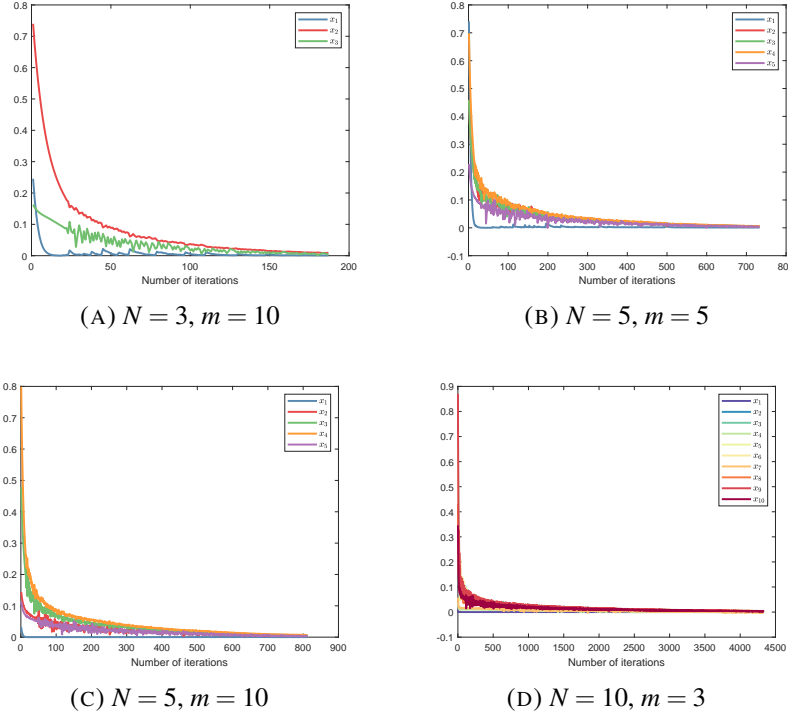(C) $N = 5$, $m = 10$

(D) $N = 10$, $m = 3$

FIGURE 8. Numerical behaviors of each component of x_n for Example 4.2.

different dimensions. On the other hand, we also display the numerical behaviors of each component of the generated sequence $\{x_n\}$ with different initial values and different dimensions, see Figure 2, Figure 4, Figure 6 and Figure 8. The numerical results show that the generated sequence $\{x_n\}$ is convergent to 0 as $n \to \infty$. From which, it implies that the sequence $\{x_n\}$ converges to the solution of problems.

(ii) Based on the provided data in Example 4.1 and Example 4.2, we conclude that our proposed Algorithms (3.1) and (3.21) are efficient for solving optimization problems in finite dimensional Euclidean spaces. In addition, our algorithms are robust because the choice of initial values and dimensions do not affect the computational performance of the algorithms.

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