J. Nonlinear Var. Anal. 5 (2021), No. 2, pp. 201-210
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.5.2021.2.02

A NEW ITERATIVE ALGORITHM FOR SPLIT FEASIBILITY AND FIXED POINT PROBLEMS

CHARLES E. CHIDUME*, ABUBAKAR ADAMU

Mathematics Institute (An ERCE of the European Mathematical Society),
African University of Science and Technology, Abuja, Nigeria

Abstract. A new iterative algorithm for approximating a solution of a split feasibility problem and the fixed point problem of a quasi-ϕ-nonexpansive mapping is proposed and studied. It is proved that the sequence generated by the new algorithm converges strongly to a common solution of the split feasibility problem and the fixed point problem in real Banach spaces, which are more general than Hilbert spaces.
Keywords. Fixed point; Halpern method; quasi-ϕ-nonexpansive; Strong convergence; Banach space.

1. Introduction

Let C and D be nonempty closed and convex subsets of real Banach spaces E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear map. The *split feasibility problem* (SFP) consists of

$$\text{finding} \quad x^* \in C \quad \text{such that} \quad Ax^* \in D. \tag{1.1}$$

The SFP was first introduced by Censor and Elfving [1] in finite dimensional real Hilbert spaces for modeling inverse problems, which arise in image reconstructions and phase retrievals. It is now well known that the SFP and its generalizations have numerous real applications in several disciplines, such as, medical imaging [2]. Hence, they have attracted the attention and interest of many researchers; see, e.g., [3, 4, 5, 6, 7] and the references therein.

For approximating a solution of the SFP in finite dimensional real Hilbert spaces, Censor and Elfving [1] studied, based on projections, an iterative algorithm, which involves the numerical computation of the inverse of a matrix. To overcome this drawback, Byrne *et al.* [8] proposed the so-called *C-Q algorithm*, which does not involve the computations of the inverse of any matrix but involves projections onto two subsets C and Q. However, the calculation of projections is not easy for general sets.

In 2010, Moudafi [9] studied the problem of approximating a solution of the *split common fixed point problem* (SCFPP) (see [9]) for quasi-nonepxansive mappings and demi-contractive mappings in real Hilbert spaces. He proposed an iterative method which does not involve projections and proved weak convergence theorems in real Hilbert spaces.

*Corresponding author.
E-mail addresses: cchidume@aust.edu.ng (C.E. Chidume), aadamu@aust.edu.ng (A. Adamu).
Received October 12, 2020; Accepted January 30, 2021.

In 2014, Kraikaew and Saejung [10] combined the Moudafi's method [9] with the Halpern-type method and proposed a new iterative algorithm, which does not involve the projection operator for solutions of the split feasibility problem in real Hilbert spaces.

Recently, many authors investigated various iterative algorithms for approximating a solution of the SFP and its generalizations in real Hilbert spaces; see, e.g, [11, 12, 13, 14, 15] and the references therein. However, there are few results established in real Banach spaces, which is more general than Hilbert spaces.

In 2015, Tang *et al.* [16] introduced an iterative algorithm, which does not involve the projection operator to approximate a solution of the split common fixed point problem of a quasi-strict pseudocontractive mapping and an asymptotically nonexpansive mapping in two Banach spaces and one of the two Banach spaces, E_1, is uniformly convex, and 2-uniformly smooth real Banach space. They proved weak and strong convergence theorems under the assumption of the semi-compactness of the operators involved. However, it was shown (see [17]) that the Banach space E_1 studied by Tang *et al.* [16], which was supposed to be more general than Hilbert spaces, is necessarily a real Hilbert space.

Recently, in 2018, Ma, Wang and Chang [18] proposed an iterative algorithm to approximate a common solution of the split feasibility problem and the fixed point problem of quasi-ϕ-nonexpansive mappings in the setting of real Banach spaces that are 2-uniformly convex and 2-uniformly smooth. They proved that the sequence generated by their proposed algorithm converges strongly to a common solution of the split feasibility and the fixed point problem without the assumption of semi-compactness on the mappings. However, we note that the 2-uniformly convex and 2-uniformly smooth Banach space is indeed a real Hilbert space.

