A family of Dirichlet-Morrey type space $\mathcal{D}^{p,\lambda}_{p-1}$ is introduced in this paper. For any positive Borel measure $\mu$, the boundedness and compactness of the identity operator $I_d$ from $\mathcal{D}^{p,\lambda}_{p-1}$ to tent spaces $\mathcal{T}_s^p(\mu)$ are studied. As an application, the boundedness of the Volterra integral operators $T_g$ and $I_g$, and the multiplication operator $M_g$ from $\mathcal{D}^{p,\lambda}_{p-1}$ to the general function space $F(p,p-1-\lambda,s)$ are studied. The essential norm of $T_g$ and $I_g$ are also investigated.

Keywords. Dirichlet-Morrey type space; Carleson measure; Volterra integral operator.

1. INTRODUCTION

Let $H(D)$ denote the space of all analytic functions in the open unit disc $D$. The Hardy space $H^p$ $(0 < p < \infty)$ is the set of all $f \in H(D)$ with (see [1])

$$
\|f\|_{H^p} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.
$$

Let $H^\infty$ denote the space of all bounded analytic functions with the supremum norm $\|f\|_{H^\infty} = \sup_{z \in D} |f(z)|$.

Let $-1 < \alpha < \infty$ and $0 < p < \infty$. The Dirichlet type space $\mathcal{D}^p_{\alpha}$ is the set of all $f \in H(D)$ satisfying

$$
\|f\|_{\mathcal{D}^p_{\alpha}} = |f(0)| + \left( \int_D |f'(z)|^p (1-|z|^2)^\alpha dA(z) \right)^{\frac{1}{p}} < \infty,
$$

where $dA$ is the normalized area measure in $D$. The spaces $\mathcal{D}^p_{p-1}$ are closely related with Hardy spaces $H^p$. In fact, $\mathcal{D}^2_1 = H^2$. If $0 < p \leq 2$ (see [2]), then $\mathcal{D}^p_{p-1} \subseteq H^p$. If $2 \leq p < \infty$ (see [3]), then $H^p \subseteq \mathcal{D}^p_{p-1}$. If $\alpha = 0$ and $p = 2$, then $\mathcal{D}^p_{\alpha}$ is the classical Dirichlet space $\mathcal{D}$. When $\alpha > p - 1$, $\mathcal{D}^p_{\alpha}$ is the weighted Bergman space $A^p_{\alpha-p}$. 

*Corresponding author.

E-mail addresses: hl152808@163.com (L. Hu), yangrong071428@163.com (R. Yang), jyulsx@163.com (S. Li).

Received March 1, 2021; Accepted March 30, 2021.
Let $0 < p < \infty$, $-2 < q < \infty$, and $0 \leq s < \infty$. The general function space $F(p,q,s)$, which first introduced by Zhao in [4], consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)} = |f(0)| + \sup_{a \in \mathbb{D}} \left( \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) \right)^{1/p} < \infty,$$

where $\sigma_a = \frac{a - \overline{z}}{1 - a \overline{z}}$ is a Möbius mapping interchanging 0 with a. Clearly, $F(p,q,0)$ is the Dirichlet type space $D_q^p$. $F(2,0,s)$ coincides with $Q_s$ space (see [5]). $F(2,0,1)$ is the $BMOA$ space. $F(p,p-2,0)$ is just the Besov space $B_p$. If $s > 1$, then $F(p,p-2,s)$ is the Bloch space $B$, which is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_B = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) < \infty.$$ 

From [4], the norm of $f \in B$ has many equivalent forms. The little Bloch space, denoted by $B_0$, is the set of those $f \in H(\mathbb{D})$ satisfying $\lim_{|z| \to 1} |f'(z)|(1 - |z|^2) = 0$. It is well known that $B$ is a Banach space under the norm $\| \cdot \|_B$, and $B_0$ is a closed subspace of $B$.

Let $g \in H(\mathbb{D})$. The Volterra integral operator $T_g$, which was first introduced by Pommerenke in [6], is defined as

$$T_g f(z) = \int_0^z f(\xi)g'(\xi) d\xi, \quad z \in \mathbb{D}, f \in H(\mathbb{D}).$$

Its related operator $I_g$ is defined by

$$I_g f(z) = \int_0^z f'(\xi)g(\xi) d\xi, \quad z \in \mathbb{D}, f \in H(\mathbb{D}).$$

It is clear that $M_g f(z) = T_g f(z) + I_g f(z) + g(0)f(0)$, where $M_g f(z) = g(z)f(z)$ is called the multiplication operator. The Volterra integral operator $T_g$ was studied by many authors recently. For more results on operator $T_g$, we refer to [5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the references therein.

For any $I \subset \partial \mathbb{D}$, the boundary of $\mathbb{D}$, let $|I|$ be the normalized arc length of $I$. Let

$$S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |I| \leq r < 1, e^{i\theta} \in I\}$$

be the Carleson box based on $I$. Let $0 < p < \infty$, $0 \leq s < \infty$, and $\mu$ a positive Borel measure on $\mathbb{D}$. The tent space $T^p_s(\mu)$ consists of all $\mu$-measure functions $f$ satisfying

$$\|f\|_{T^p_s(\mu)} = \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f(z)|^p d\mu(z) < \infty.$$

