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DIRICHLET-MORREY TYPE SPACES AND VOLTERRA INTEGRAL OPERATORS

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Abstract. A family of Dirichlet-Morrey type space $\mathscr{D}_{p-1}^{p,\lambda}$ is introduced in this paper. For any positive Borel measure μ, the boundedness and compactness of the identity operator I_d from $\mathscr{D}_{p-1}^{p,\lambda}$ to tent spaces $\mathscr{T}_s^p(\mu)$ are studied. As an application, the boundedness of the Volttera integral operators T_g and I_g, and the multiplication operator M_g from $\mathscr{D}_{p-1}^{p,\lambda}$ to the general function space $F(p,p-1-\lambda,s)$ are studied. The essential norm of T_g and I_g are also investigated.

Keywords. Dirichlet-Morrey type space; Carleson measure; Volterra integral operator.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the open unit disc $\mathbb{D}$. The Hardy space H^p $(0<p<\infty)$ is the set of all $f\in H(\mathbb{D})$ with (see [1])

$$\|f\|_{H^p}^p=\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta<\infty.$$

Let H^∞ denote the space of all bounded analytic functions with the supremum norm $\|f\|_{H^\infty}=\sup_{z\in\mathbb{D}}|f(z)|$.

Let $-1<\alpha<\infty$ and $0<p<\infty$. The Dirichlet type space $\mathscr{D}_\alpha^p$ is the set of all $f\in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathscr{D}_\alpha^p}=|f(0)|+\left(\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^\alpha dA(z)\right)^{\frac{1}{p}}<\infty,$$

where dA is the normalized area measure in $\mathbb{D}$. The spaces $\mathscr{D}_{p-1}^p$ are closely related with Hardy spaces H^p. In fact, $\mathscr{D}_1^2=H^2$. If $0<p\le 2$ (see [2]), then $\mathscr{D}_{p-1}^p\subseteq H^p$. If $2\le p<\infty$ (see [3]), then $H^p\subseteq\mathscr{D}_{p-1}^p$. If $\alpha=0$ and $p=2$, then $\mathscr{D}_\alpha^p$ is the classical Dirichlet space $\mathscr{D}$. When $\alpha>p-1$, $\mathscr{D}_\alpha^p$ is the weighted Bergman space $A_{\alpha-p}^p$.

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Let $0 < p < \infty$, $-2 < q < \infty$, and $0 \le s < \infty$. The general function space $F(p,q,s)$, which first introduced by Zhao in [4], consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)} = |f(0)| + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) \right)^{1/p} < \infty,$$

where $\sigma_a = \frac{a-z}{1-\bar{a}z}$ is a Möbius mapping interchanging 0 with a. Clearly, $F(p,q,0)$ is the Dirichlet type space $\mathscr{D}_q^p$. $F(2,0,s)$ coincides with Q_s space (see [5]). $F(2,0,1)$ is the $BMOA$ space. $F(p,p-2,0)$ is just the Besov space B_p. If $s > 1$, then $F(p,p-2,s)$ is the Bloch space $\mathscr{B}$, which is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty.$$

From [4], the norm of $f \in \mathscr{B}$ has many equivalent forms. The little Bloch space, denoted by $\mathscr{B}_0$, is the set of those $f \in H(\mathbb{D})$ satisfying $\lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0$. It is well known that $\mathscr{B}$ is a Banach space under the norm $\|\cdot\|_{\mathscr{B}}$, and $\mathscr{B}_0$ is a closed subspace of $\mathscr{B}$.

Let $g \in H(\mathbb{D})$. The Volterra integral operator T_g, which was first introduced by Pommerenke in [6], is defined as

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \; z \in \mathbb{D}, f \in H(\mathbb{D}).$$

Its related operator I_g is defined by

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta, \; z \in \mathbb{D}, f \in H(\mathbb{D}).$$

It is clear that $M_g f(z) = T_g f(z) + I_g f(z) + g(0)f(0)$, where $M_g f(z) = g(z)f(z)$ is called the multiplication operator. The Volterra integral operator T_g was studied by many authors recently. For more results on operator T_g, we refer to [5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the references therein

For any $I \subset \partial\mathbb{D}$, the boundary of $\mathbb{D}$, let $|I|$ be the normalized arc length of I. Let

$$S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1-|I| \le r < 1, e^{i\theta} \in I\}$$

be the Carleson box based on I. Let $0 < p < \infty$, $0 \le s < \infty$, and μ a positive Borel measure on $\mathbb{D}$. The tent space $\mathscr{T}_s^p(\mu)$ consists of all μ-measure functions f satisfying

$$\|f\|_{\mathscr{T}_s^p}^p = \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^p d\mu(z) < \infty.$$

Suppose that, $0 \le \lambda \le 1$, the analytic Morrey space, denoted by $\mathscr{L}^{2,\lambda}(\mathbb{D})$, is the space of all $f \in H^2(\mathbb{D})$ such that

$$\|f\|_{\mathscr{L}^{2,\lambda}}^2 = \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_I |f(\xi) - f_I|^2 \frac{|d\xi|}{2\pi} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\xi) \frac{|d\xi|}{2\pi}.$$

Clearly, $\mathscr{L}^{2,1}(\mathbb{D})$ coincides with the $BMOA$ space. $\mathscr{L}^{2,0}(\mathbb{D})$ is just the Hardy space H^2. Moreover, $BMOA \subset \mathscr{L}^{2,\lambda} \subset H^2$ for $0 < \lambda < 1$. The space $\mathscr{L}^{2,\lambda}(\mathbb{D})$ was investigated in [10, 20, 21].