It is our purpose in this paper to propose a new iterative algorithm for approximating a common solution of the split feasibility problem and the fixed point problem of quasi-ϕ-nonexpansive mappings in the setting of two real Banach spaces, which are much more general than real Hilbert spaces. Under our setting, E_1 is assumed to be a p-uniformly convex ($p > 1$) and uniformly smooth real Banach spaces and E_2 is an arbitrary smooth real Banach space. Furthermore, the sequence generated by our algorithm is proved to converge strongly to a common solution of the split feasibility problem and the fixed point problem without any assumption of semi-compactness on the mappings. The convergence theorem proved in this paper is applicable to $L_p,\ l_p$ and the Sobolev spaces $W_p^m(\Omega)$, for $1 < p < \infty$.

2. Preliminaries

Let E be a strictly convex and smooth real Banach space. For $p > 1$, let $J_p : E \to 2^{E^*}$ be defined by

$$J_p(x) := \{u^* \in E^* : \langle x, u^* \rangle = \|x\| \|u^*\|,\ \ \|u^*\| = \|x\|^{p-1}\}.$$

J_p is called the *generalized duality mapping on E*. If $p = 2$, J_2 is called the *normalized duality mapping* and is denoted by J. In a real Hilbert space H, J is the identity map on H. It is easy to see from the definition that

$$J_p(x) = \|x\|^{p-2} Jx, \quad \text{and} \quad \langle x, J_p x \rangle = \|x\|^p,\ \ \forall x \in E.$$

It is well-known that if E is smooth, then J is single-valued and if E is strictly convex, J is one-to-one, and J is surjective if E is reflexive.

Definition 2.1. [19] Let E be a real normed space with dimension $E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) := \left\{1 - \left|\left|\frac{u+v}{2}\right|\right| : ||u|| = ||v|| = 1; \varepsilon = ||u-v||\right\}.$$

Let $p > 1$ be a real number and $\delta_E : (0,2] \to [0,1]$ be the modulus of convexity of E. Then, a normed space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$.

It is well known that L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, for $1 < p < \infty$, are p-uniformly convex and that the following estimates hold (see, e.g., [19] and [20]):

$$\delta_{l_p}(\varepsilon) = \delta_{L_p}(\varepsilon) = \delta_{W_m^p(\Omega)}(\varepsilon) = \begin{cases} \frac{p-1}{8}\varepsilon^2 + o(\varepsilon^2) > \frac{p-1}{8}\varepsilon^2, & 1 < p < 2; \\ 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}} > \frac{1}{p}\left(\frac{\varepsilon}{2}\right)^p & p \geq 2. \end{cases}$$

Definition 2.2. [21] Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty closed and convex subset of E. The map $\Pi_C : E \to C$ defined by $\tilde{x} = \Pi_C(x) \in C$ such that $\phi(\tilde{x}, x) = \inf_{y \in C} \phi(y, x)$ is called the *generalized projection* of E onto C. Clearly, in a real Hilbert space H, the generalized projection Π_C coincides with the metric projection P_C from H onto C.

Definition 2.3. [21] Let E_1 and E_2 be two reflexive, strictly convex and smooth real Banach spaces. The collection of mappings $A : E_1 \to E_2$ that are linear and continuous is a normed linear space with norm defined by $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$. The dual operator $A^* : E_2^* \longrightarrow E_1^*$ defined by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle$, $\forall\, x \in E_1$, $y^* \in E_2^*$ is called the adjoint operator of A. The adjoint operator A^* has the property $\|A^*\| = \|A\|$.

Definition 2.4. [22] Let C be a nonempty closed and convex subset of a real Banach space E and let $T : C \to C$ be a mapping. Then, T is said to be quasi-ϕ-nonexpansive if $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ and

$$\phi(x, Ty) \leq \phi(x, y)\ \forall\, x \in F(T),\ y \in C.$$

Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^*. For $p > 1$, Chidume [23] defined the following functionals: $\phi_p : E \times E \to \mathbb{R}^+$ by

$$\phi_p(x,y) := \|x\|^p - p\langle x, J_p y\rangle + \|J_p y\|^p,\ \ \forall x, y \in E;$$

$V_p : E \times E^* \to \mathbb{R}^+$ by

$$V_p(x, x^*) := \|x\|^p - p\langle x, x^*\rangle + \|x^*\|^p,\ \ \forall x \in E,\ x^* \in E^*.$$

It is clear from these definitions that

$$V_p(x, x^*) = \phi_p(x, J_p^{-1}x^*),\ \ \forall x \in E,\ x^* \in E^*. \tag{2.1}$$

Remark 2.1. If $p = 2$, we denote $\phi_2(x,y)$ simply as $\phi(x,y)$ and

$$\phi(x,y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2,\ \ \forall x, y \in E.$$

In the sequel, we need the following lemmas recently established by Chidume in [23].

