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APPROXIMATION OF FIXED POINTS FOR A SEMIGROUP OF BREGMAN QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to study the iterative scheme of the Halpern type for a commutative semigroup $\mathfrak{J} = \{S_\lambda : \lambda \in \mathscr{Q}\}$ of Bregman quasi-nonexpansive mappings on a closed and convex subset of a Banach space. A strong convergence theorem is established for finding a common fixed point solution. Our results extend and improve some related results in the current literature. In addition, we present numerical examples to illustrate the performance of our method in finite and infinite dimensional spaces.
Keywords. Fixed point; Halpern method; Bregman quasi-nonexpansive; Strong convergence; Banach space.

1. INTRODUCTION

Let X be a real reflexive Banach space endowed with the topological dual X^*. The value of the functional $\zeta \in E^*$ at $x \in X$ can be written as $\langle \zeta, x\rangle$. Let C be a nonempty closed and convex subset of X. Let S be a mapping from C into itself. Let $f : X \to (-\infty, +\infty)$ be a proper, lower semi-continuous and convex function with the effective domain $dom f := \{u \in X : f(u) < +\infty\}$. Given any $u \in dom f$, we denote by $f^\circ(u,v)$ the right-hand derivative of f at u in the direction $v \in X$, that is,

$$f^\circ(u,v) := \lim_{s\searrow 0} \frac{f(u+sv)-f(u)}{s}. \tag{1.1}$$

The function f is called Gâteaux differentiable at u if $\lim_{s\searrow 0}(f(u+sv)-f(u))/s$ exists at any $v \in X$. The function f is called Fréchet differentiable at u if limit (1.1) is attained uniformly in $\|v\| = 1$. Recall that f is uniformly Fréchet differentiable on a subset of X if limit (1.1) is attained uniformly for $\|v\| = 1$ and $u \in X$. Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The bifunction function $D_f : dom f \times int(dom f) \to [0, +\infty)$ is given by

$$D_f(v,u) = f(v) - f(u) - \langle \nabla f(u), v-u\rangle.$$

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Thus D_f is called the Bregman distance with respect to f; see [1, 2]. The Bregman projection of $u \in int(domf)$ onto the nonempty, closed and convex set $C \subset domf$ is the unique vector $Proj_C^f(u) \in C$ such that $D_f(Proj_C^f(u),u) = \inf\{D_f(v,u), v \in C\}$; see [1].

Let $f : X \to (-\infty,+\infty)$ be a function. If X is reflexive, we say that f is Legendre [3] if and only if it satisfies the following two conditions

(1) we denote by $int(domf)$ the interior of the domain of f, which is nonempty. f is Gâteaux differentiable on $int(domf)$ with $domf = int(domf)$;
(2) we denote by $int(domf^*)$ the interior of the domain of f^*, which is nonempty. f^* is Gâteaux differentiable on $int(domf^*)$ with dom $f^* = int(domf^*)$.

If X is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [4]). From this, and conditions (1), (2), we have the following facts (a1) $\nabla f = (\nabla f^*)^{-1}$; (a2) $ran\nabla f = dom\nabla f^* = int(domf^*)$; (a3) $ran\nabla f^* = dom\nabla f = int(domf)$. Moreover, conditions (1) and (2) assert that functions f and f^* are strictly convex on the interior of their respective domains; see [3].

The study of fixed point problems plays a key role in different areas of applied mathematics such as convex optimization, set-valued optimization, etc [5, 6, 7]. The problem of finding fixed points of nonlinear mappings has been the object of active research in various setting. The practical importance of fixed point problems sheds new light on the iterative methods for finding fixed points of different types of mappings in nonlinear analysis; see [8, 9, 10, 11] and the references therein. The general study of the strong convergence property is an interesting subject. Let us give K a nonempty closed and convex subspace of a Hilbert space H. In order to achieve strong convergence results, the special attention has been paid to the studies concerning the analysis of the so-called Halpern's iteration [12] acting on a Hilbert space as follows

$$\forall x_1, u \in K,\ x_{n+1} = \alpha_n u + (1-\alpha_n)Sx_n, \forall n \geq 1, \tag{1.2}$$

where $\alpha_n = n^{-a}, a \in (0,1)$ and $S : K \to K$ is nonexpansive, that is, $\|x-y\| \geq \|Sx - Sy\|$, $\forall x, y \in K$. Indeed, it was proved that $\{x_n\}$ converges strongly to a solution of $F(S)$, where $F(S)$ stands for the nonempty fixed point set of S. In recent years, because of its simple construction, this approach has been widely used by many authors in different styles. The study with the parameter sequence $\{\alpha_n\}$ provides a valuable tool to analyze the Halpern iteration. As an alternative to Halpern's iteration, Lions [13] obtained a strong convergence, when $\{\alpha_n\}$ satisfies the following conditions (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iii) $\lim_{n\to\infty}(\alpha_{n+1} - \alpha_n)/\alpha_{n+1}^2 = 0$. On the other hand, Wittmann [14] proved the strong convergence if the condition (iii) mentioned above is removed and replaced with the condition (iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. In 2002, Xu [15] introduced another control condition (v) $\lim_{n\to\infty}(\alpha_{n+1} - \alpha_n)/\alpha_{n+1} = 0$ instead of conditions (iii) or (iv) and proved the strong convergence of $\{x_n\}$. Naturally, there is a puzzling question arises in literature. Is it possible to ensure the strong convergence of the sequence $\{x_n\}$ generated by (1.2) by providing that $\{\alpha_n\}$ satisfies only condition (i) and (ii)? In this perspective, Suzuki [16], based on the following Halpern-type iteration, gave a positive answer. Based on the following algorithm

$$\forall x_1, u \in K,\ x_{n+1} = \alpha_n u + (1-\alpha_n)(\beta x_n + (1-\beta)Sx_n), \forall n \geq 1.$$