Suppose that, $0 \leq \lambda \leq 1$, the analytic Morrey space, denoted by $\mathcal{L}^{2,\lambda}(\mathbb{D})$, is the space of all $f \in H^2(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{L}^{2,\lambda}} = \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^\lambda} \int_{I} |f(\xi) - f_I|^2 \frac{|d\xi|}{2\pi} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\xi) \frac{|d\xi|}{2\pi}.$$ 

clearly, $\mathcal{L}^{2,1}(\mathbb{D})$ coincides with the $BMOA$ space. $\mathcal{L}^{2,0}(\mathbb{D})$ is just the Hardy space $H^2$. Moreover, $BMOA \subset \mathcal{L}^{2,\lambda} \subset H^2$ for $0 < \lambda < 1$. The space $\mathcal{L}^{2,\lambda}(\mathbb{D})$ was investigated in [10, 20, 21].
Recently, Galanopoulos, Merchán and Siskakis [8] defined the Dirichlet-Morrey space $\mathcal{D}_p^{2,\lambda}$, which consists of all functions $f \in \mathcal{D}_p^2$ such that
\[
\|f\|_{\mathcal{D}_p^{2,\lambda}} = |f(0)| + \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\lambda(1-\lambda)/2} \|f \circ \sigma_a - f\|_{\mathcal{D}_p^2} < \infty,
\]
where $0 \leq p, \lambda \leq 1$. It is easy to check that $\mathcal{D}_p^{2,\lambda} = \mathcal{D}_{p,1}^{2,\lambda}$, $\mathcal{D}_p^{2,1} = Q_p$, $\mathcal{D}_p^{2,0} = D_p^2$, and
\[
Q_p \subset \mathcal{D}_p^{2,\lambda} \subset D_p^2, \quad 0 < \lambda < 1.
\]
They studied the boundedness and compactness of the Volterra operator $T_g$ on the space $\mathcal{D}_p^{2,\lambda}$ (see [8]). For example, if $T_g$ is bounded on $\mathcal{D}_p^{2,\lambda}$, then $g \in Q_p$, while if $g \in W_p$, then $T_g$ is bounded on $\mathcal{D}_p^{2,\lambda}$.

Let $0 < p < \infty$, and $0 < \lambda < 1$. In this paper, we define a new class space $\mathcal{D}_p^{p,\lambda}$, called the Dirichlet-Morrey type space. Let $f \in \mathcal{D}_p^{p-1}$. We say that the function $f$ belongs to $\mathcal{D}_p^{p,\lambda}$ if
\[
\|f\|_{\mathcal{D}_p^{p,\lambda}} = |f(0)| + \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\lambda(1-\lambda)/p} \|f \circ \sigma_a - f\|_{\mathcal{D}_p^{p-1}} < \infty.
\]
It is obvious that $\mathcal{D}_p^{p,\lambda}$ is a Banach space under the above norm when $p \geq 1$. Clearly, $\mathcal{D}_p^{p,\lambda} = \mathcal{D}_{p,1}^{2,\lambda}$ when $p = 2$. By a simple calculation, we have $\mathcal{D}_p^{p,0} = \mathcal{D}_p^{p-1}$, $\mathcal{D}_p^{p,1} = F(p, p-2, 1)$, and
\[
F(p, p-2, 1) \subset \mathcal{D}_p^{p,\lambda} \subset \mathcal{D}_p^{p-1}, \quad 0 < \lambda < 1.
\]

In this paper, we first study some basic properties of the Dirichlet-Morrey type space $\mathcal{D}_p^{p,\lambda}$ in Section 2. The boundedness and compactness of the identity operator $I_d$ from $\mathcal{D}_p^{p,\lambda}$ to the tent space $\mathcal{T}^p(\mu)$ are studied in Section 3. Using the embedding theorem, we study the boundedness of operators $T_g, I_g$ and $M_g$ from $\mathcal{D}_p^{p,\lambda}$ to the space $F(p, p-1-\lambda, s)$ in Section 4. Finally, in Section 5, we investigate the essential norm and compactness of $T_g$ and $I_g$.

In this paper, we write $F \approx G$ between two functions if $F \leq G \leq F$, where $G \leq F$ means that there exists a nonnegative constant $C$ such that $G \leq CF$.

2. SOME AUXILIARY PROPERTIES

We begin this section with the definition of the Carleson measure. Let $\mu$ be a positive Borel measure on $\mathbb{D}$, and $0 < \alpha < \infty$. $\mu$ is called an $\alpha$-Carleson measure (see [13]) if
\[
\|\mu\|_{CM_{\alpha}} = \sup_{I \subset \mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty.
\]
$\mu$ is the classical Carleson measure when $\alpha = 1$. $\mu$ is called a vanishing $\alpha$-Carleson measure if
\[
\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^\alpha} = 0.
\]
The following result gives an equivalent characterization of $\alpha$-Carleson measure (see [13]).
Lemma 2.1. Let $0 < \alpha, t < \infty$, and let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then $\mu$ is an $\alpha$-Carleson measure if and only if

$$\sup_{b \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |b|^2)^t}{|1 - b^2|^{\alpha + t}} d\mu(z) < \infty.$$  

Moreover,

$$\sup_{I \subset \partial \mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} \approx \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |b|^2)^t}{|1 - b^2|^{\alpha + t}} d\mu(z).$$

The following result gives an equivalent characterization for the Dirichlet-Morrey type space $\mathcal{D}^{p, \lambda}_{p-1}$.