Lemma 2.1. *Let E be a reflexive, strictly convex and smooth real Banach space. Then, for $p > 1$,*

$$V_p(u, u^*) + p\langle J_p^{-1}u^* - u, v^*\rangle \leq V_p(u, u^* + v^*),\ \ \forall u \in E,\ \ u^*, v^* \in E^*. \tag{2.2}$$

Lemma 2.2. *For $p > 1$, let E be a p-uniformly convex and smooth real Banach space. Let D be a nonempty closed and convex subset of E. Let $x_1 \in E$ be arbitrary and $P_D : E \to D$ be the metric projection of E onto D. Then*

$$x^* = P_D x_1 \;\Leftrightarrow\; \langle x^* - z, J_p(x_1 - x^*)\rangle \geq 0, \;\; \forall z \in D.$$

Lemma 2.3. [24] *Let E be a p-uniformly convex and smooth real Banach space. Then, there exists a constant $c_2 > 0$ such that, for every $x, y \in E$,*

$$\langle x - y, J_p x - J_p y\rangle \geq c_2 \|x - y\|^p.$$

Lemma 2.4. [23] *Let E be a p-uniformly convex and smooth real Banach space with dual space E^*. For $p > 1$, let $J_p : E \to E^*$ be the generalized duality map. Then,*

$$\|J_p^{-1}x - J_p^{-1}y\| \leq \kappa_p \|x - y\|^{\frac{1}{p-1}}, \quad \forall x, y \in E, \tag{2.3}$$

where $\kappa_p = \left(\frac{1}{c_2}\right)^{\frac{1}{p-1}}$ with c_2 being the constant appearing in Lemma 2.3

Lemma 2.5. [23] *Let E be a reflexive, strictly convex and smooth real Banach space. Then, for $p > 1$,*

$$\phi_p\big(x, J_p^{-1}(\lambda J_p u + (1-\lambda) J_p v)\big) \leq \lambda \phi_p(x, u) + (1-\lambda)\phi_p(x, v), \quad \forall x, u, v \in E. \tag{2.4}$$

We also need the following well known lemmas.

Lemma 2.6. [21] *Let D be a nonempty closed and convex subset of a reflexive, strictly convex and smooth real Banach space E. Then,*

$$\phi(u, \Pi_D y) + \phi(\Pi_D y, y) \leq \phi(u, y), \;\; \forall\, u \in D, y \in E, \tag{2.5}$$

where Π_D is the generalized projection of E onto D.

Lemma 2.7. [25] *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$, as $n \to \infty$.*

Lemma 2.8. [24] *For $p > 1$, let E be a p-uniformly convex real Banach space. Then, there exists a constant $c_p > 0$ such that, for all $x, y \in E$,*

$$\|\alpha x + (1-\alpha) y\|^p \leq \alpha \|x\|^p + (1-\alpha)\|y\|^p - c_p w_p(\alpha)\|x - y\|^p, \tag{2.6}$$

where $w_p(\alpha) := (1-\alpha)^p \alpha + (1-\alpha)\alpha^p$, $\alpha \in (0,1)$

2.1. **Analytical representations of generalized duality maps in L_p, l_p, and W_m^p, spaces with $1 < p < \infty$.** Using the analytic representation of the normalized duality mappings in L_p, l_p, and W_m^p, $1 < p < \infty$ (see, e.g., Lindenstrauss and Tzafriri [20]) and the relation $J_p(x) = ||x||^{p-2}J(x)$, we obtain the analytical representations of generalized duality mappings in these spaces as follows:

$$\begin{aligned}
Jz &= y \in l_q,\;\; y = \{|z_1|^{p-2}z_1, |z_2|^{p-2}z_2, \dots\},\;\; z = \{z_1, z_2, \dots\},\\
J^{-1}z &= y \in l_p,\;\; y = \{|z_1|^{q-2}z_1, |z_2|^{q-2}z_2, \dots\},\;\; z = \{z_1, z_2, \dots\},\\
Jz &= \|z\|_{L_p}^{2-p}|z(s)|^{p-2}z(s) \in L_q(G),\;\; s \in G,\\
J^{-1}z &= |z(s)|^{q-2}z(s) \in L_p(G),\;\; s \in G, \text{ and}\\
Jz &= \sum_{|\alpha|\le m}(-1)^{|\alpha|}D^{\alpha}(|D^{\alpha}z(s)|^{p-2}D^{\alpha}z(s)) \in W^q_{-m}(G), m > 0, s \in G,
\end{aligned}$$

