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MATHEMATICAL JUSTIFICATION OF A GENERALIZED EQUILIBRIUM PROBLEM

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Abstract. Generalizing the quasivariational inequality, Joly and Mosco [A propos de l'existence et de la régularité des solutions de certaines inéquations quasi-variationnelles, J. Funct. Anal. 34 (1979), 107-137] introduced an abstract formulation of this concept. The aim of this short note is to outline the abstract problem and to point out its relevance by exhibiting some examples which show interesting uses in deriving existence results for quasiequilibrium problems.
Keywords. Ky Fan minimax inequality; Quasiequilibrium problem; Generalized equilibrium problem.

1. Introduction

Ky Fan [1] established his famous minimax inequality result, which concerns the existence of solutions for an inequality of minimax type that nowadays is called in literature equilibrium problem:

$$\text{find } x \in C \text{ such that } f(x,y) \geq 0 \text{ for all } y \in C,$$

where $C \subseteq X$ is a nonempty set of a topological vector space and $f : C \times C \to \mathbb{R}$ is a bifunction. Such a model has gained a lot interest in the last decades because it has been used in different contexts as economics, engineering, physics, chemistry and so on [2].

A more general setting where the constraint set depends on the current analyzed point was studied for the first time in the context of impulse control problem and it was subsequently used by several authors for describing a lot of problems that arise in different fields: equilibrium problems in mechanics, Nash equilibrium problems, equilibria in economics, network equilibrium problems and so on. This more general setting, commonly called quasiequilibrium problem, reads

$$\text{find } x \in K(x) \text{ such that } f(x,y) \geq 0 \text{ for all } y \in K(x), \tag{1.1}$$

where the constraint $K : C \rightrightarrows C$ is a set-valued map.

Joly and Mosco [3] studied a class of variational problems involving an extended valued bifunction $\varphi : C \times C \to (-\infty, +\infty]$, which captures the nature of the constraint. This problem asks to

$$\text{find } x \in C \text{ such that } f(x,y) + \varphi(x,y) \geq \varphi(x,x) \text{ for all } y \in C, \tag{1.2}$$

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where the domain of $\varphi(x,\cdot)$

$$D_\varphi(x) = \{y \in C : \varphi(x,y) < +\infty\}$$

is assumed to be nonempty for every $x \in C$. Problem (1.1) with K nonempty-valued can be obtained as particular case of (1.2) if $\varphi(x,y) = \delta(y,K(x))$, where δ is the function defined by

$$\delta(x,A) = \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{if } x \notin A, \end{cases}$$

for all $x \in X$ and $A \subseteq X$. Furthermore, if $f \equiv 0$ and $\varphi(x,y) = \delta(y,K(x))$, then problem (1.2) consists of finding the fixed points of the set-valued map K.

An equivalent formulation of (1.2) consists of finding $x \in C$ such that

$$\begin{cases} x \in D_\varphi(x) \\ f(x,y) + \varphi(x,y) - \varphi(x,x) \geq 0, \quad \forall y \in D_\varphi(x). \end{cases} \tag{1.3}$$

System (1.3) is not a quasiequilibrium problem since the bifunction $f(x,y)+\varphi(x,y)-\varphi(x,x)$ is not well-defined on $C \times C$ but only on the graph of D_φ where it assumes the value $-\infty$ when $x \notin D_\varphi(x)$.

Anyway, each variational problem (1.2) may be reformulated as a quasiequilibrium problem where the bifunction and the feasibility set-valued map are chosen appropriately. Let $\widehat{C} = C \times \mathbb{R}$, $\widehat{x} = (x,a)$ and $\widehat{y} = (y,b)$, and define the set-valued map $K : \widehat{C} \rightrightarrows \widehat{C}$ as

$$K(\widehat{x}) = \{(y,b) \in \widehat{C} : b \geq \varphi(x,y)\}$$

and the bifunction $\widehat{f} : \widehat{C} \times \widehat{C} \to \mathbb{R}$ as $\widehat{f}(\widehat{x},\widehat{y}) = f(x,y) - a + b$.