Proposition 2.1. Let $0 < \lambda < 1$, $0 < p < \infty$, and $f \in H(\mathbb{D})$. Then $f \in \mathcal{D}^{p, \lambda}_{p-1}$ if and only if

$$\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^p (1 - |z|^2)^{p-1} dA(z) < \infty. \quad (2.1)$$

Moreover,

$$\|f\|_{\mathcal{D}^{p, \lambda}_{p-1}} \approx \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^p (1 - |z|^2)^{p-1} dA(z).$$

Proof. First, assume that $f \in \mathcal{D}^{p, \lambda}_{p-1}$. For any interval $I \subset \partial \mathbb{D}$, let $\xi$ be the midpoint of interval $I$, and $b = (1 - |I|)\xi$. Then $|I| = 1 - |b| \approx 1 - |b|^2 \approx |1 - \bar{b}z|$ for $z \in S(I)$. Using the change of variables, we deduce that

$$\infty > \|f\|_{\mathcal{D}^{p, \lambda}_{p-1}}^p \approx \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \|f \circ \sigma_b - f(b)\|_{\mathcal{D}^{p, \lambda}_{p-1}}^p$$

$$= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |(f \circ \sigma_b)'(z)|^p (1 - |z|^2)^{p-1} dA(z)$$

$$= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(\sigma_b(z))|^p \frac{(1 - |z|^2)^{p-1}(1 - |b|^2)^p}{|1 - \bar{b}z|^2} dA(z)$$

$$= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{2-\lambda} \int_{\mathbb{D}} |f'(w)|^p \frac{(1 - |w|^2)^{p-1}}{|1 - bw|^2} dA(w)$$

$$\approx \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(w)|^p (1 - |w|^2)^{p-1} dA(w).$$

Hence inequality (2.1) holds.

Conversely, suppose that inequality (2.1) holds. Then

$$\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^p (1 - |z|^2)^{p-1} dA(z) = \sup_{I \subset \partial \mathbb{D}} \frac{\mu_f(S(I))}{|I|^\lambda} < \infty,$$
where $d\mu_f(z) = |f'(z)|^p (1 - |z|^2)^{p-1} dA(z)$. So we see that $\mu_f$ is a $\lambda$-Carleson measure. Then Lemma 4.1 implies that

$$
\sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-\lambda} \left\| f \circ \sigma_b - f(b) \right\|_{\mathcal{D}^p_{p-1}}^p
= \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} (1 - |b|^2)^{1-\lambda} |f'(z)|^p (1 - |z|^2)^{p-1} \frac{(1 - |b|^2)}{|1 - bz|^2} dA(z)
= \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} (1 - |b|^2)^{2-\lambda} \frac{d\mu_f(z)}{|1 - bz|^2} < \infty.
$$

So $f \in \mathcal{D}^p_{p-1}$. This completes the proof. $\square$

**Proposition 2.2.** Let $0 < \lambda < 1$, $0 < p < \infty$ and $f \in \mathcal{D}^p_{p-1}$. Then

$$|f(w)| \leq \frac{\left\| f \right\|_{\mathcal{D}^p_{p-1}}}{(1 - |w|^2)^{\frac{1-\lambda}{p}}}, \quad w \in \mathbb{D}.
$$

**Proof.** Assume $f \in \mathcal{D}^p_{p-1}$. For any $a \in \mathbb{D}$, using [22, Lemma 4.12], we have

$$
|f'(a)|^p (1 - |a|^2)^p \leq p \int_{\mathbb{D}} |(f \circ \sigma_a)'(z)|^p (1 - |z|^2)^{p-1} dA(z)
= \frac{p}{(1 - |a|^2)^{1-\lambda}} (1 - |a|^2)^{1-\lambda} \left\| f \circ \sigma_a - f(a) \right\|_{\mathcal{D}^p_{p-1}}^p
\leq \frac{p \left\| f \right\|_{\mathcal{D}^p_{p-1}}^p}{(1 - |a|^2)^{1-\lambda}}.
$$

Therefore,

$$
|f'(a)| \leq \frac{p^{1/p}}{(1 - |a|^2)^{\frac{1-\lambda}{p} + 1}} \left\| f \right\|_{\mathcal{D}^p_{p-1}}.
$$

Integrating both sides of the last inequality from 0 to $a$, we get the desired result immediately. $\square$

**Lemma 2.2.** [13, Corollary 2.5] Let $a, b \in \mathbb{D}$ and $r > -1, s, t > 0$ such that $0 < s + t - r - 2 < s$. Then

$$
\int_{\mathbb{D}} \frac{(1 - |z|^2)^r}{|1 - \bar{a}z|^s |1 - \bar{b}z|^t} dA(z) \leq \frac{1}{(1 - |a|^2)^{s+t-r-2}}.
$$

**Proposition 2.3.** Let $0 < p < \infty$, and $0 < \lambda < 1$. Then the function

$$
f_b(z) = \frac{1}{(1 - \bar{b}z)^{\frac{1-\lambda}{p}}}, \quad b \in \mathbb{D},
$$

belongs to $\mathcal{D}^p_{p-1}$.
Proof. From Lemma 2.2, we get
\[
\|f_b\|_{p,p}^p \approx \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \|f_b \circ \sigma_a - f(a)\|_{p,p,\lambda}^p
\]
\[
= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f_b'(z)|^p (1 - |z|^2)^{p-1} \frac{(1 - |a|^2)}{|1 - \overline{a}z|^2} \, dA(z)
\]
\[
\leq \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{(1 - |z|^2)^{p-1}}{|1 - \overline{a}z|^2 |1 - \overline{b}z|^{1-\lambda+p}} \, dA(z)
\]
< \infty.

This finishes the proof. \qed

3. Embedding \( \mathcal{D}^{p,\lambda}_{p-1} \) into \( \mathcal{T}^D_p(\mu) \)

In this section, we discuss the boundedness and compactness of the identity operator \( I_d : \mathcal{D}^{p,\lambda}_{p-1} \to \mathcal{T}^D_p(\mu) \).