3. MAIN RESULT

Before prove our main convergence theorem, we list the following assumptions:

1. E_1 is a p-uniformly convex and uniformly smooth real Banach space, and E_2 is an arbitrary smooth real Banach space;
2. $T : E_1 \to E_1$ is a closed quasi-ϕ-nonepxansive mapping;
3. $A : E_1 \to E_2$ is a bounded linear mapping with adjoint A^*;
4. $\alpha \in (0,1)$ and γ are constants such that $0 < \gamma < \frac{1}{\kappa_p^{p-1}\|A\|^p}$;
5. Q is a nonempty closed and convex subset of E_2, and $F(T) := \{w \in E_1 :\; Tw = w\}$. P_Q denotes the metric projection of E_2 onto Q, and $\Pi_{C_{n+1}}x_1$ denotes the generalized projection of x_1 onto C_{n+1}, $\forall\, n \ge 1$;
6. $J_i,\; i = 1,2$ denotes the generalized duality mapping on $E_i,\; i = 1,2$, respectively; and $J_i^{-1},\; i = 1,2$ denotes the generalized duality mapping on $E_i^*,\; i = 1,2$, respectively; where $E_i^*,\; i = 1,2$, represents the dual space for each i.

Algorithm 3.1.

$$\begin{cases}
x_1 \in E_1,\; C_1 = E_1 \;\; e_n = J_2(P_Q - I)Ax_n, \;\; n \ge 1,\\
z_n = J_1^{-1}(J_1x_n + \gamma A^*e_n),\\
y_n = J_1^{-1}\big((1-\alpha)J_1z_n + \alpha J_1Tz_n\big),\\
C_{n+1} = \{v \in C_n : \phi_p(v,z_n) \le \phi_p(v,x_n);\;\; \phi_p(v,y_n) \le \phi_p(v,x_n)\},\\
x_{n+1} = \Pi_{C_{n+1}}x_1.
\end{cases}$$

We now in a position to prove our main convergence theorem.

Theorem 3.1. *If $\Gamma := \{w \in F(T) : Aw \in Q\} \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to some $x^* \in \Gamma$ with $x^* = \Pi_\Gamma x_1$.*

Proof. We divide the proof into 5 steps.
Step 1. Show that C_n is closed and convex for each $n \ge 1$.

Clearly, $C_1 = E_1$ is closed and convex. Assume that C_n is closed and convex for some $n \ge 1$. For any $v \in C_n$, we have

$$\phi_p(v,z_n) \le \phi_p(v,x_n) \Leftrightarrow p\langle v, J_1(x_n) - J_1(z_n)\rangle \le \|J_1(x_n)\|^p - \|J_1(z_n)\|^p,$$

$$\phi_p(v,y_n) \le \phi_p(v,x_n) \Leftrightarrow p\langle v, J_1(x_n) - J_1(y_n)\rangle \le \|J_1(x_n)\|^p - \|J_1(y_n)\|^p.$$

From these conditions, we conclude that C_{n+1} is closed and convex.
Step 2. Show that $\Gamma \subset C_n$, for each $n \ge 1$.