Recently, Galanopoulos, Merchán and Siskakis [8] defined the Dirichlet-Morrey space $\mathscr{D}_p^{2,\lambda}$, which consists of all functions $f \in \mathscr{D}_p^2$ such that

$$\|f\|_{\mathscr{D}_p^{2,\lambda}} = |f(0)| + \sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{p(1-\lambda)}{2}} \|f\circ\sigma_a - f(a)\|_{\mathscr{D}_p^2} < \infty,$$

where $0 \le p, \lambda \le 1$. It is easy to check that $\mathscr{D}_1^{2,\lambda} = \mathscr{L}^{2,\lambda}, \mathscr{D}_p^{2,1} = Q_p, \mathscr{D}_p^{2,0} = \mathscr{D}_p^2$, and

$$Q_p \subset \mathscr{D}_p^{2,\lambda} \subset D_p^2,\ 0 < \lambda < 1.$$

They studied the boundedness and compactness of the Volterra operator T_g on the space $\mathscr{D}_p^{2,\lambda}$ (see [8]). For example, if T_g is bounded on $\mathscr{D}_p^{2,\lambda}$, then $g \in Q_p$, while if $g \in W_p$, then T_g is bounded on $\mathscr{D}_p^{2,\lambda}$.

Let $0 < p < \infty$, and $0 < \lambda < 1$. In this paper, we define a new class space $\mathscr{D}_{p-1}^{p,\lambda}$, called the Dirichlet-Morrey type space. Let $f \in \mathscr{D}_{p-1}^p$. We say that the function f belongs to $\mathscr{D}_{p-1}^{p,\lambda}$ if

$$\|f\|_{\mathscr{D}_{p-1}^{p,\lambda}} = |f(0)| + \sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{1-\lambda}{p}} \|f\circ\sigma_a - f(a)\|_{\mathscr{D}_{p-1}^p} < \infty.$$

It is obvious that $\mathscr{D}_{p-1}^{p,\lambda}$ is a Banach space under the above norm when $p \ge 1$. Clearly, $\mathscr{D}_{p-1}^{p,\lambda} = \mathscr{L}^{2,\lambda}$ when $p = 2$. By a simple calculation, we have $\mathscr{D}_{p-1}^{p,0} = \mathscr{D}_{p-1}^p$, $\mathscr{D}_{p-1}^{p,1} = F(p, p-2, 1)$, and

$$F(p, p-2, 1) \subset \mathscr{D}_{p-1}^{p,\lambda} \subset \mathscr{D}_{p-1}^p,\ 0 < \lambda < 1.$$

In this paper, we first study some basic properties of the Dirichlet-Morrey type space $\mathscr{D}_{p-1}^{p,\lambda}$ in Section 2. The boundedness and compactness of the identity operator I_d from $\mathscr{D}_{p-1}^{p,\lambda}$ to the tent space $\mathscr{T}_s^p(\mu)$ are studied in Section 3. Using the embedding theorem, we study the boundedness of operators T_g, I_g and M_g from $\mathscr{D}_{p-1}^{p,\lambda}$ to the space $F(p, p-1-\lambda, s)$ in Section 4. Finally, in Section 5, we investigate the essential norm and compactness of T_g and I_g.

In this paper, we write $F \approx G$ between two functions if $F \preceq G \preceq F$, where $G \preceq F$ means that there exists a nonnegative constant C such that $G \le CF$.

2. Some Auxiliary Properties

We begin this section with the definition of the Carleson measure. Let μ be a positive Borel measure on $\mathbb{D}$, and $0 < \alpha < \infty$. μ is called an α-Carleson measure (see [13]) if

$$\|\mu\|_{CM_\alpha} = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$

μ is the classical Carleson measure when $\alpha = 1$. μ is called a vanishing α-Carleson measure if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^\alpha} = 0.$$

The following result gives an equivalent characterization of α-Carleson measure (see [13]).

Lemma 2.1. *Let $0 < \alpha, t < \infty$, and let μ be a positive Borel measure on $\mathbb{D}$. Then μ is an α-Carleson measure if and only if*

$$\sup_{b\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|b|^2)^t}{|1-\bar{b}z|^{\alpha+t}}d\mu(z)<\infty.$$

Moreover,

$$\sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^{\alpha}}\approx\sup_{b\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|b|^2)^t}{|1-\bar{b}z|^{\alpha+t}}d\mu(z).$$

The following result gives an equivalent characterization for the Dirichlet-Morrey type space $\mathscr{D}^{p,\lambda}_{p-1}$.