Theorem 3.1. Let \( 0 < p < \infty, 0 < \lambda < 1, \) and \( \lambda < s < \infty \). Let \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then the identity operator \( I_d : \mathcal{D}^{p,\lambda}_{p-1} \to \mathcal{T}^D_p(\mu) \) is bounded if and only if \( \mu \) is a \( s + 1 - \lambda \)-Carleson measure.

Proof. Assume first that \( I_d : \mathcal{D}^{p,\lambda}_{p-1} \to \mathcal{T}^D_p(\mu) \) is bounded. For any \( b \in \mathbb{D} \), set
\[
f_b(z) = \frac{1 - |b|^2}{(1 - \overline{b}z)^{1+p}}, \quad z \in \mathbb{D}.
\]
Proposition 2.3 yields that \( f_b \in \mathcal{D}^{p,\lambda}_{p-1} \). For any interval \( I \subset \partial \mathbb{D} \), let \( \xi \) be the midpoint of \( I \). Set \( b = (1 - |I|)\xi \). Then
\[
|I| = 1 - |b| \approx 1 - |b|^2 \approx |1 - \overline{b}z|
\]
for \( z \in S(I) \). Moreover \( |f_b(z)| \approx \frac{1}{|I|^{1+p}} \), \( z \in S(I) \). Hence,
\[
\frac{\mu(S(I))}{|I|^{s+1-\lambda}} \approx \frac{1}{|I|^{s}} \int_{S(I)} |f_b(z)|^p d\mu(z) \leq \|f_b\|_{p,p,\lambda}^p < \infty,
\]
which implies that \( \mu \) is a \( s + 1 - \lambda \)-Carleson measure.

Conversely, suppose that \( \mu \) is a \( s + 1 - \lambda \)-Carleson measure. Let \( f \in \mathcal{D}^{p,\lambda}_{p-1} \). For any \( I \subset \partial \mathbb{D} \), let \( \xi \) be the midpoint of \( I \). Set \( a = (1 - |I|)\xi \). Then
\[
\frac{1}{|I|^{s}} \int_{S(I)} |f(z)|^p d\mu(z) \lesssim \frac{1}{|I|^{s}} \int_{S(I)} |f(a)|^p d\mu(z) + \frac{1}{|I|^{s}} \int_{S(I)} |f(z) - f(a)|^p d\mu(z)
\]
\[= E + F.\]

Proposition 2.2 yields that
\[
E \lesssim \|f\|_{p,p,\lambda}^p \frac{\mu(S(I))}{|I|^{s+1-\lambda}} \lesssim \|f\|_{p,p,\lambda}^p.
\]
By the assumption that $\mu$ is a $s + 1 - \lambda$-Carleson measure, we see that $I_d : A^p_{s-1-\lambda} \to L^p(\mu)$ is bounded (see [23] or [22]). Hence, by the fact that $D_p - \partial$ is compact, we have

$$F = \frac{1}{|I|^s} \int_{S(I)} |f(z) - f(a)|^p d\mu(z)$$

$$\approx (1 - |a|^2)^{1-\lambda} \int_{S(I)} |f(z) - f(a)|^p \frac{(1 - |a|^2)^{s+1-\lambda}}{|1 - \bar{a}z|^{2s+2-2\lambda}} d\mu(z)$$

$$\leq (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f(z) - f(a)|^p \frac{(1 - |z|^2)^{s+1-\lambda} (1 - |a|^2)^{s+1-\lambda}}{|1 - \bar{a}z|^{2s+2-2\lambda}} dA(z)$$

$$= (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} \frac{f \circ \sigma_a(w) - f(a)}{|(1 - |a|^2)^{s+1-\lambda}} dA(w)$$

$$\leq (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |(f \circ \sigma_a)'(w)|^p (1 - |w|^2)^{p+1-\lambda} dA(w)$$

$$\leq ||f||^p_{\mathcal{D}^p_{p-1}}.$$

So the identity operator $I_d : \mathcal{D}^p_{p-1} \to \mathcal{T}^p_s(\mu)$ is bounded. This completes the proof. \[ \square \]

We say that the identity operator $I_d : \mathcal{D}^p_{p-1} \to \mathcal{T}^p_s(\mu)$ is compact if

$$\lim_{k \to \infty} \frac{1}{|I|^s} \int_{S(I)} |f_k(z)|^p d\mu(z) = 0,$$

where $I \subset \partial \mathbb{D}$, $\{f_k\}$ is a bounded sequence in $\mathcal{D}^p_{p-1}$, and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$.

**Theorem 3.2.** Let $\mu$ be a positive Borel measure on $\mathbb{D}$. Let $0 < p < \infty$, $0 < \lambda < 1$, and $\lambda < s < \infty$ such that point evaluation functional is bounded on $\mathcal{T}^p_s(\mu)$. Then the identity operator $I_d : \mathcal{D}^p_{p-1} \to \mathcal{T}^p_s(\mu)$ is compact if and only if the measure $\mu$ is a vanishing $s + 1 - \lambda$-Carleson measure.