Let $w^* \in \Gamma$. Using Lemma 2.1, we compute as follows

$$\begin{aligned}\phi_p(w^*,z_n) &= \phi_p(w^*,J_1^{-1}(J_1x_n+\gamma A^*e_n))\\ &= V_p(w^*,J_1x_n+\gamma A^*e_n)\\ &\leq V_p(w^*,J_1x_n) - p\langle J_1^{-1}(J_1x_n+\gamma A^*e_n)-w^*,-\gamma A^*e_n\rangle\\ &= \phi_p(w^*,x_n) - p\gamma\langle Aw^*-Az_n,e_n\rangle\\ &= \phi_p(w^*,x_n) - p\gamma\langle Aw^*-Ax_n,e_n\rangle - p\gamma\langle Ax_n-Az_n,e_n\rangle. \end{aligned} \tag{3.1}$$

It follows from Lemma 2.2 that

$$\begin{aligned}\langle Aw^*-Ax_n,e_n\rangle &= \langle Aw^*-Ax_n,J_2(P_Q-I)Ax_n\rangle\\ &= \langle Aw^*-P_QAx_n,J_2(P_QAx_n-Ax_n)\rangle + \|(P_Q-I)Ax_n\|^p\\ &\geq \|(P_Q-I)Ax_n\|^p. \end{aligned} \tag{3.2}$$

Using Lemma 2.4 yields that

$$\begin{aligned}-\langle Ax_n-Az_n,e_n\rangle &\leq \|A\|.\|z_n-x_n\|.\|J_2(P_Q-I)Ax_n\|\\ &\leq \|A\|.\|J_1^{-1}(J_1x_n+\gamma A^*J_2(P_Q-I)Ax_n)-J_1^{-1}J_1x_n\|.\|J_2(P_Q-I)Ax_n\|\\ &\leq \|A\|\kappa_p\|\gamma A^*J_2(P_Q-I)Ax_n\|^{\frac{1}{p-1}}\|J_2(P_Q-I)Ax_n\|\\ &\leq \|A\|\kappa_p\gamma^{\frac{1}{p-1}}\|A\|^{\frac{1}{p-1}}\|J_2(P_Q-I)Ax_n\|^{\frac{p}{p-1}}\\ &= \kappa_p\gamma^{\frac{1}{p-1}}\|A\|^{\frac{p}{p-1}}\|J_2(P_Q-I)Ax_n\|^{\frac{p}{p-1}}. \end{aligned} \tag{3.3}$$

Substituting (3.2) and (3.3) in (3.1), we obtain

$$\begin{aligned}\phi_p(w^*,z_n) &\leq \phi_p(w^*,x_n) - p\gamma\|(P_Q-I)Ax_n\|^p + p\gamma^{\frac{p}{p-1}}\kappa_p\|A\|^{\frac{p}{p-1}}\|(P_Q-I)Ax_n\|^p\\ &= \phi_p(w^*,x_n) - p\gamma\Big(1-\gamma^{\frac{1}{p-1}}\kappa_p\|A\|^{\frac{p}{p-1}}\Big)\|(P_Q-I)Ax_n\|^p. \end{aligned} \tag{3.4}$$

Using the condition that $0<\gamma<\frac{1}{\kappa_p^{p-1}\|A\|^p}$, inequality (3.4), we arrive at

$$\phi_p(w^*,z_n) \leq \phi_p(w^*,x_n). \tag{3.5}$$

Furthermore, using Lemma 2.5 and inequality (3.5), we obtain that

$$\begin{aligned}\phi_p(w^*,y_n) &= \phi_p\big(w^*,J_1^{-1}\big[(1-\alpha)J_1z_n+\alpha J_1Tz_n\big]\big)\\ &\leq (1-\alpha)\phi_p(w^*,z_n) + \alpha\phi_p(w^*,Tz_n)\\ &\leq \phi_p(w^*,z_n) \leq \phi_p(w^*,x_n). \end{aligned} \tag{3.6}$$

From inequalities (3.5) and (3.6), we obtain that $w^* \in C_{n+1}$, which further implies that $\Gamma \subset C_n$, for all $n \geq 1$. Hence, $\{x_n\}$ is well defined.

Step 3. Show that $\{x_n\}$ is a Cauchy sequence.