He established the strong convergence of the iterative sequence $\{x_n\}$ when conditions (i) and (ii) are imposed on the parameter sequence $\{\alpha_n\}$ only. Recall that S is said to be quasi-nonexpansive with respect to a nonempty $F(S)$ if $\|x-y\| \geq \|Sx-y\|$, for all $x \in K$ and $y \in F(S)$. When we try to extend some results from Hilbert spaces to Banach spaces, our study may encounter some difficulties since many of useful examples for (quasi) nonexpansive mappings in Hilbert spaces that are no longer (quasi) nonexpansive in Banach spaces (for example, the metric projection $Proj_N$ onto a nonempty, closed and convex subset K of a Hilbert space H). In that case, as one of the most effective tools to overcome these difficulties is to use the Bregman distance function D_f instead of the norm $\|\cdot\|$ and Bregman (quasi) nonexpansive mappings instead of (quasi) nonexpansive mappings. It paves the way to further studies concerning the analysis of iterative algorithms for solving variational inequalities and optimization problems with the help of the Bregman's technique; see [17, 18, 19, 20]. An element $p \in C$ is said to be an asymptotically fixed point of S if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Sx_n\| \to 0$ as $n \to \infty$; see [21]. We denote the asymptotical fixed point set of S by $\hat{F}(S)$. We say that a mapping T is Bregman strongly nonexpansive (BSNE, for short) if $\hat{F}(S) \neq \emptyset$ and $D_f(p,x) \geq D_f(p,Sx)$, $\forall x \in C,\ p \in \hat{F}(S)$, and if whenever $\{x_n\} \subset C$ is bounded, $\lim_{n\to\infty}(D_f(p,x_n) - D_f(p,Sx_n)) = 0$, $\forall p \in \hat{F}(S)$, it yields that $\lim_{n\to\infty} D_f(Sx_n,x_n) = 0$, see [22]. Recently, Halpern-type method has been developed for finding a common fixed point of a given family of operators.

In this direction, Zegeye [23] improved the Halpern method and presented the following iterative scheme

$$\forall x_1, u \in C,\ x_{n+1} = Proj_C^f(\nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(S_N S_{N-1} \cdots S_1 x_n))),$$

where $S_m : C \to C$ are BSNE mappings with $F(S_m) = \hat{F}(S_m)$, for all $m = 1, 2, \cdots, N$ and $\{\alpha_n\} \subset (0,1)$ satisfies conditions (i) and (ii). If $\mathscr{F} = \bigcap_{m=1}^N F(S_m) \neq \emptyset$, it was proved that $\{x_n\}$ converges to $Proj_{\mathscr{F}}^f(u)$ in norm.

Recently, numerical methods for the fixed point problems with BSNE mappings have been studied in various ways; see, e.g., [22, 24, 25] and the references therein. We say that a mapping $S : C \to C$ is Bregman quasi-nonexpansive (BQNE, for short) with respect to a nonempty fixed point set $F(S)$ [26] if $D_f(p,Sx) \leq D_f(p,x), \forall x \in C, p \in F(S)$. This class of mappings is of particular significance in convex optimization theories and fixed point problems since it encompasses other noteworthy classes of Bregman nonexpansive type operators (for example, all BSNE mappings are BQNE, in uniformly convex Banach spaces); see [17]. Closely related to this field is the problem of finding a common fixed point solution of a communicative semigroup of nonexpansive mappings (i.e., $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$, where C is a closed and convex subset of a reflexive Banach space). In [27], Yao and Noor proposed a viscosity approximation method for solving this problem and proved the following theorem.

Theorem 1.1. *(VI-Method) Let C be a nonempty closed convex subset of a reflexive Banach space X. Let $\tilde{C}$ be any bounded subset of C. Suppose that $\mathscr{Q}$ is an unbounded subset of $\mathbb{R}^+$ such that $r + t \in \mathscr{Q}$, $\forall r, t \in \mathscr{Q}$. Let $\mathfrak{J} = \{S_t : t \in \mathscr{Q}\}$ be a commutative semigroup of nonexpansive self-mappings defined on C such that $S_r S_t = S_{r+t}$, $\forall r, t \in \mathscr{Q}$. Let $g : C \to C$ be a contraction (i.e., $\exists \chi \in (0,1)$ such that $\chi\|x - y\| \geq \|g(x) - g(y)\|$, $\forall x, y \in C$). Let us denote by $\mathscr{O}$ the set of common fixed point of $\mathfrak{J}$, i.e., $\mathscr{O} = \{x \in C : x = S_\gamma x, \gamma \in \mathscr{Q}\}$. Assume that $\mathscr{O} \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ be three sequences in $(0,1)$. Let $\{\gamma_n\}$ be a sequence in $\mathscr{Q}$. Additionally, assume that*