**Proof.** Assume first that $I_d : \mathcal{D}^p_{p-1} \to \mathcal{T}^p_s(\mu)$ is compact. Let $\{I_k\}$ be a sequence of interval of $\partial \mathbb{D}$ with $\lim_{k \to \infty} |I_k| = 0$. Let $\xi_n$ be the midpoint of $I_k$ and $b_k = (1 - |I_k|) \bar{\xi}_n$. Then, for any $z \in S(I_k)$, $1 - |b_k|^2 \approx 1 - |\bar{b}_k z| \approx |I_k|$. Set

$$f_k(z) = \frac{1 - |b_k|^2}{(1 - |\bar{b}_k z|)^{1+1-\lambda}}, z \in \mathbb{D}.$$

Proposition 2.3 yields that the sequence $\{f_k\}$ is bounded in $\mathcal{D}^p_{p-1}$. Moreover, $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Then

$$\frac{\mu(S(I_k))}{|I_k|^{s+1-\lambda}} \approx \frac{1}{|I_k|^s} \int_{S(I_k)} |f_k(z)|^p d\mu(z) \leq ||f_k||^p_{\mathcal{D}^p_{p-1}} \to 0,$$
as \( k \to \infty \). Therefore, \( \mu \) is a vanishing \( s+1-\lambda \)-Carleson measure.

Conversely, suppose that \( \mu \) is a vanishing \( s+1-\lambda \)-Carleson measure. Then \( \mu \) is a \( s+1-\lambda \)-Carleson measure. So the identity operator \( I_d : \mathcal{D}_{p-1}^{\lambda} \to \mathcal{D}_{p}^{\mu} \) is bounded. Let \( \mu_r(z) = 0 \) for \( r \leq |z| < 1 \) and \( \mu_r(z) = \mu(z) \) for \( |z| < r \). Then as \( r \to 1 \), we have

\[
\| \mu - \mu_r \|_{CM_{s+1-\lambda}} \to 0.
\]

Let \( \{ f_k \} \) be a bounded sequence in \( \mathcal{D}_{p-1}^{\lambda} \) with \( \sup_{k \in \mathbb{N}} \| f_k \|_{\mathcal{D}_{p-1}^{\lambda}} \leq 1 \) and \( f_k \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \). We obtain

\[
\frac{1}{|I|^s} \int_{S(l)} |f_k(z)|^p d\mu(z) \leq \frac{1}{|I|^s} \int_{S(l)} |f_k(z)|^p d\mu_r(z) + \frac{1}{|I|^s} \int_{S(l)} |f_k(z)|^p d(\mu - \mu_r)(z)
\]

\[
\leq \frac{1}{|I|^s} \int_{S(l)} |f_k(z)|^p d\mu_r(z) + \| \mu - \mu_r \|_{CM_{s+1-\lambda}} \| f_k \|_{\mathcal{D}_{p-1}^{\lambda}}^p
\]

\[
\leq \frac{1}{|I|^s} \int_{S(l)} |f_k(z)|^p d\mu_r(z) + \| \mu - \mu_r \|_{CM_{s+1-\lambda}}.
\]

As \( k \to \infty \) and \( r \to 1 \), we obtain \( \lim_{k \to \infty} \| f_k \|_{\mathcal{D}^p} = 0 \). So the identity operator \( I_d : \mathcal{D}_{p-1}^{\lambda} \to \mathcal{D}_{p}^{\mu} \) is compact. This completes the proof. \( \square \)

4. The Boundedness of Integral Operators

In this section, we study the boundedness of the operators \( T_g, I_g, \) and \( M_g \) from the space \( \mathcal{D}_{p-1}^{\lambda} \) to \( F(p, p-1-\lambda, s) \).

**Lemma 4.1.** Let \( 0 < p < \infty \), \( 0 < \lambda < 1 \), \( \lambda < s < \infty \), and \( f \in F(p, p-1-\lambda, s) \). Then

\[
|f(z)| \leq \frac{\| f \|_{F(p, p-1-\lambda, s)}}{(1 - |z|^2)^{\frac{1-\lambda}{p}}}, \quad z \in \mathbb{D}.
\]

**Proof.** Suppose that \( f \in F(p, p-1-\lambda, s) \). For each \( a \in \mathbb{D} \), using Lemma 4.12 in [22], we get

\[
\infty > \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^s dA(z)
\]

\[
= \int_{\mathbb{D}} |f'(\sigma_a(z))|^p (1 - |\sigma_a(z)|^2)^{p-1-\lambda} (1 - |z|^2)^s |\sigma'_a(z)|^2 dA(z)
\]

\[
= \int_{\mathbb{D}} |(f \circ \sigma_a)'(z)|^p (1 - |z|^2)^{p-1-\lambda+s} (1 - |a|^2)^{1-\lambda} |1 - \bar{a}z|^{2-\lambda} dA(z)
\]

\[
\geq \int_{\mathbb{D}} |(f \circ \sigma_a)'(z)|^p (1 - |a|^2)^{1-\lambda} (1 - |z|^2)^{p-1-\lambda+s} dA(z)
\]

\[
\geq |f'(a)|^p (1 - |a|^2)^{p+1-\lambda}.
\]

So

\[
|f'(a)| \leq \frac{\| f \|_{F(p, p-1-\lambda, s)}}{(1 - |a|^2)^{1+\frac{1-\lambda}{p}}}, \quad a \in \mathbb{D}.
\]
Since \( f(z) - f(0) = \int_0^1 f'(w)dw \), by integrating both sides of the last inequality, we obtain the desired result immediately. \( \square \)

**Theorem 4.1.** Let \( 0 < p < \infty, 0 < \lambda < 1, \lambda < s < \infty \), and \( g \in H(\mathbb{D}) \). Then \( T_g : \mathcal{D}^{p,\lambda}_{p-1} \to F(p, p-1 - \lambda, s) \) is bounded if and only if \( g \in \mathcal{B} \). Moreover,

\[
\|T_g\|_{\mathcal{D}^{p,\lambda}_{p-1} \to F(p, p-1 - \lambda, s)} \approx \|g\|_{\mathcal{B}}. \tag{4.1}
\]