We observe that since E_1 is p-uniformly convex, which means that it is reflexive and strictly convex. So, Lemmas 2.6 and 2.7 are applicable. Hence, we can use the functional ϕ in these lemmas instead of the generalized functional ϕ_p. Using these lemmas, the proof that $\{x_n\}$ is a Cauchy sequence is standard. However, for the completeness, we sketch the short proof here. Let $w \in \Gamma$ be arbitrary. We have $x_n = \Pi_{C_n}x_1$, $\forall n \geq 1$. It follows from Lemma 2.6 that

$$\phi(x_n,x_1) \leq \phi(w,x_1),\ \forall n \geq 1,\ w \in \Gamma.$$

This implies that $\{\phi(x_n,x_1)\}$ is bounded. Consequently, $\{x_n\}$ is bounded. Also, since $x_n=\Pi_{C_n}x_1$ and $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}\subset C_n$, it follows from Lemma 2.6 and the definition of C_{n+1} that $\phi(x_n,x_1)\le\phi(x_{n+1},x_1)$, so that $\{\phi(x_n,x_1)\}$ is non-decreasing and bounded. Hence, $\lim\limits_{n\to\infty}\phi(x_n,x_1)$ exists. For any positive integers m,n, without loss of generality, let $n>m$. Then, from $x_n=\Pi_{C_n}x_1\in C_n\subset C_m$ and Lemma 2.6, we have

$$\begin{aligned}\phi(x_m,x_n)&=\phi(x_m,\Pi_{C_n}x_1)\le\phi(x_m,x_1)-\phi(\Pi_{C_n}x_1,x_1)\\&=\phi(x_m,x_1)-\phi(x_n,x_1)\to 0,\ \ as\ \ m,n\to\infty.\end{aligned}$$

By using Lemma 2.7, we obtain that $\lim\limits_{n,m\to\infty}\|x_n-x_m\|=0$. Hence, $\{x_n\}$ is a Cauchy sequence.

Step 4. Show that $(i)\ \lim\limits_{n\to\infty}\|(P_Q-I)Ax_n\|=0$ and $(ii)\ \lim\limits_{n\to\infty}\|z_n-Tz_n\|=0$.

Since $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_n$, it follows from the definition of C_{n+1} that

$$\phi(x_{n+1},z_n)\le\phi(x_{n+1},x_n)\quad\text{and}\quad\phi(x_{n+1},y_n)\le\phi(x_{n+1},x_n).$$

From these inequalities, we obtain that $\lim_{n\to\infty}\phi(x_{n+1},z_n)=0$ and $\lim_{n\to\infty}\phi(x_{n+1},y_n)=0$. From Lemma 2.7, we conclude that

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0,\quad\lim_{n\to\infty}\|x_{n+1}-y_n\|=0\quad\text{and thus,}\quad\lim_{n\to\infty}\|y_n-z_n\|=0.\tag{3.7}$$

Since $\{x_n\}$ is a Cauchy sequence in E_1, we find that there exists $x^*\in E_1$ such that $x_n\to x^*\in E$ as $n\to\infty$. It then follows from equations (3.7) that $z_n\to x^*$ and $y_n\to x^*$ as $n\to\infty$. Using inequality (3.4), we obtain,

$$p\gamma\Big[1-\gamma^{\frac{1}{p-1}}\kappa_p\|A\|^{\frac{p}{p-1}}\Big]\|(P_Q-I)Ax_n\|^p\le\phi_p(w^*,x_n)-\phi_p(w^*,z_n)\to 0,\ \ as\ \ n\to\infty.$$

Since

$$p\gamma[1-\gamma^{\frac{1}{p-1}}\kappa_p\|A\|^{\frac{p}{p-1}}]>0,$$

it follows that $\lim_{n\to\infty}\|(P_Q-I)Ax_n\|=0$, which complete the proof of (i).

For (ii), we compute as follows:

$$\begin{aligned}\phi_p(w,y_n)&=\phi_p(w,J_1^{-1}[(1-\alpha)J_1z_n+\alpha J_1Tz_n])\\&=\|w\|^p-p\langle w,(1-\alpha)J_1z_n\rangle-p\langle w,\alpha J_1Tz_n\rangle+(1-\alpha)\|J_1z_n\|^p\\&\quad+\alpha\|J_1Tz_n\|^p-c_pw_p(\alpha)\|J_1z_n-J_1Tz_n\|^p\\&=\alpha\phi_p(w,Tz_n)+(1-\alpha)\phi_p(w,z_n)-c_pw_p(\alpha)\|J_1z_n-J_1Tz_n\|^p.\end{aligned}$$

This implies

$$\begin{aligned}c_pw_p(\alpha)\|J_1z_n-J_1Tz_n\|&\le\alpha\phi_p(w,Tz_n)+(1-\alpha)\phi_p(w,z_n)-\phi_p(w,y_n)\\&\le\alpha\phi_p(w,z_n)+(1-\alpha)\phi_p(w,z_n)-\phi_p(w,y_n)\\&=\phi_p(w,z_n)-\phi_p(w,y_n)\to 0\ \ as\ \ n\to\infty.\end{aligned}$$