(1) $\gamma_n \to \infty$ *as* $n \to \infty$;
(2) $\alpha_n + \beta_n + \lambda_n = 1$;
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(4) the family $\mathfrak{J}$ satisfies the uniformly asymptotically regularity condition

$$\lim_{r\in\mathscr{Q},r\to\infty}\sup_{x\in\tilde{C}}\|S_sS_rx-S_rx\|=0,\ \textit{uniformly in}\ s\in\mathscr{Q}.$$

For any $x_1\in C$, define a sequence $\{x_n\}$ by

$$x_{n+1}=\alpha_n g(x_n)+\beta_n x_n+\lambda_n S_{\gamma_n}x_n,\ \ n\geq 1.$$

Then $\{x_n\}$ converges to a common fixed point of the family $\mathfrak{J}$ in norm.

As a significant refinement of the research work discussed above, we deal with a new Halpern type method for finding a common fixed point corresponding to a semigroup of BQNE mappings in real reflexive Banach spaces. We prove the strong convergence of the sequences generated by the proposed algorithm with the weak restrictions, that is, the control variable $\{\alpha_n\}$ satisfies conditions (i) and (ii) only. Our paper is organized as follows. Section 2 presents certain essential preliminary definitions and propositions, which will be useful for establishing our main result. Section 3 is devoted to the study of our iterative method. In Section 4, some examples are considered. Section 5, the last section, ends this paper with a concluding remark.

2. PRELIMINARIES

This section contains some definitions, notations and propositions, which will be used in the proof of our main result in the next section.

Give a Legendre and Gâteaux differentiable function $f:X\to\mathbb{R}$. Based on the ideas from [2, 28], we restrict our attention to the function $V_f:X\times X\to[0,+\infty)$ with respect of f given by

$$V_f(z,z^*)=f(z)+f^*(z^*)-\langle z,z^*\rangle,\ \forall z\in X,z^*\in X^*.$$

Thus, one sees that V_f is nonnegative and $V_f(z,z^*)=D_f(z,\nabla f^*(z^*))$, for all $z\in X,z^*\in X^*$. With the help of the subdifferential inequality from [29], we obtain that

$$\langle p^*,\nabla f^*(z^*)-z\rangle\leq V_f(z,z^*+p^*)-V_f(z,z^*),\ \forall z\in X,\ z^*,p^*\in X^*.$$

If $f:X\to(-\infty,+\infty]$ is a proper, lower semi-continuous and convex function, then $f^*:X^*\to(-\infty,+\infty]$ is a proper, weak* lower semi-continuous and convex function; see [30]. In this case, V_f is convex in the second variable. Hence,

$$\sum_{i=1}^{N}t_iD_f(z,\nabla f(x_i))\geq D_f\left(z,\nabla f^*\left(\sum_{i=1}^{N}t_i\nabla f(x_i)\right)\right),\ \forall z\in X.$$

As similar to the equivalent properties of metric projections in Hilbert spaces, the Bregman projection with respect to totally convex and differentiable functions has the following results.

Proposition 2.1 ([18])**.** *Let f be totally convex on $int(domf)$. Let C be a nonempty, closed and convex subset of $int(domf)$. Given $x\in int(domf)$, $\hat{x}\in C$, the following conditions are equivalent*

(i) $\hat{x}$ is the Bregman projection of x onto C with respect to f;
(ii) $\hat{x}$ is the unique solution of the variational inequality, $\langle\nabla f(z)-\nabla f(x),y-z\rangle\geq 0,\ \forall y\in C$;
(iii) $\hat{x}$ is the unique solution of the inequality, $D_f(z,x)+D_f(y,z)\leq D_f(y,x),\ \forall y\in C$.

Let X be a Banach space and let $B_r := \{z \in X : \|z\| \leq r\}$, $\forall r > 0$. Then a function $f : X \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of X [31] if $\rho_r(s) > 0$, $\forall r, s > 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is given by

$$\rho_r(s) = \inf_{x,y \in B_r, \|x-y\|=s, \lambda \in (0,1)} \frac{\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y)}{\lambda(1-\lambda)},$$

for all $s \geq 0$. The function ρ_r is called the gauge of uniform convexity of f.

Proposition 2.2 ([32]). *Let X be a Banach space. Let $f : X \to \mathbb{R}$ be uniformly convex on bounded subsets of X. Let r be a positive constant. Then*

$$f\big(\sum_{k=1}^{n} \lambda_n x_n\big) \leq \sum_{k=1}^{n} \lambda_n f(x_n) - \lambda_i \lambda_j \rho_r(\|x_i - x_j\|), \ \forall i, j \in \{0, 1, 2, \cdots, n\},$$

where $\lambda_n \in (0,1)$, $(x_k)_{k=1}^n \in B_r$, $\sum_{k=1}^{n} \lambda_k = 1$, and ρ_r denotes the gauge of uniform convexity of f.

A function $f : X \to (-\infty, +\infty]$ is said to be strongly coercive if $\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty$, see [31].

Proposition 2.3 ([31]). *Let X be a reflexive Banach space and let $f : X \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent*

(i) f is bounded on bounded subsets and uniformly smooth on bounded subsets of X;
(ii) f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of X^*;
(iii) $dom f = X^*$, f^* is strongly coercive and uniformly convex on bounded subsets of X^*.

Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. We say that the function f is totally convex at a point $x \in int(dom f)$ if its modulus of total convexity at x is positive, in other words, the function, $v_f : int(dom f) \times [0, +\infty)$, defined by

$$v_f(x, s) = \inf\{D_f(y, x) : y \in dom f, \|y - x\| = s\},$$

is positive whenever $s > 0$; see [33]. Moreover, the function f is said to be totally convex when it is totally convex at each point $x \in int(dom f)$. Let a function $v_f : ind(dom f) \times [0, +\infty) \to [0, +\infty]$ be the modulus of total convexity associated with the function f on the set X, given by $v_f(X, s) = \inf\{v_f(x, s) : x \in X \cap int(dom f)\}$, $\forall s > 0$. Then f is called totally convex on bounded sets if $v_f(X, s)$ is positive for any nonempty bounded subset of X, for any $s > 0$.

Proposition 2.4 ([34]). *Let $x \in int(dom f)$. The following statements are equivalent:*

(i) the function f is totally convex at x;
(ii) for any sequence $\{y_n\} \subset dom f$, $\lim_{n\to+\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x\| = 0$.

Proposition 2.5 ([25]). *Let $f : X \to \mathbb{R}$ be proper, convex, lower-semicontinuous, Gâteaux differentiable on $int(dom f)$ and totally convex at a point $x \in int(dom f)$. If $D_f(x, x_n)$ is bounded, then $\{x_n\}$ is also bounded.*

Proposition 2.6 ([26]). *Let $f : X \to \mathbb{R}$ be a Legendre and totally convex function. If $x_1 \in X$ and the sequence $\{D_f(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

We say that the function f is sequentially consistent [18] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $int(domf)$ and $domf$, respectively such that the first one is bounded and

$$\lim_{n\to+\infty} D_f(y_n,x_n)=0 \Rightarrow \lim_{n\to+\infty}\|y_n-x_n\|=0.$$

Proposition 2.7 ([33]). *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Proposition 2.8 ([35]). *If $f: X\to\mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

Proposition 2.9 ([36]). *Let $f: X\to(-\infty,+\infty)$ be a Legendre function. Let C be a nonempty closed convex subset of $int(domf)$. Let $S: C\to C$ be a Bregman nonexpansive mapping with respect to f. Then $F(S)$ is closed and convex.*

Proposition 2.10 ([37]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*

$$a_{n+1}\le(1-\alpha_n)a_n+\alpha_n\delta_n,\ n\ge n_0,$$

where $\{\alpha_n\}\subset(0,1)$ and $\{\delta_n\}\subset\mathbb{R}$ satisfy the following conditions $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, and $\limsup_{n\to\infty}\delta_n\le 0$. Then, $\lim_{n\to\infty}a_n=0$.

Proposition 2.11 ([38]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{k_i\}$ of $\{k\}$ such that $a_{k_i}<a_{k_i+1}$ for all $i\in\mathbb{N}$. Then there exists an increasing sequence $\{m_n\}\subset\mathbb{N}$ such that $m_n\to\infty$ as $n\to\infty$ and the following properties are satisfied by all (sufficiently large) numbers $n\in\mathbb{N}$*

$$a_{m_n}\le a_{m_n+1},\quad a_n\le a_{m_n+1}.$$

Indeed, m_n is the largest number k in the set $\{1,2,\cdots,n\}$ such that $a_n\le a_{k+1}$ holds.

3. Main Result

Theorem 3.1. *Let C be a nonempty closed and convex subset of a reflexive Banach space X with a weakly sequentially continuous duality mapping. Let $\mathscr{Q}$ be an unbounded subset of $\mathbb{R}^+$ such that $r+t\in\mathscr{Q}$ whenever $r,t\in\mathscr{Q}$. Let $\mathfrak{J}=\{S_\lambda:\lambda\in\mathscr{Q}\}$ be a commutative family of Bregman quasi-nonexpansive self-mappings on C with $\mathscr{O}=\{x\in C: S_\lambda x=x,\lambda\in\mathscr{Q}\}\neq\emptyset$. Additionally, assume that*

(1) $(I-S_\lambda)$ is demiclosed at zero, for $\lambda\in\mathscr{Q}$;
(2) $\mathfrak{J}$ is a semigroup such that $S_rS_t=S_{r+t}$, $\forall\ r,t\in\mathscr{Q}$;
(3) $\mathfrak{J}$ is a uniformly asymptotically regular semigroup on C, i.e.,

$$\lim_{r\in\mathscr{Q}}\sup_{x\in\tilde{C}}\|S_tS_rx-S_rx\|=0 \,\textit{uniformly in}\, t\in\mathscr{Q},$$

where $\tilde{C}$ is any bounded subset of C, see [39];
(4) $\mathfrak{J}$ is a $L(\lambda)$-Lipschizian semigroup on C, i.e., there exists a bounded measurable function $\lambda\to L(\lambda):[0,\infty)\to(0,+\infty)$ such that

$$\|S(\lambda)x-S(\lambda)y\|\le L(\lambda)\|x-y\|,\ \forall x,y\in C,\lambda\ge 0,$$

for any $S\in\mathfrak{J}$, see [39].