**Proof.** Assume first that \( g \in \mathcal{B} \). From [4, Theorem 1.3], we get

\[
\int_0^\infty \|g\|^p_{\mathcal{B}} \approx \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |g'(z)|^p (1 - |z|^2)^{p-2} (1 - |\sigma_a(z)|^2)^{s+1-\lambda} dA(z)
\]

\[
\approx \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{s+1-\lambda}} \int_{S(I)} |g'(z)|^p (1 - |z|^2)^{s+p-1-\lambda} dA(z)
\]

\[
\approx \sup_{I \subset \partial \mathbb{D}} \frac{\mu_g(S(I))}{|I|^{s+1-\lambda}} = \|g\|_{CM_{s+1-\lambda}},
\]

where \( d\mu_g(z) = |g'(z)|^p (1 - |z|^2)^{s+p-1-\lambda} dA(z) \). So \( \mu_g \) is a \( s + 1 - \lambda \)-Carleson measure. Theorem 3.1 yields that \( I_d : \mathcal{D}^{p,\lambda}_{p-1} \to \mathcal{D}_s^p(\mu) \) is bounded. Let \( f \in \mathcal{D}^{p,\lambda}_{p-1} \). We deduce that

\[
\|T_g f\|_{F(p, p-1 - \lambda, s)} \approx \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p |g'(z)|^p (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^s dA(z)
\]

\[
= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p |g'(z)|^p (1 - |z|^2)^{s+p-1-\lambda} \frac{(1 - |a|^2)^s}{|1 - az|^{2s}} dA(z)
\]

\[
\approx \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^p d\mu_g(z)
\]

\[
\leq \|g\|_{CM_{s+1-\lambda}} \|f\|_{\mathcal{D}^{p,\lambda}_{p-1}}^p
\]

\[
\approx \|g\|^p_{\mathcal{B}} \|f\|^p_{\mathcal{D}^{p,\lambda}_{p-1}} < \infty.
\]

So \( T_g : \mathcal{D}^{p,\lambda}_{p-1} \to F(p, p-1 - \lambda, s) \) is bounded.

Conversely, suppose that \( T_g : \mathcal{D}^{p,\lambda}_{p-1} \to F(p, p-1 - \lambda, s) \) is bounded. For \( r > 0 \) and any \( b \in \mathbb{D} \), let \( \mathbb{D}(b, r) \) denote the Bergman metric disc centered at \( b \) with radius \( r \), that is, \( \mathbb{D}(b, r) = \{ z \in \mathbb{D} : \beta(b, z) < r \} \). From [22], we obtain

\[
\frac{(1 - |b|^2)^2}{|1 - bz|^4} \approx \frac{1}{(1 - |b|^2)^2} \approx \frac{1}{(1 - |z|^2)^2}, \quad z \in \mathbb{D}(b, r).
\]

Let \( f_b \) be defined as in Theorem 3.1. Using [22, Proposition 4.13], we see that

\[
\int_{\mathbb{D}(b, r)} |g'(z)|^p (1 - |b|^2)^{p+s} (1 - |z|^2)^{p-1-\lambda+s} |1 - bz|^{p+1-\lambda+2s} dA(z)
\]

\[
\approx \int_{\mathbb{D}(b, r)} |g'(z)|^p (1 - |z|^2)^{p-2} dA(z)
\]

\[
\approx |g'(b)|^p (1 - |b|^2)^p.
\]
Theorem 4.2. Let $0 < p < \infty$, $0 < \lambda < 1$, $\lambda < s < \infty$, and $g \in H(\mathbb{D})$. Then $I_g : \mathcal{D}^{p, \lambda}_{p-1} \to F(p, p - 1 - \lambda, s)$ is bounded if and only if $g \in H^\infty$.

Proof. Suppose first that $g \in H^\infty$. Since $(I_g f)(z) = f'(z) g(z)$, for each $f \in \mathcal{D}^{p, \lambda}_{p-1}$, we have

$$
\|I_g f\|_{p, (p, p-1-\lambda, s)}^{p} \approx \sup_{a \in \mathbb{D}} \int_{D}(f'(z))^{p} |g(z)|^{p} (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^{s} dA(z)
$$

$$
\leq \|g\|_{p, (p, p-1-\lambda, s)}^{p} \sup_{a \in \mathbb{D}} \int_{D} |f'(z)|^{p} (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^{s} dA(z)
$$

$$
\leq \|g\|_{p=1}^{p} \sup_{a \in \mathbb{D}} \int_{\overline{D}} |f'(z)|^{p} (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^{s} dA(z)
$$

$$
\approx \|g\|_{p=1}^{p} \sup_{a \in \mathbb{D}} \int_{\overline{D}} |f'(z)|^{p} (1 - |z|^2)^{p-1-\lambda} dA(z)
$$

$$
\leq \|g\|_{H^\infty}^{p} \|f\|_{\mathcal{D}^{p, \lambda}_{p-1}}^{p}.
$$

So $I_g : \mathcal{D}^{p, \lambda}_{p-1} \to F(p, p - 1 - \lambda, s)$ is bounded.