Proposition 2.1. *Let $0<\lambda<1$, $0<p<\infty$, and $f\in H(\mathbb{D})$. Then $f\in\mathscr{D}^{p,\lambda}_{p-1}$ if and only if*

$$\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^{\lambda}}\int_{S(I)}|f'(z)|^p(1-|z|^2)^{p-1}dA(z)<\infty. \tag{2.1}$$

Moreover,

$$\|f\|_{\mathscr{D}^{p,\lambda}_{p-1}}\approx\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^{\lambda}}\int_{S(I)}|f'(z)|^p(1-|z|^2)^{p-1}dA(z).$$

Proof. First, assume that $f\in\mathscr{D}^{p,\lambda}_{p-1}$. For any interval $I\in\partial\mathbb{D}$, let ξ be the midpoint of interval I, and $b=(1-|I|)\xi$. Then $|I|=1-|b|\approx 1-|b|^2\approx|1-\bar{b}z|$ for $z\in S(I)$. Using the change of variables, we deduce that

$$\begin{aligned}
\infty>\|f\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}&\approx\sup_{b\in\mathbb{D}}(1-|b|^2)^{1-\lambda}\|f\circ\sigma_b-f(b)\|^p_{\mathscr{D}^p_{p-1}}\\
&=\sup_{b\in\mathbb{D}}(1-|b|^2)^{1-\lambda}\int_{\mathbb{D}}|(f\circ\sigma_b)'(z)|^p(1-|z|^2)^{p-1}dA(z)\\
&=\sup_{b\in\mathbb{D}}(1-|b|^2)^{1-\lambda}\int_{\mathbb{D}}|f'(\sigma_b(z))|^p\frac{(1-|z|^2)^{p-1}(1-|b|^2)^p}{|1-\bar{b}z|^{2p}}dA(z)\\
&=\sup_{b\in\mathbb{D}}(1-|b|^2)^{2-\lambda}\int_{\mathbb{D}}|f'(w)|^p\frac{(1-|w|^2)^{p-1}}{|1-\bar{b}w|^2}dA(w)\\
&\succeq\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^{\lambda}}\int_{S(I)}|f'(w)|^p(1-|w|^2)^{p-1}dA(w).
\end{aligned}$$

Hence inequality (2.1) holds.

Conversely, suppose that inequality (2.1) holds. Then

$$\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^{\lambda}}\int_{S(I)}|f'(z)|^p(1-|z|^2)^{p-1}dA(z)=\sup_{I\subset\partial\mathbb{D}}\frac{\mu_f(S(I))}{|I|^{\lambda}}<\infty,$$

where $d\mu_f(z) = |f'(z)|^p(1-|z|^2)^{p-1}dA(z)$. So we see that μ_f is a λ-Carleson measure. Then Lemma 4.1 implies that

$$\begin{aligned}
&\sup_{b\in\mathbb{D}}(1-|b|^2)^{1-\lambda}\|f\circ\sigma_b - f(b)\|^p_{\mathscr{D}^{p,\lambda}_{p-1}} \\
=&\sup_{b\in\mathbb{D}}\int_{\mathbb{D}}(1-|b|^2)^{1-\lambda}|f'(z)|^p(1-|z|^2)^{p-1}\frac{(1-|b|^2)}{|1-\bar{b}z|^2}dA(z) \\
=&\sup_{b\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|b|^2)^{2-\lambda}}{|1-\bar{b}z|^2}d\mu_f(z) < \infty.
\end{aligned}$$

So $f \in \mathscr{D}^{p,\lambda}_{p-1}$. This completes the proof. □

Proposition 2.2. *Let* $0<\lambda<1$, $0<p<\infty$ *and* $f\in\mathscr{D}^{p,\lambda}_{p-1}$. *Then*

$$|f(w)| \preceq \frac{\|f\|_{\mathscr{D}^{p,\lambda}_{p-1}}}{(1-|w|^2)^{\frac{1-\lambda}{p}}},\ w\in\mathbb{D}.$$

Proof. Assume $f\in\mathscr{D}^{p,\lambda}_{p-1}$. For any $a\in\mathbb{D}$, using [22, Lemma 4.12], we have

$$\begin{aligned}
|f'(a)|^p(1-|a|^2)^p &\le p\int_{\mathbb{D}}|(f\circ\sigma_a)'(z)|^p(1-|z|^2)^{p-1}dA(z) \\
&= \frac{p}{(1-|a|^2)^{1-\lambda}}(1-|a|^2)^{1-\lambda}\|f\circ\sigma_a - f(a)\|^p_{\mathscr{D}^p_{p-1}} \\
&\le \frac{p\|f\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}}{(1-|a|^2)^{1-\lambda}}.
\end{aligned}$$

Therefore,

$$|f'(a)| \le \frac{p^{1/p}}{(1-|a|^2)^{\frac{1-\lambda}{p}+1}}\|f\|_{\mathscr{D}^{p,\lambda}_{p-1}}.$$

Integrating both sides of the last inequality from 0 to a, we get the desired result immediately. □

Lemma 2.2. [13, Corollary 2.5] *Let* $a,b\in\mathbb{D}$ *and* $r>-1, s,t>0$ *such that* $0<s+t-r-2<s$. *Then*

$$\int_{\mathbb{D}}\frac{(1-|z|^2)^r}{|1-\bar{a}z|^s|1-\bar{b}z|^t}dA(z) \preceq \frac{1}{(1-|a|^2)^{s+t-r-2}}.$$

Proposition 2.3. *Let* $0<p<\infty$, *and* $0<\lambda<1$. *Then the function*

$$f_b(z) = \frac{1}{(1-\bar{b}z)^{\frac{1-\lambda}{p}}},\ b\in\mathbb{D},$$

belongs to $\mathscr{D}^{p,\lambda}_{p-1}$.