Let $\{\gamma_n\}$ be a sequence in $\mathscr{Q}$ such that $\gamma_n \to \infty$, as $n \to \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. For any $x_1 \in C, u \in X$, define a sequence $\{x_n\}$ by

$$\begin{cases} y_n = \nabla f^* \big(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(S_{\gamma_n} x_n)\big), \\ x_{n+1} = Proj_C^f \nabla f^* \big(\beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n)\big). \end{cases} \tag{3.1}$$

Then the sequence $\{x_n\}$ generated by (3.1) converges, as $n \to \infty$, to $z = Proj_{\mathscr{O}}^f u$ in norm.

Proof. From Proposition 2.9, it ensues that $\mathscr{F}$ is closed and convex. As a consequence, $Proj_{\mathscr{F}}^f$ is well-defined. According to Proposition 2.3, we deduce that f^* is uniformly convex on bounded subsets of X^*. We define a mapping $\rho_r^* : [0,+\infty) \to [0,\infty]$ as the gauge of uniform convexity of f^*, associated with a constant $r > 0$. Let us now fix any $z \in \mathscr{F}$. From (3.1), and Proposition 2.2, we get that

$$\begin{aligned} D_f(z, y_n) &= V_f(z, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(S_{\gamma_n} x_n)) \\ &= (f(z) - \langle z, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(S_{\gamma_n} x_n)\rangle \\ &\quad + f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(S_{\gamma_n} x_n))) \\ &\leq (f(z) - \alpha_n \langle z, \nabla f(x_n)\rangle - (1-\alpha_n)\langle z, \nabla f(S_{\gamma_n} x_n)\rangle + \alpha_n f^*(\nabla f(x_n)) \\ &\quad + (1-\alpha_n) f^*(\nabla f(S_{\gamma_n} x_n)) - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\|)) \\ &= \alpha_n V_f(z, \nabla f(x_n)) + (1-\alpha_n) V_f(z, \nabla f(S_{\gamma_n} x_n)) \\ &\quad - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\|)) \\ &= \alpha_n D_f(z, x_n) + (1-\alpha_n) D_f(z, S_{\gamma_n} x_n) \\ &\quad - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\|)) \\ &\leq D_f(z, x_n) - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\|). \end{aligned} \tag{3.2}$$

By combining successively (3.1) with (3.2), one clearly derives that

$$\begin{aligned} D_f(z, x_{n+1}) &\leq D_f(z, \nabla f^*(\beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n))) \\ &\leq \beta_n D_f(z, u) + (1-\beta_n) D_f(z, y_n) \\ &\leq \beta_n D_f(z, u) + (1-\beta_n) D_f(z, x_n) \\ &\quad - \alpha_n(1-\beta_n)(1-\alpha_n)\rho_r^*(\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\|). \end{aligned} \tag{3.3}$$

As a classical result, we observe that

$$\begin{aligned} &\alpha_n(1-\beta_n)(1-\alpha_n)\rho_r^*(\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\|) \\ &\leq \beta_n D_f(z, u) + (1-\beta_n) D_f(z, x_n) - D_f(z, x_{n+1}). \end{aligned} \tag{3.4}$$

By neglecting the nonnegative term $\alpha_n(1-\beta_n)(1-\alpha_n)\rho_r^*(\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\|)$ in (3.3), we additionally obtain that

$$\begin{aligned} D_f(z, x_{n+1}) &\leq \beta_n D_f(z, u) + (1-\beta_n) D_f(z, x_n) \\ &\leq \max\{D_f(z, u), D_f(z, x_n)\} \\ &\leq \cdots \leq \max\{D_f(z, u), D_f(z, z_0)\}. \end{aligned}$$

From this and Proposition 2.5, one sees that $\{x_n\}$ is bounded. Then, we assert that there exists a subsequence $\{z_{k_i}\}$ of $\{x_n\}$ such that $z_{k_i} \rightharpoonup p$ as $i \to \infty$. For the sake of simplicity, we set

$$\delta_n := \langle \nabla f(u) - \nabla f(z), \nabla f^*(\beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n)) - z\rangle. \tag{3.5}$$

With the help of (3.5), we calculate that

$$\begin{aligned} D_f(z, x_{n+1}) &\leq V_f(z, (1-\beta_n)\nabla f(y_n) + \beta_n \nabla f(u)) \\ &\leq V_f(z, (1-\beta_n)\nabla f(y_n) + \beta_n \nabla f(u) - \beta_n(\nabla f(u) - \nabla f(z))) + \beta_n \delta_n \\ &= D_f(z, \nabla f^*(\beta_n \nabla f(z) + (1-\beta_n)\nabla f(y_n)) + \beta_n \delta_n \\ &\leq (1-\beta_n) D_f(z, x_n) + \beta_n \delta_n. \end{aligned} \tag{3.6}$$

Next, let us consider two possible cases on $\{D_f(z, x_n)\}$.