Conversely, assume that $I_g : \mathcal{D}^{p, \lambda}_{p-1} \to F(p, p - 1 - \lambda, s)$ is bounded. Set

$$
f_b(z) = \frac{1 - |b|^2}{\overline{b}(1 - \overline{b}z)^{1 + \frac{1-\lambda}{p}}}, 0 \neq b \in \mathbb{D}.
$$

It is obvious that

$$
\|I_g f_b\|_{p, (p, p-1-\lambda, s)} \leq \|I_g\|_{\mathcal{D}^{p, \lambda}_{p-1} \to F(p, p-1-\lambda, s)} \|f_b\|_{\mathcal{D}^{p, \lambda}_{p-1}} < \infty
$$

due to Proposition 2.3. For each $b \in \mathbb{D}$ and $r > 0$, we have

$$
\|I_g f_b\|_{p, (p, p-1-\lambda, s)}^{p} = \sup_{a \in \mathbb{D}} \int_{D} |(I_g f_b)'(z)|^{p} (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^{s} dA(z)
$$

$$
\geq \int_{D(b, r)} |f'_b(z)|^{p} |g(z)|^{p} (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_b(z)|^2)^{s} dA(z)
$$

$$
\geq \int_{D(b, r)} |g(z)|^{p} \frac{(1 - |z|^2)^{p-1-\lambda+s}(1 - |b|^2)^{p+s}}{|1 - \overline{b}z|^{2p+1-\lambda+2s}} dA(z)
$$

$$
\geq \frac{1}{(1 - |b|^2)^{p+1}} \int_{D(b, r)} |g(z)|^{p} (1 - |z|^2)^{p-1} dA(z)
$$

$$
\geq |g(b)|^{p}.
$$

The last inequality is due to [22, Proposition 4.13]. By the arbitrariness of $b$, we see that $g \in H^\infty$. This completes the proof. \hfill \square

Theorem 4.3. Let $0 < p < \infty$, $0 < \lambda < 1$, $\lambda < s < \infty$, and $g \in H(\mathbb{D})$. Then $M_g : \mathcal{D}^{p, \lambda}_{p-1} \to F(p, p - 1 - \lambda, s)$ is bounded if and only if $g \in H^\infty$. 

Lemma 5.1. See [24].

In this section, we estimate the essential norm of the operators $T_g$ and $I_g$ from the space $D_{p-1}$ to $F(p, p - \lambda, s)$. Recall that the essential norm of a bounded linear operator $L : W \to Q$ is defined as

$$
\|L\|_{e,W \to Q} = \inf_{S} \{\|L - S\|_{W \to Q} : S \text{ is compact from } W \text{ to } Q \}.
$$

Here $(W, \| \cdot \|_W)$, $(Q, \| \cdot \|_Q)$ are two Banach spaces. It is known that $L : W \to Q$ is compact if and only if $\|L\|_{e,W \to Q} = 0$.

Let $B$ and $Y$ be Banach spaces such that $B \subset Y$. Given $f \in Y$, the distance of $f$ to $B$ denoted by $\text{dist}_Y(f, B)$, is defined as $\text{dist}_Y(f, B) = \inf_{g \in B} \|f - g\|_Y$. Set $g_r(z) = g(rz)$, $0 < r < 1$, $z \in \mathbb{D}$.

The following lemma gives the distance from the Bloch space $B$ to the little Bloch space $B_0$. See [24].

**Lemma 5.1.** If $g \in B$, then

$$
\text{dist}_B(g, B_0) \approx \limsup_{|z| \to 1^{-}} (1 - |z|^2) |g'(z)| \approx \limsup_{r \to 1^{-}} \|g - g_r\|_{B}.
$$

**Lemma 5.2.** Let $g \in B$, $1 \leq p < \infty$, $0 < r, \lambda < 1$ and $\lambda < s < \infty$. Then $T_g : D_{p-1} \to F(p, p - 1 - \lambda, s)$ is compact.
Proof. Let \( \{f_k\} \) be a bounded sequence in \( \mathcal{D}_{p-1}^{p,\lambda} \), and converge to zero uniformly on compact subsets of \( \mathbb{D} \). Using the fact that \( \mathcal{D}_{p-1}^{p,\lambda} = F(p, p - 1 - \lambda, s) \), we obtain that

\[
\| T_{gr} f_k \|_{F(p, p - 1 - \lambda, s)}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p |g_r(z)|^p (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^s dA(z)
\]

\[
\leq \frac{\|g\|_{\mathcal{D}_p^{p,\lambda}}^p}{(1 - r^2)^p} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^s dA(z)
\]

\[
\leq \frac{\|g\|_{\mathcal{D}_p^{p,\lambda}}^p}{(1 - r^2)^p} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p (1 - |z|^2)^{2p-1-\lambda} (1 - |\sigma_a(z)|^2)^s dA(z)
\]

\[
\leq \frac{\|g\|_{\mathcal{D}_p^{p,\lambda}}^p}{(1 - r^2)^p} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^s dA(z)
\]

\[
\leq \frac{\|g\|_{\mathcal{D}_p^{p,\lambda}}^p}{(1 - r^2)^p} \| f_k \|_{\mathcal{D}_p^{p,\lambda}}^p.
\]

Employing the Dominated Convergence Theorem, we obtain that

\[
\lim_{k \to \infty} \| T_{gr} f_k \|_{F(p, p - 1 - \lambda, s)}^p \leq \limsup_{k \to \infty} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^s dA(z)
\]

\[
\leq \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \lim_{k \to \infty} |f_k(z)|^p (1 - |z|^2)^{p-1-\lambda} (1 - |\sigma_a(z)|^2)^s dA(z)
\]

\[
= 0.
\]

Hence \( T_{gr} : \mathcal{D}_{p-1}^{p,\lambda} \to F(p, p - 1 - \lambda, s) \) is compact. This finishes the proof. \(\square\)

**Theorem 5.1.** Let \( g \in H(\mathbb{D}), \ 1 \leq p < \infty, \ 0 < \lambda < 1 \) and \( \lambda < s < \infty \). If \( T_g : \mathcal{D}_{p-1}^{p,\lambda} \to F(p, p - 1 - \lambda, s) \) is bounded, then

\[
\| T_g \|_{\mathcal{D}_{p-1}^{p,\lambda} \to F(p, p - 1 - \lambda, s)} \approx \text{dist}_{\mathcal{D}}(g, \mathcal{B}_0) \approx \limsup_{|z| \to 1^-} (1 - |z|^2)|g'(z)|.
\]