Consider the following quasiequilibrium problem

$$\text{find } \widehat{x} \in K(\widehat{x}) \text{ such that } \widehat{f}(\widehat{x},\widehat{y}) \geq 0 \text{ for all } \widehat{y} \in K(\widehat{x}). \tag{1.4}$$

Theorem 1.1. *If $x \in C$ solves (1.2), then $(x,\varphi(x,x))$ solves (1.4). Vice versa, if $(x,a) \in \widehat{C}$ solves (1.4), then x solves (1.2).*

Proof. If $x \in C$ solves (1.2), then $\varphi(x,x) \in \mathbb{R}$ and hence $\widehat{x} = (x,\varphi(x,x)) \in K(\widehat{x})$. Moreover, for each $\widehat{y} \in K(\widehat{x})$, i.e., $y \in D_\varphi(x)$ and $b \geq \varphi(x,y)$, we have

$$\widehat{f}(\widehat{x},\widehat{y}) = f(x,y) - \varphi(x,x) + b \geq f(x,y) - \varphi(x,x) + \varphi(x,y) \geq 0.$$

For the converse, if $\widehat{x} = (x,a) \in K(\widehat{x})$ solves (1.4), then $x \in D_\varphi(x)$. For each $y \in D_\varphi(x)$, choosing $b = \varphi(x,y)$, we get

$$f(x,y) - \varphi(x,x) + \varphi(x,y) \geq f(x,y) - a + b \geq 0.$$

This completes the proof. □

It is worth mentioning that $f(x,x) = 0$ for all $x \in D_\varphi(x)$, which is a fairly common assumption, guarantees that $x \mapsto (x,\varphi(x,x))$ is a one-to-one correspondence between the solutions of (1.2) and the solutions of (1.4). Indeed, if (x,a) solves (1.4), then $a \geq \varphi(x,x)$. Taking $\widehat{y} = (x,\varphi(x,x)) \in K(\widehat{x})$, we have

$$0 \leq \widehat{f}(\widehat{x},\widehat{y}) = f(x,x) - a + \varphi(x,x) = -a + \varphi(x,x).$$

Therefore, it follows that $a = \varphi(x,x)$.

The reformulation (1.4) was proposed in [3]. We point out its major drawback, which consists in the fact that the domain of new problem (1.4) is not compact even if C is. For this reason, it could be not convenient to establish the solvability of (1.2) passing through (1.4). Anyway, we can rewrite (1.2) as a quasiequilibrium problem without modifying C, the domain. Indeed, to solve (1.2) is equivalent to solve the following problem

$$\text{find } x \in C \text{ such that } f(x,y) + \widehat{\varphi}(x,y) + \delta(y, D_\varphi(x)) \geq \widehat{\varphi}(x,x) + \delta(x, D_\varphi(x)) \text{ for all } y \in C,$$

where

$$\widehat{\varphi}(x,y) = \begin{cases} \varphi(x,y), & \text{if } y \in D_\varphi(x), \\ \psi(x,y), & \text{otherwise,} \end{cases}$$

and ψ is an arbitrary finite bifunction. Clearly, we get the quasiequilibrium problem

$$\text{find } x \in D_\varphi(x) \text{ such that } f(x,y) + \widehat{\varphi}(x,y) - \widehat{\varphi}(x,x) \geq 0 \text{ for all } y \in D_\varphi(x).$$

Since ((1.2)) can be equivalently reformulated as quasiequilibrium problem (1.1), the question is: why do we need to study problem (1.2)? In the next section, we focus on the question of the solvability of quasiequilibrium problems and we provide convincing evidence that the study of problem (1.2) would be worthwhile.

2. Main insights

We start recalling one of the most classical and cited results for the existence of solutions of quasiequilibrium problem (1.1).

Theorem 2.1 (Theorem 6.4.21 in [4])**.** *Let C be a compact convex subset of a Hilbert space. Assume that*

- (a_1) *K is upper semicontinuous with nonempty closed convex values,*
- (a_2) *$f(x,x) \geq 0$, for all $x \in C$,*
- (a_3) *$f(x,\cdot)$ is convex, for all $x \in C$,*
- (a_4) *$f(\cdot,y)$ is upper semicontinuous, for all $y \in C$,*
- (a_5) *the set*

$$\left\{ x \in C : \inf_{y \in K(x)} f(x,y) \geq 0 \right\}$$

is closed.

Then problem (1.1) has a solution.

Assumption (a_5) is a consistency hypothesis between f and K, and it is guaranteed by the upper semicontinuity of f and the lower semicontinuity of K.

The next example shows that it may be helpful to use the more general format (1.2) in studying the existence of solutions of a quasiequilibrium problem.