Case 1: Suppose that there exists $k_0 \in \mathbb{N}$ such that $\{D_f(z, x_n)\}$ is non-increasing for all $n \geq n_0$. In such case, we directly obtain that $\{D_f(z, x_n)\}$ is convergent. From this and (3.3), one deduces that

$$\lim_{n\to\infty} \alpha_n (1-\lambda_n)(1-\alpha_n)\rho_r^*(\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\|) = 0.$$

By using the property of ρ_r^*, we directly obtain that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\| = 0. \tag{3.7}$$

Since f is a strongly coercive and uniformly convex on bounded subsets of X, it shows that f^* is uniformly Frechet differentiable on bounded subsets of X^* and ∇f^* is uniformly continuous. In light of (3.7), we immediately get that

$$\begin{aligned} \|\nabla f(y_n) - \nabla f(x_n)\| &= \|\nabla f(\nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(S_{\gamma_n} x_n))) - \nabla f(x_n)\| \\ &= (1-\alpha_n)\|\nabla f(x_n) - \nabla f(S_{\gamma_n} x_n)\| \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.8}$$

Keeping in mind that f is strongly coercive, uniformly convex on bounded subsets of X, it shows that f^* is uniformly Fréchet differentiable on bounded subsets of X^*. From this and Proposition 2.8, we observe that ∇f^* is uniformly continuous. By putting together (3.7) and (3.8), it ensues that

$$\lim_{n\to\infty} \|S_{\gamma_n} x_n - x_n\| = 0, \tag{3.9}$$

and

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.10}$$

Since S_λ satisfies $L(\lambda)$-Lipschizian for any $\lambda \in \mathscr{Q}$, we immediately deduce that

$$\begin{aligned} \|x_n - S_\lambda x_n\| &\leq \|x_n - S_{\gamma_n} x_n\| + \|S_{\gamma_n} x_n - S_\lambda S_{\gamma_n} x_n\| + \|S_\lambda S_{\gamma_n} x_n - S_\lambda x_n\| \\ &\leq \sup_{x \in \tilde{C}} \|S_\lambda S_{\gamma_n} x - S_{\gamma_n} x\| + \|S_{\gamma_n} x_n - x_n\| + L(\lambda)\|x_n - S_{\gamma_n} x_n\|, \end{aligned} \tag{3.11}$$

where $\tilde{C}$ is a bounded subset of C, which contains $\{x_n\}$. Due to the fact that $\mathfrak{J} = \{S_\lambda : \lambda \in \mathscr{Q}\}$ satisfies the uniformly asymptotically regularity condition, it follows from (3.9) with (3.11) that

$$\lim_{n\to\infty} \|S_\lambda x_n - x_n\| = 0.$$

By invoking the fact that $(I-S_\lambda)$ is demiclosed at zero for $\lambda \in \mathscr{Q}$, we check that $p \in \mathscr{O}$. Observing that $z = Proj_{\mathscr{O}}u$, $\lim_{n\to\infty}\beta_n = 0$, and the fact that $z_{n_i} \rightharpoonup p$ as $i \to \infty$, one successively finds from (3.10) that

$$\limsup_{n\to\infty}\delta_n = \lim_{i\to\infty}\delta_{n_i} = \langle \nabla f(u) - \nabla f(z), p - z\rangle \leq 0.$$

From $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\beta_n = \infty$, Proposition 2.10 leads to $\lim_{n\to\infty} D_f(z, x_n) = 0$. Since f is totally convex on bounded sets of X, we conclude from Proposition 2.4 that $\lim_{n\to\infty} x_n = z$ immediately.

Case 2: Suppose that there is no $n_0 \in \mathbb{N}$ such that $\{D_f(z, z_n)\}_{n=n_0}^{\infty}$ is monotonically decreasing. Accordingly, by using Proposition 2.11, there exists a strictly increasing sequence $\{m_j\}$ of positive integers such that

$$D_f(z, z_{m_j}) \leq D_f(z, z_{m_j+1}),\ D_f(z, z_j) \leq D_f(z, z_{m_j+1}),\ \forall j \in \mathbb{N}.$$

We go back to (3.4) to obtain that

$$\alpha_{m_j}(1-\beta_{m_j})(1-\alpha_{m_j})\rho_r^*(\|\nabla f(z_{m_j}) - \nabla f(S_{\gamma_{m_j}} z_{m_j})\|) \leq \beta_{m_j} D_f(z, u) - \beta_{m_j} D_f(z, z_{m_j}).$$

On the other hand, by using the conditions

$$0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$$

and $\lim_{n\to\infty}\beta_n = 0$, one sees that

$$\lim_{j\to\infty}\|\nabla f(z_{m_j}) - \nabla f(S_{\gamma_{m_j}} z_{m_j})\| = 0.$$

The same argument as **Case 1** shows that

$$\lim_{j\to\infty}\|y_{m_j} - x_{m_j}\| = 0,$$

which further implies that

$$\limsup_{j\to\infty}\delta_{m_j} = \langle \nabla f(u) - \nabla f(z), p - z\rangle \leq 0. \tag{3.12}$$

Collecting the above results (3.6), and (3.12), one arrives at

$$D_f(z, x_j) \leq D_f(z, x_{m_j+1}) \leq \delta_{m_j}.$$

Consequently, we deduce that $\lim_{j\to\infty} D_f(z, x_j) = 0$. By assumption, we find that $\lim_{j\to\infty} x_j = z$. Therefore, we can conclude that $\{x_n\}$ converges to $z = Proj_{\mathscr{O}}^f u$, as $n \to \infty$, in norm. This completes the proof. □

4. Numerical examples

In this section, we present two numerical examples to test our main result. All the codes were written in Python 3.7 and run on PC with Intel (R) Core (TM) i5-8250U CPU @1.60GHz. We apply our proposed algorithm to solve the problems and compare numerical results with Algorithm VI-Method mentioned in Theorem 1.1.