Proof. From Lemma 2.2, we get

$$\begin{aligned}\|f_b\|^p_{\mathscr{D}^{p,\lambda}_{p-1}} &\approx \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\|f_b\circ\sigma_a - f(a)\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}\\ &= \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}|f_b'(z)|^p(1-|z|^2)^{p-1}\frac{(1-|a|^2)}{|1-\bar{a}z|^2}dA(z)\\ &\preceq \sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p-1}}{|1-\bar{a}z|^2|1-\bar{b}z|^{1-\lambda+p}}dA(z)\\ &<\infty.\end{aligned}$$

This finishes the proof. □

3. Embedding $\mathscr{D}^{p,\lambda}_{p-1}$ Into $\mathscr{T}_s^p(\mu)$

In this section, we discuss the boundedness and compactness of the identity operator $I_d : \mathscr{D}^{p,\lambda}_{p-1} \to \mathscr{T}_s^p(\mu)$.

Theorem 3.1. *Let $0 < p < \infty$, $0 < \lambda < 1$, and $\lambda < s < \infty$. Let μ be a positive Borel measure on $\mathbb{D}$. Then the identity operator $I_d : \mathscr{D}^{p,\lambda}_{p-1} \to \mathscr{T}_s^p(\mu)$ is bounded if and only if μ is a $s+1-\lambda$-Carleson measure.*

Proof. Assume first that $I_d : \mathscr{D}^{p,\lambda}_{p-1} \to \mathscr{T}_s^p(\mu)$ is bounded. For any $b \in \mathbb{D}$, set

$$f_b(z) = \frac{1-|b|^2}{(1-\bar{b}z)^{1+\frac{1-\lambda}{p}}},\ z\in\mathbb{D}.$$

Proposition 2.3 yields that $f_b \in \mathscr{D}^{p,\lambda}_{p-1}$. For any interval $I \subset \partial\mathbb{D}$, let ξ be the midpoint of I. Set $b = (1-|I|)\xi$. Then

$$|I| = 1-|b| \approx 1-|b|^2 \approx |1-\bar{b}z|$$

for $z \in S(I)$. Moreover $|f_b(z)| \approx \frac{1}{|I|^{\frac{1-\lambda}{p}}}$, $z \in S(I)$. Hence,

$$\frac{\mu(S(I))}{|I|^{s+1-\lambda}} \approx \frac{1}{|I|^s}\int_{S(I)}|f_b(z)|^p d\mu(z) \le \|f_b\|^p_{\mathscr{D}^{p,\lambda}_{p-1}} < \infty,$$

which implies that μ is a $s+1-\lambda$-Carleson measure.

Conversely, suppose that μ is a $s+1-\lambda$-Carleson measure. Let $f \in \mathscr{D}^{p,\lambda}_{p-1}$. For any $I \subset \partial\mathbb{D}$, let ξ be the midpoint of I. Set $a = (1-|I|)\xi$. Then

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^p d\mu(z) &\preceq \frac{1}{|I|^s}\int_{S(I)}|f(a)|^p d\mu(z) + \frac{1}{|I|^s}\int_{S(I)}|f(z)-f(a)|^p d\mu(z)\\ &:= E + F.\end{aligned}$$

Proposition 2.2 yields that

$$E \preceq \|f\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}\frac{\mu(S(I))}{|I|^{s+1-\lambda}} \preceq \|f\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}.$$

By the assumption that μ is a $s+1-\lambda$-Carleson measure, we see that $I_d : A^p_{s-1-\lambda} \to L^p(\mu)$ is bounded (see [23] or [22]). Hence, by the fact that $\mathscr{D}^{p,\lambda}_{p-1} \subset \mathscr{D}^p_{p-1} \subset A^p_{s-1-\lambda}$, we have

$$
\begin{aligned}
F =&\frac{1}{|I|^s}\int_{S(I)} |f(z)-f(a)|^p d\mu(z)\\
\approx&(1-|a|^2)^{1-\lambda}\int_{S(I)} |f(z)-f(a)|^p \frac{(1-|a|^2)^{s+1-\lambda}}{|1-\bar{a}z|^{2s+2-2\lambda}} d\mu(z)\\
\preceq&(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}} |f(z)-f(a)|^p \frac{(1-|a|^2)^{s+1-\lambda}}{|1-\bar{a}z|^{2s+2-2\lambda}} d\mu(z)\\
\preceq&(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}} |f(z)-f(a)|^p \frac{(1-|z|^2)^{s-1-\lambda}(1-|a|^2)^{s+1-\lambda}}{|1-\bar{a}z|^{2s+2-2\lambda}} dA(z)\\
=&(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}} |f\circ\sigma_a(w)-f(a)|^p (1-|w|^2)^{s-1-\lambda} dA(w)\\
\preceq&(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}} |(f\circ\sigma_a)'(w)|^p (1-|w|^2)^{p+s-1-\lambda} dA(w)\\
\preceq&(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}} |(f\circ\sigma_a)'(w)|^p (1-|w|^2)^{p-1} dA(w)\\
\preceq&\|f\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}.
\end{aligned}
$$

So the identity operator $I_d : \mathscr{D}^{p,\lambda}_{p-1} \to \mathscr{T}_s^p(\mu)$ is bounded. This completes the proof. □

We say that the identity operator $I_d : \mathscr{D}^{p,\lambda}_{p-1} \to \mathscr{T}_s^p(\mu)$ is compact if

$$\lim_{k\to\infty} \frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^p d\mu(z) = 0,$$

where $I \subset \partial\mathbb{D}$, $\{f_k\}$ is a bounded sequence in $\mathscr{D}^{p,\lambda}_{p-1}$, and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$.