Example 2.1. Let us consider the quasiequilibrium problem (1.1) with $C = [-5,5]$, $K : C \rightrightarrows C$ defined by

$$K(x) = \left\{ y \in C : x^2/5 - 24/5 \leq y \leq x^2/25 - 26/25 \right\},$$

and $f : C \times C \to \mathbb{R}$ defined by $f(x,y) = 2y^2 - xy - x^2 + h(x,y)$, where

$$h(x,y) = \begin{cases} -1, & \text{if } (x,y) = (4,1), \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $f(4,\cdot)$ is not continuous at $y = 1$ and $f(\cdot,1)$ is not upper semicontinuous at $x = 4$. Therefore, the assumptions (a_3) and (a_4) of Theorem 2.1 fail.

Now, we transform the problem considered into (1.2):

$$\text{find } x \in C \text{ such that } 2y^2 - xy - x^2 + h(x,y) + \delta(y,K(x)) \geq \delta(x,K(x)) \text{ for all } y \in C.$$

There are few existence results for problem (1.2). One of the most cited paper is [5] where the author provides a characterization of the nonemptiness of the solution set when the problem is defined in a reflexive Banach space and the objective bifunction is monotone.

For the sake of simplicity, we specialize a result in [5] to the case when C is compact and the space is Euclidean.

Theorem 2.2. *Let C be a compact convex subset of $\mathbb{R}^n$. Assume that*

(b_1) $f(x,x) = 0$ *for all* $x \in C$,

(b_2) f *is monotone,*

(b_3) $f(x,\cdot) + \varphi(z,\cdot)$ *is lower semicontinuous and strictly quasi-convex for each* $x,z \in C$,

(b_4) $\varphi(x,\cdot)$ *is lower semicontinuous with* $D_\varphi(x)$ *convex for each* $x \in C$,

(b_5) *the function* $t \in [0,1] \mapsto f(ty + (1-t)x, y)$ *is upper semicontinuous at* $t = 0$ *for each* $x,y \in C$,

(b_6) *for each* $(x_k,z_k) \to (x,z)$ *and*

$$f(y,z_k) + \varphi(x_k,z_k) \leq \varphi(x_k,y), \quad \forall y \in C$$

one has

$$f(y,z) + \varphi(x,z) \leq \varphi(x,y), \quad \forall y \in C.$$

Then problem (1.2) has a solution.

We recall that a bifunction f is said to be monotone if

$$f(x,y) + f(y,x) \leq 0, \quad \forall x,y \in C$$

and a function $\phi : C \to (-\infty,+\infty]$ is strictly quasiconvex if, given any $x_1,x_2 \in \operatorname{dom}\phi$ with $\phi(x_1) \neq \phi(x_2)$,

$$\phi(tx_1 + (1-t)x_2) < \max\{\phi(x_1),\phi(x_2)\}, \quad \forall t \in (0,1).$$

Unfortunately, the bifunction $f(x,y) = 2y^2 - xy - x^2 + h(x,y)$ is not monotone, even not quasi-monotone, since $f(-1,1) = f(1,-1) = 2$ and Theorem 2.2 seems not useful for our problem. Nevertheless, by decomposing f into a sum of

$$g_1(x,y) = xy - x^2$$

and

$$g_2(x,y) = 2y^2 - 2xy + h(x,y),$$

and taking into account that $g_2(x,x) = 0$, we can reformulate the problem as follows

$$\text{find } x \in C \text{ such that } g_1(x,y) + \varphi(x,y) \geq \varphi(x,x) \text{ for all } y \in C, \tag{2.1}$$

where

$$\varphi(x,y) = g_2(x,y) + \delta(y,K(x)).$$

Now, it is easy to verify that the reformulated problem (2.1) satisfies all the six assumptions of Theorem 2.2 and the unique solution of the problem is $x = -1$.

Notice that, in the previous example, $x = -2$ is a fixed point of K and

$$g_2(-2,-2)+\delta(-2,K(-2)) = 0 > -1 = \inf_{y\in C}[g_2(-2,y)+\delta(y,K(-2))].$$

It is a pity, because if

$$\varphi(x,x) = \inf_{y\in C}\varphi(x,y), \quad \forall x \in D_\varphi(x), \tag{2.2}$$

then each solution $x \in C$ of the quasiequilibrium problem

$$\text{find } x \in D_\varphi(x) \text{ such that } f(x,y) \geq 0 \text{ for all } y \in D_\varphi(x) \tag{2.3}$$

is a solution of (1.2) since

$$f(x,y)+\varphi(x,y) \geq f(x,y)+\inf_{y\in C}\varphi(x,y) \geq \inf_{y\in C}\varphi(x,y) = \varphi(x,x).$$

Therefore, with the additional assumption (2.2), which is verified in the case when $\varphi(x,y) = \delta(y,K(x))$, we could obtain existence results for (1.2) as corollaries to the existence results for quasiequilibrium problems applied to problem (2.3).