Example 4.1. Let $C = \mathbb{R}^M$ be an M-dimensional Euclidean space. For any sequence $\{\gamma_n\} \subset [1,+\infty)$, we define an $M \times M$ square matrix by $S_{\gamma_n} := (a_{t_{ij}})_{1 \leq i,j \leq M}$ whose terms are given by

$$a_{t_{ij}} = \begin{cases} 1/\gamma_n, & \text{if } j = M+1-i \text{ and } j < i; \\ -1/\gamma_n, & \text{if } j = M+1-i \text{ and } j > i; \\ 0, & \text{otherwise.} \end{cases}$$

With the given C and $\mathfrak{J} = \{S_{\gamma_n} : \gamma_n \in [1,+\infty)\}$, we use Algorithm (3.1) and Algorithm VI-Method to solve the fixed point problem. The contraction mapping can be defined by $g(x) = 0.5x$, $\forall x \in C$ for Algorithm VI-Method. Without loss of generality, we ssume that

(1) $\alpha_n = 0.05$, $\beta_n = 1/(n+1)$, $\gamma_n = 10+n$ ($\forall n \in \mathbb{N}$) for Algorithm (3.1);
(2) $\alpha_n = 0.05$, $\beta_n = 1/(n+1)$, $\lambda_n = 1/(5+n)$ and $\gamma_n = 10+n$ ($\forall n \in \mathbb{N}$) for Algorithm VI-Method.

The initial point x_1 is randomly choose from $[0,1]^M$. Fix the anchor point $u = x_1$. We take the number of iteration $n = 1000$ as the stopping criterion.

Figure 1 shows a comparison between Algorithm (3.1) and Algorithm VI-Method in two cases respectively when $M = 30$ and $M = 50$. Based on the provided data, we see that our proposed algorithm dramatically reduces iterations and the running time needed to find a fixed point of the semigroup $\mathfrak{J}$ over the feasible set C, compared with Algorithm VI-Method.

As displayed in Figure 2, we know that the convergent point $x = (0,0,0,0)^{\mathsf{T}}$, which is also the common fixed point of $\mathfrak{J}$.

Example 4.2. Let $X = L^2([0,1])$ with the inner product

$$\langle x,y \rangle := \int_0^1 x(p)y(p)\,\mathrm{d}p, \quad \forall x,y \in X,$$

and the induced norm

$$\|x\|_{L^2} := \Big(\int_0^1 |x(p)|^2\,\mathrm{d}p \Big)^{\frac{1}{2}}, \quad \forall x \in X.$$

Suppose that $C := \{x \in X : \|x\| \leq 1\}$ is a unit ball. Give a sequence $\{\gamma_n\} \subseteq [1,+\infty)$. We define mappings $S_{\gamma_n} : C \to X$ by

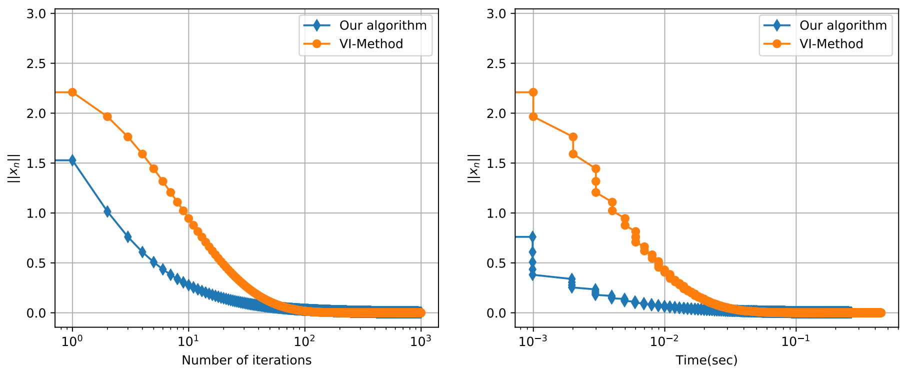
$$S_{\gamma_n} x(p) = \max\{0, x(p)/\gamma_n\}.$$

We check that $\mathfrak{J} = \{S_{\gamma_n} : \gamma_n \in [1,+\infty)\}$ is a semigroup of 1-Lipschitz continuous and quasi-nonexpansive mappings on C. The solution set of the fixed point problem with respect to $\mathfrak{J}$ is given by $\Gamma = \{0\} \neq \emptyset$. According to [40], we have that

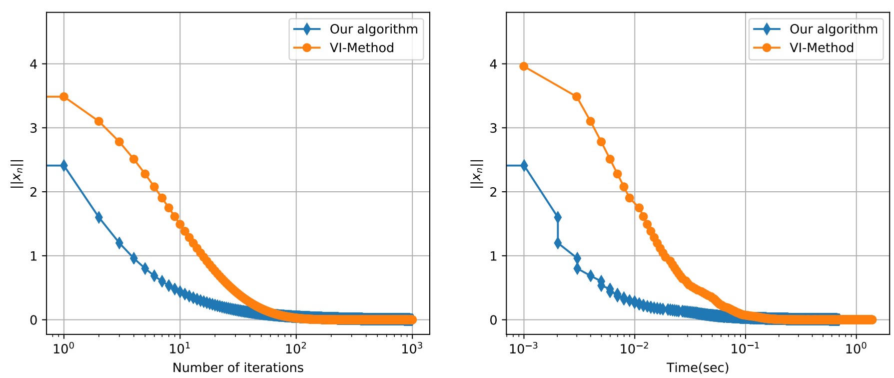
$$P_C(x) = \begin{cases} \dfrac{x}{\|x\|_{L^2}}, & \text{if } \|x\|_{L^2} \geq 1; \\ x, & \text{if } \|x\|_{L^2} \leq 1. \end{cases}$$