Since $c_pw_p(\alpha)>0$, we obtain that

$$\lim_{n\to\infty}\|J_1z_n-J_1Tz_n\|=0.\tag{3.8}$$

Since E_1^* is uniformly smooth, we obtain that J_1^{-1} is uniformly continuous on bounded sets. Hence, equation (3.8) yields that $\lim_{n\to\infty}\|z_n-Tz_n\|=0$, establishing (ii).

Step 5. Show that $Tx^*=x^*$ and $x^*=\Pi_\Gamma x_1$.

Since $z_n \to x^*$ and $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$, it follows from the fact that T is closed that $Tx^* = x^*$. Furthermore, from step 4, we proved that $\|(P_Q - I)Ax_n\| \to 0$ and $x_n \to x^*$, as $n \to \infty$. Since $(P_Q - I)A$ is continuous, it follows that $\|(P_Q - I)Ax^*\| = 0$, so $Ax^* \in Q$. Hence, $x^* \in \Gamma$. Let $y^* := \Pi_\Gamma x_1 \in \Gamma$. From $x_n = \Pi_{C_n} x_1$ and $y^* \in \Gamma \subset C_n$, we have

$$\phi(\Pi_\Gamma x_1, \Pi_{C_n} x_1) + \phi(\Pi_{C_n} x_1, x_1) \le \phi(y^*, x_1),$$

which implies that $\phi(x_n, x_1) \le \phi(y^*, x_1)$. This implies that

$$\phi(x^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \le \phi(y^*, x_1) = \phi(\Pi_\Gamma x_1, x_1) \le \phi(x^*, x_1).$$

This implies that $\phi(x^*, x_1) = \phi(y^*, x_1)$, so that $y^* = x^*$. Hence, $\{x_n\}$ converges strongly to $x^* \in \Gamma$, where $x^* = \Pi_\Gamma x_1$. The proof is complete. □

Remark 3.1. Theorem 3.1 holds if the constant $\alpha \in (0,1)$ is replaced with a sequence $\{\alpha_n\}$ in $(0,1)$ such that $0 < \delta \le \alpha_n < 1$, $\forall n \ge 1$ and some $\delta > 0$. The proof follows directly as in the proof of Theorem 3.1.

Remark 3.2. Theorem 3.1 holds if $E_1 = L_p$, l_p or the Sobolev spaces, $W_p^m(\Omega)$, for $1 < p < \infty$. These spaces are p-uniformly convex and uniformly smooth (see, e.g., Linderstraus and Tzaferri [20], see also Chidume [19], page 44).

Remark 3.3. The condition on γ in the proof of theorem 3.1 depends on the norm, $||A||$, of A. This is not a drawback on implementing the algorithm because, *one does not need to compute this norm* to use the algorithm. For computational purposes, this norm can be replaced with a constant associated with the mapping A and this is easy to compute as follows. To assert that a linear mapping A is bounded, one has to show that the following inequality holds:

$$||Ax|| \le K||x||, \quad \forall x \in E, \tag{3.9}$$

for some constant $K > 0$. This constant $K > 0$ is an upper bound for $||A||$ and is generally easy to obtain (since it is not unique) for any bounded linear mapping.

It is easy to see from the proof of Theorem 3.1 that the condition

$$0 < \gamma < \left[\frac{1}{\kappa_p ||A||^{\frac{p}{p-1}}}\right]^{p-1},$$

can then be replaced with the condition

$$0 < \gamma < \left[\frac{1}{\kappa_p K^{\frac{p}{p-1}}}\right]^{p-1},$$

where K is easily obtained from inequality (3.9).

We note that in the special case with $1 < p \le 2$, the spaces L_p, l_p and $W_p^m(\Omega)$ are 2-uniformly convex spaces. In this case, Algorithm 3.1 is reduced to the following algorithm.