Theorem 3.2. *Let μ be a positive Borel measure on $\mathbb{D}$. Let $0<p<\infty$, $0<\lambda<1$, and $\lambda < s < \infty$ such that point evaluation functional is bounded on $\mathscr{T}_s^p(\mu)$. Then the identity operator $I_d : \mathscr{D}^{p,\lambda}_{p-1} \to \mathscr{T}_s^p(\mu)$ is compact if and only if the measure μ is a vanishing $s+1-\lambda$-Carleson measure.*

Proof. Assume first that $I_d : \mathscr{D}^{p,\lambda}_{p-1} \to \mathscr{T}_s^p(\mu)$ is compact. Let $\{I_k\}$ be a sequence of interval of $\partial\mathbb{D}$ with $\lim_{k\to\infty} |I_k| = 0$. Let ξ_n be the midpoint of I_k and $b_k = (1-|I_k|)\xi_n$. Then, for any $z \in S(I_k)$, $1-|b_k|^2 \approx |1-\bar{b}_k z| \approx |I_k|$. Set

$$f_k(z) = \frac{1-|b_k|^2}{(1-\bar{b}_k z)^{1+\frac{1-\lambda}{p}}},\ z\in\mathbb{D}.$$

Proposition 2.3 yields that the sequence $\{f_k\}$ is bounded in $\mathscr{D}^{p,\lambda}_{p-1}$. Moreover, $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Then

$$\frac{\mu(S(I_k))}{|I_k|^{s+1-\lambda}} \approx \frac{1}{|I_k|^s}\int_{S(I_k)} |f_k(z)|^p d\mu(z) \preceq \|f_k\|^p_{\mathscr{T}_s^p} \to 0,$$

as $k \to \infty$. Therefore, μ is a vanishing $s+1-\lambda$-Carleson measure.

Conversely, suppose that μ is a vanishing $s+1-\lambda$-Carleson measure. Then μ is a $s+1-\lambda$-Carleson measure. So the identity operator $I_d : \mathscr{D}_{p-1}^{p,\lambda} \to \mathscr{T}_s^p(\mu)$ is bounded. Let $\mu_r(z) = 0$ for $r \le |z| < 1$ and $\mu_r(z) = \mu(z)$ for $|z| < r$. Then as $r \to 1$, we have

$$\|\mu - \mu_r\|_{CM_{s+1-\lambda}} \to 0.$$

Let $\{f_k\}$ be a bounded sequence in $\mathscr{D}_{p-1}^{p,\lambda}$ with $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathscr{D}_{p-1}^{p,\lambda}} \preceq 1$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. We obtain

$$\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^p d\mu(z)\\
\preceq&\frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^p d\mu_r(z) + \frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^p d(\mu-\mu_r)(z)\\
\preceq&\frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^p d\mu_r(z) + \|\mu-\mu_r\|_{CM_{s+1-\lambda}} \|f_k\|^p_{\mathscr{D}_{p-1}^{p,\lambda}}\\
\preceq&\frac{1}{|I|^s}\int_{S(I)} |f_k(z)|^p d\mu_r(z) + \|\mu-\mu_r\|_{CM_{s+1-\lambda}}.
\end{aligned}$$

As $k \to \infty$ and $r \to 1$, we obtain $\lim_{k\to\infty} \|f_k\|_{\mathscr{T}_s^p} = 0$. So the identity operator $I_d : \mathscr{D}_{p-1}^{p,\lambda} \to \mathscr{T}_s^p(\mu)$ is compact. This completes the proof. □

4. The Boundedness of Integral Operators

In this section, we study the boundedness of the operators T_g, I_g, and M_g from the space $\mathscr{D}_{p-1}^{p,\lambda}$ to $F(p, p-1-\lambda, s)$.

Lemma 4.1. *Let $0 < p < \infty$, $0 < \lambda < 1$, $\lambda < s < \infty$, and $f \in F(p, p-1-\lambda, s)$. Then*

$$|f(z)| \preceq \frac{\|f\|_{F(p,p-1-\lambda,s)}}{(1-|z|^2)^{\frac{1-\lambda}{p}}},\ z \in \mathbb{D}.$$

Proof. Suppose that $f \in F(p, p-1-\lambda, s)$. For each $a \in \mathbb{D}$, using Lemma 4.12 in [22], we get

$$\begin{aligned}
\infty >&\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-1-\lambda} (1-|\sigma_a(z)|^2)^s dA(z)\\
=&\int_{\mathbb{D}} |f'(\sigma_a(z))|^p (1-|\sigma_a(z)|^2)^{p-1-\lambda} (1-|z|^2)^s |\sigma_a'(z)|^2 dA(z)\\
=&\int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|^p \frac{(1-|z|^2)^{p-1-\lambda+s}(1-|a|^2)^{1-\lambda}}{|1-\bar{a}z|^{2-\lambda}} dA(z)\\
\succeq&\int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|^p (1-|a|^2)^{1-\lambda} (1-|z|^2)^{p-1-\lambda+s} dA(z)\\
\succeq&|f'(a)|^p (1-|a|^2)^{p+1-\lambda}.
\end{aligned}$$

So

$$|f'(a)| \preceq \frac{\|f\|_{F(p,p-1-\lambda,s)}}{(1-|a|^2)^{1+\frac{1-\lambda}{p}}},\ a \in \mathbb{D}.$$

Since $f(z)-f(0)=\int_0^z f'(w)dw$, by integrating both sides of the last inequality, we obtain the desired result immediately. □

Theorem 4.1. *Let* $0<p<\infty$, $0<\lambda<1$, $\lambda<s<\infty$, *and* $g\in H(\mathbb{D})$. *Then* $T_g:\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)$ *is bounded if and only if* $g\in\mathscr{B}$. *Moreover,*

$$\|T_g\|_{\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)}\approx\|g\|_{\mathscr{B}}. \tag{4.1}$$