We set $\alpha_n = 0.5$, $\beta_n = 1/(n+3)$, $\gamma_n = 2+n$ ($\forall n \in \mathbb{N}$) for Algorithm (3.1). We give the starting point $x_1 = 1$. Fix the anchor point $u = x_1$. We take the number of iterations $n = 1000$ as the stopping criterion. The numerical result is described in Figure 3.

In Figure 3, the value of errors $\{\|x_n\|_{L^2}\}$, $\{\|x_n - y_n\|_{L^2}\}$ and $\{\|x_{n+1} - x_n\|_{L^2}\}$ is represented by the y-axis, and the number of iterations and the running time are represented by the x-axis. The convergence of $\|x_n\|_{L^2}$ to 0 indicates that the generated sequence $\{x_n\}$ converges to the solution 0, as $n \to \infty$.

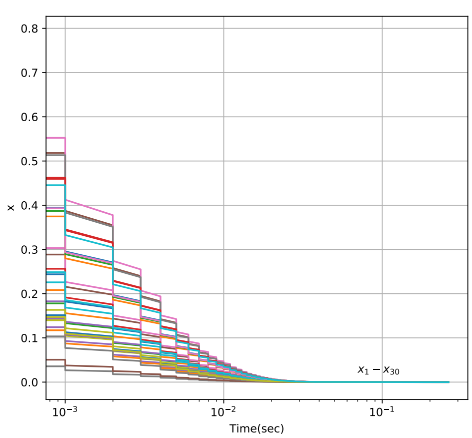


(a) $M = 30$

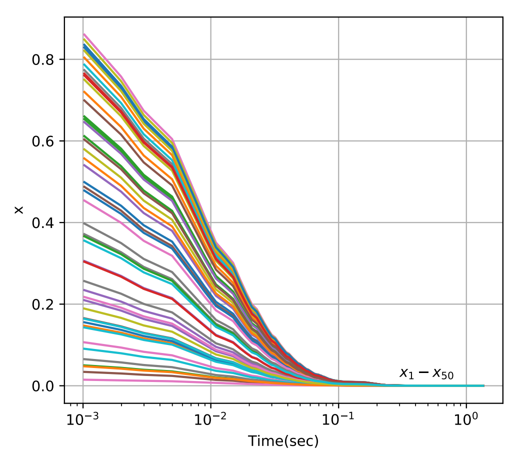


(b) $M = 50$

FIGURE 1. Convergence behavior of the sequence $\{\|x_n\|\}$ for Example 4.1



(a) $M = 30$



(b) $M = 50$

FIGURE 2. Convergence behavior of each component of x_n for Example 4.1.

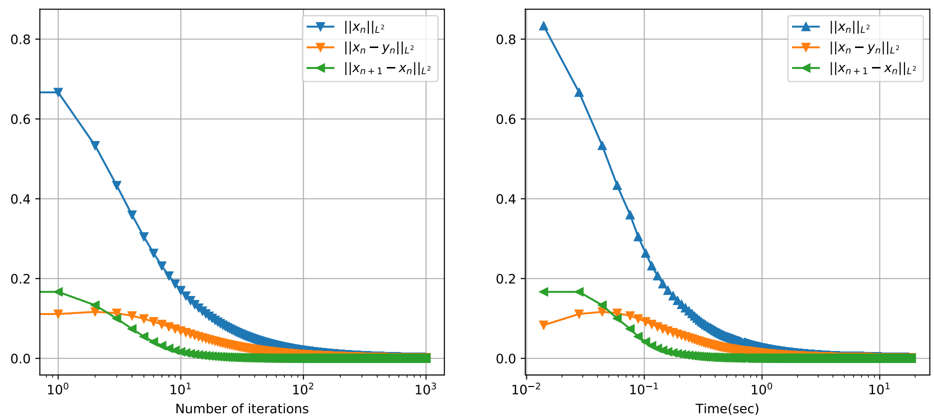


FIGURE 3. Convergence behavior of sequences $\{\|x_n\|_{L^2}\}$, $\{\|x_n - y_n\|_{L^2}\}$ and $\{\|x_{n+1} - x_n\|_{L^2}\}$ for Example 4.2.

5. CONCLUSIONS

In this paper, we investigated BQNE mappings in our convergence analysis. One of the highlights of this paper is that our proposed iterative scheme is extremely general since BQNE mappings have some remarkable properties not shared by some other operators. In addition, our fixed point results of BQNE mappings can be applied to many optimization problems in finite dimensional spaces. Numerical experiments approved the convergence analysis of the algorithm. The results presented in this paper also showed the computational advantages of our proposed algorithm, compared with the previous algorithm [27].

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