Algorithm 3.2.

$$\begin{cases} x_1 \in E_1,\ C_1 = E_1 \quad e_n = J_2(P_Q - I)Ax_n, \quad n \geq 1, \\ z_n = J_1^{-1}(J_1 x_n + \gamma A^* e_n), \\ y_n = J_1^{-1}\big((1-\alpha)J_1 z_n + \alpha J_1 T z_n\big) \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, x_n); \quad \phi(v, y_n) \leq \phi(v, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_1. \end{cases}$$

where J_i is the usual normalized duality mapping on E_i and ϕ is the Lyapunov functional of Alber [21].

Acknowledgments

The authors appreciate the support from their institution. The authors would also like to thank the referees for their esteemed comments and suggestions.

References

[1] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algo. 8 (1994), 221-239.

[2] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, Phys. Med. Biol. 51 (2006), 2353-2365.

[3] X. Qin, L. Wang, J.C. Yao, Inertial splitting method for maximal monotone mappings, J. Nonlinear Convex Anal. 21 (2020), 2325-2333.

[4] W. Takahahsi, H.K. Xu, J.C. Yao, Iterative methods for generalized split feasibility problems in Hilbert spaces, Set-Valued Var. Anal. **23** (2015), 205–221.

[5] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009), 587-600.

[6] Y. Shehu, F.U. Ogbuisi, O.S. Iyiola, Strong convergence theorem for solving split equality fixed point problem which does not involve the prior knowledge of operator norms, Bull. Iranian Math. Soc. 43 (2017), 349–371.

[7] X. Qin, J.C. Yao, A viscosity iterative method for a split feasibility problem, J. Nonlinear Convex Anal. 20 (2019), 1497-1506.

[8] C. Byrne, Y. Censor, A. Gibali, S. Reich, The split common null point problem, J. Nonlinear Convex Anal. 13 (2012), 759-775.

[9] A. Moudafi, A note on the split common fixed-point problem for quasi-nonexpansive operators, Nonlinear Anal. 74 (2011), 4083–4087.

[10] R. Kraikaew, S. Saejung, On split common fixed point problems, J. Math. Anal. Appl. 415 (2014), 513–524.

[11] S.Y. Cho, A monotone Bregan projection algorithm for fixed point and equilibrium problems in a reflexive Banach space, Filomat, 34 (2020), 1487–1497.

[12] Y. Shehu, J.C. Yao, Rate of convergence for inertial iterative method for countable family of certain quasi-nonexpansive mappings, J. Nonlinear Convex Anal. 21 (2020), 533-541.

[13] C.E. Chidume, O.M. Romanus, U.V. Nnyaba, An iterative algorithm for solving split equality fixed point problems for a class of nonexpansive-type mappings in Banach spaces, Numer. Algo. 82 (2019), 987–1007.

[14] C.E. Chidume, A. Adamu, On split generalized mixed equality equilibrium and split equality fixed point problems, Appl. Set-Valued Anal. Optim. 2 (2020), 273-283.

[15] A. Moudafi, The split common fixed point problem for demi-contractive mappings, Inverse Probl. 26 (2020), 055007.

[16] J.F. Tang, S.S. Chang, L. Wang, X.R. Wang, On the split common fixed point problem for strict pseudo-contractive and asymptotically nonexpansive mappings in Banach spaces, J. Inequal. Appl. 2015 (2015), 305.

[17] C.E. Chidume, Remarks on the paper entitled, "On the split common fixed point problem for strict psedocontractive and asymptotically nonexpansive mappings in Banach spaces", to appear.
[18] Z. Ma, L. Wang, S.S. Chang, On the split feasibility problem and fixed point problem of quasi-ϕ-nonexpansive mapping in Banach spaces, Numer. Algor. 80 (2019), 1203–1218.
[19] C.E. Chidume, Geometric properties of Banach spaces and nonlinear iterations, Lecture Notes in Mathematics, vol. 1965. Springer, London, 2009.
[20] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II: Function Spaces, Ergebnisse Math. Grenzgebiete Bd., vol. 97. Springer, Berlin, 1979.
[21] Y. Alber Metric and generalized projection operators in Banach spaces: properties and applications, Theory and Applications of Nonlininear Operators of Accretive and Monotone Type (A. G. Kartsatos, Ed.), pp. 15-50, Marcel Dekker, New York, 1996.
[22] X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225 (2009), 20-30.
[23] C.E. Chidume, Geometric Inequalities in Real Banach Spaces with Applications, to appear.
[24] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16 (1991), 1127–1138.
[25] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002), 938-945.