Proof. Assume first that $g\in\mathscr{B}$. From [4, Theorem 1.3], we get

$$\begin{aligned}\infty>\|g\|^p_{\mathscr{B}}&\approx\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^{s+1-\lambda}dA(z)\\&\approx\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^{s+1-\lambda}}\int_{S(I)}|g'(z)|^p(1-|z|^2)^{s+p-1-\lambda}dA(z)\\&\approx\sup_{I\subset\partial\mathbb{D}}\frac{\mu_g(S(I))}{|I|^{s+1-\lambda}}=\|\mu_g\|_{CM_{s+1-\lambda}},\end{aligned}$$

where $d\mu_g(z)=|g'(z)|^p(1-|z|^2)^{s+p-1-\lambda}dA(z)$. So μ_g is a $s+1-\lambda$-Carleson measure. Theorem 3.1 yields that $I_d:\mathscr{D}^{p,\lambda}_{p-1}\to\mathscr{T}^p_s(\mu)$ is bounded. Let $f\in\mathscr{D}^{p,\lambda}_{p-1}$. We deduce that

$$\begin{aligned}\|T_gf\|^p_{F(p,p-1-\lambda,s)}&\approx\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(z)|^p|g'(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(z)|^p|g'(z)|^p(1-|z|^2)^{s+p-1-\lambda}\frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}}dA(z)\\&\approx\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^p d\mu_g(z)\\&\preceq\|\mu_g\|_{CM_{s+1-\lambda}}\|f\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}\\&\approx\|g\|^p_{\mathscr{B}}\|f\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}<\infty.\end{aligned}$$

So $T_g:\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)$ is bounded.

Conversely, suppose that $T_g:\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)$ is bounded. For $r>0$ and any $b\in\mathbb{D}$, let $\mathbb{D}(b,r)$ denote the Bergman metric disc centered at b with radius r, that is, $\mathbb{D}(b,r)=\{z\in\mathbb{D}:\beta(b,z)<r\}$. From [22], we obtain

$$\frac{(1-|b|^2)^2}{|1-\bar{b}z|^4}\approx\frac{1}{(1-|b|^2)^2}\approx\frac{1}{(1-|z|^2)^2},\ z\in\mathbb{D}(b,r).$$

Let f_b be defined as in Theorem 3.1. Using [22, Proposition 4.13], we see that

$$\begin{aligned}\infty>\|T_gf_b\|^p_{F(p,p-1-\lambda,s)}&\succeq\int_{\mathbb{D}}|f_b(z)|^p|g'(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_b(z)|^2)^s dA(z)\\&\succeq\int_{\mathbb{D}(b,r)}|g'(z)|^p\frac{(1-|b|^2)^{p+s}(1-|z|^2)^{p-1-\lambda+s}}{|1-\bar{b}z|^{p+1-\lambda+2s}}dA(z)\\&\approx\int_{\mathbb{D}(b,r)}|g'(z)|^p(1-|z|^2)^{p-2}dA(z)\\&\succeq|g'(b)|^p(1-|b|^2)^p.\end{aligned}$$

Using this and the arbitrariness of b, we have that $g \in \mathscr{B}$. From the above proof, we see that (4.1) holds. This completes the proof. $\square$

Theorem 4.2. *Let* $0 < p < \infty$, $0 < \lambda < 1$, $\lambda < s < \infty$, *and* $g \in H(\mathbb{D})$. *Then* $I_g : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$ *is bounded if and only if* $g \in H^\infty$.

Proof. Suppose first that $g \in H^\infty$. Since $(I_g f(z))' = f'(z)g(z)$, for each $f \in \mathscr{D}^{p,\lambda}_{p-1}$, we have

$$\begin{aligned}
\|I_g f\|^p_{F(p,p-1-\lambda,s)} &\approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p |g(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z) \\
&\preceq \|g\|^p_{H^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z) \\
&\preceq \|g\|^p_{H^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^\lambda dA(z) \\
&\approx \|g\|^p_{H^\infty} \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^p (1-|z|^2)^{p-1} dA(z) \\
&\preceq \|g\|^p_{H^\infty} \|f\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}.
\end{aligned}$$

So $I_g : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$ is bounded.

Conversely, assume that $I_g : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$ is bounded. Set

$$f_b(z) = \frac{1-|b|^2}{\bar{b}(1-\bar{b}z)^{1+\frac{1-\lambda}{p}}}, \; 0 \neq b \in \mathbb{D}.$$

It is obvious that

$$\|I_g f_b\|_{F(p,p-1-\lambda,s)} \preceq \|I_g\|_{\mathscr{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)} \|f_b\|_{\mathscr{D}^{p,\lambda}_{p-1}} < \infty$$

due to Proposition 2.3. For each $b \in \mathbb{D}$ and $r > 0$, we have

$$\begin{aligned}
\|I_g f_b\|^p_{F(p,p-1-\lambda,s)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(I_g f_b)'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z) \\
&\succeq \int_{\mathbb{D}(b,r)} |f_b'(z)|^p |g(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_b(z)|^2)^s dA(z) \\
&\succeq \int_{\mathbb{D}(b,r)} |g(z)|^p \frac{(1-|z|^2)^{p-1-\lambda+s}(1-|b|^2)^{p+s}}{|1-\bar{b}z|^{2p+1-\lambda+2s}} dA(z) \\
&\succeq \frac{1}{(1-|b|^2)^{p+1}} \int_{\mathbb{D}(b,r)} |g(z)|^p (1-|z|^2)^{p-1} dA(z) \\
&\succeq |g(b)|^p.
\end{aligned}$$

The last inequality is due to [22, Proposition 4.13]. By the arbitrariness of b, we see that $g \in H^\infty$. This completes the proof. $\square$

Theorem 4.3. *Let* $0 < p < \infty$, $0 < \lambda < 1$, $\lambda < s < \infty$, *and* $g \in H(\mathbb{D})$. *Then* $M_g : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$ *is bounded if and only if* $g \in H^\infty$.