**Proof.** Let \( a_k \in \mathbb{D} \) such that \( |a_k| \to 1 \) as \( k \to \infty \). Set

\[
f_k(z) = \frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^{1 + 1/p}}, \ z \in \mathbb{D}.
\]

Then \( \{f_k\} \) is a bounded sequence in \( \mathcal{D}_{p-1}^{p,\lambda} \), and \( f_k \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \). For every compact operator \( S : \mathcal{D}_{p-1}^{p,\lambda} \to F(p, p - 1 - \lambda, s) \), by [25, Lemma 2.10], we
see that $\lim_{k \to \infty} \|Sf_k\|_{F(p,p-1-\lambda,s)} = 0$. From [22, Proposition 4.13], we have
\[
\|T_g - S\|_{\mathcal{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \\
\geq \limsup_{k \to \infty} \| (T_g - S)(f_k)\|_{F(p,p-1-\lambda,s)} \\
\geq \limsup_{k \to \infty} (\|T_gf_k\|_{F(p,p-1-\lambda,s)} - \|Sf_k\|_{F(p,p-1-\lambda,s)}) \\
= \limsup_{k \to \infty} \|T_gf_k\|_{F(p,p-1-\lambda,s)} \\
\geq \limsup_{k \to \infty} \left( \int_{\mathbb{D}} |f_k(z)|^p |g'(z)|^p (1 - |z|^2)^{p-1-\lambda} \sigma^{2}\lambda_{\lambda}(z)^2 |dA(z)| \right)^{1/p} \\
\geq \limsup_{k \to \infty} \left( \int_{\mathbb{D}(a_k,r)} |g'(z)|^p (1 - |z|^2)^{p-2} |dA(z)| \right)^{1/p} \\
\geq \limsup_{k \to \infty} |g'(a_k)|(1 - |a_k|^2).
\]
By the arbitrariness of $a_k$, we obtain
\[
\|T_g\|_{\mathcal{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \geq \limsup_{|z| \to 1^-} (1 - |z|^2) |g'(z)|.
\]
Conversely, Lemma 5.2 yields that $T_{gr} : \mathcal{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)$ is compact when $0 < r < 1$. It follows that
\[
\|T_g\|_{\mathcal{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \leq \|T_g - T_{gr}\|_{\mathcal{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \\
= \|T_{g - gr}\|_{\mathcal{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \\
\leq \|g - gr\|_{\mathcal{B}}.
\]
Employing Lemma 5.1, we get
\[
\|T_g\|_{\mathcal{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \leq \limsup_{r \to 1^-} \|g - gr\|_{\mathcal{B}} \approx \limsup_{|z| \to 1^-} (1 - |z|^2) |g'(z)|.
\]
This completes the proof. \[\square\]

It is easy to get the following result.

**Corollary 5.1.** Let $g \in H(\mathbb{D})$, $1 \leq p < \infty$, $0 < \lambda < 1$ and $\lambda < s < \infty$. Then $T_g : \mathcal{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)$ is compact if and only if $g \in \mathcal{B}_0$.

**Theorem 5.2.** Let $g \in H(\mathbb{D})$, $1 \leq p < \infty$, $0 < \lambda < 1$ and $\lambda < s < \infty$. If $I_g : \mathcal{D}^{p,\lambda}_{p-1} \to F(p,p-1-\lambda,s)$ is bounded, then
\[
\|I_g\|_{\mathcal{D}^{p,\lambda}_{p-1}\to F(p,p-1-\lambda,s)} \approx \|g\|_{H^s}.
\]

**Proof.** We define $S$ and $\{a_k\}$ as in the proof of Theorem 5.1. Set
\[
F_k(z) = \frac{1 - |a_k|^2}{\bar{a}_k(1 - \bar{a}_k z)^{1+\frac{1-\lambda}{p}}}, \quad z \in \mathbb{D}, \quad a_k \neq 0.
\]
Then by Proposition 2.2, we get that \( \|F_k\|_{\mathcal{F}^{p,\lambda}_{p-1}} \leq 1 \). Since \( S: \mathcal{F}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s) \) is compact. It follows from [25, Lemma 2.10] that \( \lim_{k \to \infty} \|SF_k\|_{F(p, p-1-\lambda, s)} = 0 \). Hence
\[
\|I_g - S\|_{\mathcal{F}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)} \leq \limsup_{k \to \infty} \|SF_k\|_{F(p, p-1-\lambda, s)} = 0.
\]
From the proof of Theorem 4.2, we get that \( \|I_g F_k\|_{F(p, p-1-\lambda, s)} \leq \|g\|_{H^s} \).

Conversely, by Theorem 4.2, we have
\[
\|I_g\|_{\mathcal{F}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)} = \inf_{S} \|I_g - S\|_{\mathcal{F}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s)} \leq \|g\|_{H^s}.
\]

This completes the proof. \( \square \)

**Corollary 5.2.** Let \( g \in H(D) \), \( 1 \leq p < \infty \), \( 0 < \lambda < 1 \), and \( \lambda < s < \infty \). Then \( I_g: \mathcal{F}^{p,\lambda}_{p-1} \to F(p, p-1-\lambda, s) \) is compact if and only if \( g = 0 \).

**Acknowledgments**

This paper was supported by National Natural Science Foundation of China (No. 11720101003).

**References**