Proof. Suppose first that $g \in H^\infty$. Employing Theorem 4.1, Theorem 4.2, and the fact that $H^\infty \subset \mathscr{B}$, we obtain that both T_g and I_g are bounded from $\mathscr{D}^{p,\lambda}_{p-1}$ to $F(p,p-1-\lambda,s)$. Therefore, $M_g : \mathscr{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)$ is bounded.

Conversely, suppose that $M_g : \mathscr{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)$ is bounded. For $a \in \mathbb{D}$, set

$$f_a(z) = \frac{1}{(1-\bar{a}z)^{\frac{1-\lambda}{p}}},\ z \in \mathbb{D}.$$

By Proposition 2.3, f_a is bounded in $\mathscr{D}^{p,\lambda}_{p-1}$. Using the assumption, we get that $M_g f_a \in F(p,p-1-\lambda,s)$. By Lemma 4.1, we obtain

$$|g(z)f_a(z)| = |M_g f_a(z)| \preceq \frac{\|M_g f_a\|_{F(p,p-1-\lambda,s)}}{(1-|z|^2)^{\frac{1-\lambda}{p}}} \preceq \frac{\|M_g\|_{\mathscr{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)}}{(1-|z|^2)^{\frac{1-\lambda}{p}}}.$$

In view of the arbitrariness of a, we get

$$|g(z)| \preceq \|M_g\|_{\mathscr{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)},$$

which means that $g \in H^\infty$. This completes the proof. □

5. Essential Norm of Integral Operators

In this section, we estimate the essential norm of the operators T_g and I_g from the space $\mathscr{D}^{p,\lambda}_{p-1}$ to $F(p,p-1-\lambda,s)$. Recall that the essential norm of a bounded linear operator $L : W \to Q$ is defined as

$$\|L\|_{e,W\to Q} = \inf_S \{\|L-S\|_{W\to Q} : S \text{ is compact from } W \text{ to } Q\}.$$

Here $(W, \|\cdot\|_W)$, $(Q, \|\cdot\|_Q)$ are two Banach spaces. It is known that $L : W \to Q$ is compact if and only if $\|L\|_{e,W\to Q} = 0$.

Let B and Y be Banach spaces such that $B \subset Y$. Given $f \in Y$, the distance of f to B denoted by $\mathrm{dist}_Y(f,B)$, is defined as $\mathrm{dist}_Y(f,B) = \inf_{g\in B} \|f-g\|_Y$. Set $g_r(z) = g(rz)$, $0<r<1, z \in \mathbb{D}$.

The following lemma gives the distance from the Bloch space $\mathscr{B}$ to the little Bloch space $\mathscr{B}_0$. See [24].

Lemma 5.1. *If $g \in \mathscr{B}$, then*

$$\mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0) \approx \limsup_{|z|\to 1^-} (1-|z|^2)|g'(z)| \approx \limsup_{r\to 1^-} \|g-g_r\|_{\mathscr{B}}.$$

Lemma 5.2. *Let $g \in \mathscr{B}$, $1 \le p < \infty$, $0 < r, \lambda < 1$ and $\lambda < s < \infty$. Then $T_{g_r} : \mathscr{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)$ is compact.*

Proof. Let $\{f_k\}$ be a bounded sequence in $\mathscr{D}^{p,\lambda}_{p-1}$, and converge to zero uniformly on compact subsets of $\mathbb{D}$. Using the fact that $\mathscr{D}^{p,\lambda}_{p-1} = F(p, p-1-\lambda, \lambda)$, we obtain that

$$\begin{aligned}\|T_{g_r}f_k\|^p_{F(p,p-1-\lambda,s)} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f_k(z)|^p|g_r'(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\preceq \frac{\|g\|^p_{\mathscr{B}}}{(1-r^2)^p}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f_k(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\preceq \frac{\|g\|^p_{\mathscr{B}}}{(1-r^2)^p}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f_k'(z)|^p(1-|z|^2)^{2p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\preceq \frac{\|g\|^p_{\mathscr{B}}}{(1-r^2)^p}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f_k'(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^\lambda dA(z)\\ &\preceq \frac{\|g\|^p_{\mathscr{B}}}{(1-r^2)^p}\|f_k\|^p_{\mathscr{D}^{p,\lambda}_{p-1}}.\end{aligned}$$

Employing the Dominated Convergence Theorem, we obtain that

$$\begin{aligned}\lim_{k\to\infty}\|T_{g_r}f_k\|^p_{F(p,p-1-\lambda,s)} &\preceq \lim_{k\to\infty}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f_k(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\preceq \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\lim_{k\to\infty} |f_k(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &= 0.\end{aligned}$$

Hence $T_{g_r} : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$ is compact. This finishes the proof. □

Theorem 5.1. *Let $g \in H(\mathbb{D})$, $1 \le p < \infty$, $0 < \lambda < 1$ and $\lambda < s < \infty$. If $T_g : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$ is bounded, then*

$$\|T_g\|_{e,\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \approx \mathrm{dist}_{\mathscr{B}}(g, \mathscr{B}_0) \approx \limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)|.$$

Proof. Let $a_k \in \mathbb{D}$ such that $|a_k| \to 1$ as $k \to \infty$. Set

$$f_k(z) = \frac{1-|a_k|^2}{(1-\bar{a}_k z)^{1+\frac{1-\lambda}{p}}},\ z \in \mathbb{D}.$$

Then $\{f_k\}$ is a bounded sequence in $\mathscr{D}^{p,\lambda}_{p-1}$, and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. For every compact operator $S : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$, by [25, Lemma 2.10], we see that $\lim_{k\to\infty}\|Sf_k\|_{F(p,p-1-\lambda,s)}=0$. From [22, Proposition 4.13], we have

$$\begin{aligned}
&\|T_g-S\|_{\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)}\\
&\succeq \limsup_{k\to\infty}\|(T_g-S)(f_k)\|_{F(p,p-1-\lambda,s)}\\
&\succeq \limsup_{k\to\infty}(\|T_gf_k\|_{F(p,p-1-\lambda,s)}-\|Sf_k\|_{F(p,p-1-\lambda,s)})\\
&=\limsup_{k\to\infty}\|T_gf_k\|_{F(p,p-1-\lambda,s)}\\
&\succeq \limsup_{k\to\infty}\left(\int_{\mathbb{D}}|f_k(z)|^p|g'(z)|^p(1-|z|^2)^{p-1-\lambda}(1-|\sigma_{a_k}(z)|^2)^s dA(z)\right)^{1/p}\\
&\succeq \limsup_{k\to\infty}\left(\int_{\mathbb{D}(a_k,r)}|g'(z)|^p(1-|z|^2)^{p-2}dA(z)\right)^{1/p}\\
&\succeq \limsup_{k\to\infty}|g'(a_k)|(1-|a_k|^2).
\end{aligned}$$

By the arbitrariness of a_k, we obtain

$$\|T_g\|_{e,D^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)}\succeq \limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)|.$$

Conversely, Lemma 5.2 yields that $T_{g_r}:\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)$ is compact when $0<r<1$. It follows that

$$\begin{aligned}
\|T_g\|_{e,\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} &\le \|T_g-T_{g_r}\|_{\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)}\\
&=\|T_{g-g_r}\|_{\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)}\\
&\preceq \|g-g_r\|_{\mathscr{B}}.
\end{aligned}$$

Employing Lemma 5.1, we get

$$\|T_g\|_{e,\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)}\preceq \limsup_{r\to 1}\|g-g_r\|_{\mathscr{B}}\approx \limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)|.$$

This completes the proof. □

It is easy to get the following result.

Corollary 5.1. *Let $g\in H(\mathbb{D})$, $1\le p<\infty$, $0<\lambda<1$ and $\lambda<s<\infty$. Then $T_g:\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)$ is compact if and only if $g\in\mathscr{B}_0$.*

Theorem 5.2. *Let $g\in H(\mathbb{D})$, $1\le p<\infty$, $0<\lambda<1$ and $\lambda<s<\infty$. If $I_g:\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)$ is bounded, then*

$$\|I_g\|_{e,\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)}\approx \|g\|_{H^\infty}.$$

Proof. We define S and $\{a_k\}$ as in the proof of Theorem 5.1. Set

$$F_k(z)=\frac{1-|a_k|^2}{\bar{a}_k(1-\bar{a}_kz)^{1+\frac{1-\lambda}{p}}},\ z\in\mathbb{D},\ a_k\ne 0.$$

Then by Proposition 2.2, we get that $\|F_k\|_{\mathscr{D}^{p,\lambda}_{p-1}} \preceq 1$. Since $S : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$ is compact. It follows from [25, Lemma 2.10] that $\lim_{k\to\infty} \|SF_k\|_{F(p,p-1-\lambda,s)} = 0$. Hence

$$\begin{aligned}\|I_g - S\|_{\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} &\succeq \limsup_{k\to\infty} \|(I_g - S)(F_k)\|_{F(p,p-1-\lambda,s)} \\ &\succeq \limsup_{k\to\infty} (\|I_g F_k\|_{F(p,p-1-\lambda,s)} - \|SF_k\|_{F(p,p-1-\lambda,s)}) \\ &= \limsup_{k\to\infty} \|I_g F_k\|_{F(p,p-1-\lambda,s)}.\end{aligned}$$

From the proof of Theorem 4.2, we get that $\|I_g F_k\|_{F(p,p-1-\lambda,s)} \succeq |g(a_k)|$. Then

$$\|I_g\|_{e,\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \succeq \|g\|_{H^\infty}.$$

Conversely, by Theorem 4.2, we have

$$\begin{aligned}\|I_g\|_{e,\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} &= \inf_S \|I_g - S\|_{\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \\ &\preceq \|I_g\|_{\mathscr{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \preceq \|g\|_{H^\infty}.\end{aligned}$$

This completes the proof. □

Corollary 5.2. *Let $g \in H(\mathbb{D})$, $1 \le p < \infty$, $0 < \lambda < 1$, and $\lambda < s < \infty$. Then $I_g : \mathscr{D}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)$ is compact if and only if $g = 0$.*

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