We furnish an example of how this simple fact may be used for proving the existence of solutions.

Consider a quasiequilibrium problem in which the bifunction can be decomposed as sum of $g_1, g_2 : C \times C \to \mathbb{R}$:

$$\text{find } x \in K(x) \text{ such that } g_1(x,y)+g_2(x,y) \geq 0 \text{ for all } y \in K(x). \tag{2.4}$$

Assume that K is continuous with nonempty closed convex values, $g_1 + g_2$ is upper semicontinuous and nonnegative on the diagonal of $C \times C$, $g_1(x,\cdot)$ is convex for each $x \in C$ but $g_1(x,\cdot)+g_2(x,\cdot)$ is not. All the assumptions of Theorem 2.1 but (a_3) are satisfied and the theorem is not applicable to this situation. By means of δ, let us reformulate (2.4) as an equivalent generalized quasivariational problem in the following way: find $x \in C$ such that

$$[g_1(x,y)+g_2(x,x)]+[g_2(x,y)+\delta(y,K(x))] \geq [g_2(x,x)+\delta(x,K(x))], \quad \forall y \in C. \tag{2.5}$$

Problem (2.5) can be viewed as a generalized variational problem (1.2) with

$$f(x,y) = g_1(x,y)+g_2(x,x)$$

and

$$\varphi(x,y) = g_2(x,y)+\delta(y,K(x)).$$

Notice that D_φ coincides with K and condition (2.2) becomes

$$g_2(x,x) = \inf_{y\in K(x)} g_2(x,y), \quad \forall x \in K(x). \tag{2.6}$$

If condition (2.6) holds, we can apply Theorem 2.1 to the following quasiequilibrium problem

$$\text{find } x \in K(x) \text{ such that } g_1(x,y)+g_2(x,x) \geq 0 \text{ for all } y \in K(x),$$

where the objective bifunction is convex with respect to the second variable. In other words, we have been able to remove the bad part of objective bifunction, at the price of adding a further condition (2.2). We illustrate this fact with an example.

Example 2.2. Let $C = [-2,2]$. Let $g_1, g_2 : C \times C \to \mathbb{R}$ be defined as follows

$$g_1(x,y) = 2y^4 - x^4, \qquad g_2(x,y) = \min\{1 - \sqrt{x^2+y^2}, 0\},$$

and let $K : C \rightrightarrows C$ be given by $K(x) = [-1+|x|/2, 1-|x|/2]$. The feasible set-valued map is continuous, and g_1 and g_2 are continuous with $g_1(x,\cdot)$ being convex. Unfortunately, $g_1(x,\cdot) + g_2(x,\cdot)$ is not convex when $|x| > 1$ and Theorem 2.1 does not work. The fixed point set of K is $[-2/3, 2/3]$. An easy calculation shows that $g_2(x,y) = 0$ for all $x \in [-2/3, 2/3]$ and $y \in K(x)$ since $x^2 + y^2 \le 1$. Hence, (2.6) is satisfied and the quasiequilibrium problem has a solution. The unique solution is $x = 0$.

A more general approach is to relate (1.2) with the inequality problem

$$\text{find } x \in D_\varphi(x) \text{ such that } f(x,y) \ge \alpha \text{ for all } y \in D_\varphi(x),$$

where α is a suitable constant depending on φ. This is possible provided that function $(x,y) \mapsto \varphi(x,x) - \varphi(x,y)$ is upper bounded on the set $\{(x,y) \in C \times C : x, y \in D_\varphi(x)\}$. Indeed, in such a case, letting

$$\alpha = \sup\{\varphi(x,x) - \varphi(x,y) : x, y \in D_\varphi(x)\},$$

if x is a fixed point of D_φ such that $f(x,y) \ge \alpha$, for all $y \in D_\varphi(x)$, then x solves (1.2). Notice that $\alpha \ge 0$ and that $\alpha = 0$ is equivalent to condition (2.2). Following this, we get a generalization of Theorem 2.1.

Theorem 2.3. *Let C be a compact convex subset of a Hilbert space and*

$$\alpha = \sup\{\varphi(x,x) - \varphi(x,y) : x, y \in D_\varphi(x)\} < +\infty.$$

Moreover, assume that

(c_1) *D_φ is upper semicontinuous with nonempty closed convex values,*
(c_2) *$f(x,x) \ge \alpha$, for all $x \in C$,*
(c_3) *$f(x,\cdot)$ is convex, for all $y \in C$,*
(c_4) *$f(\cdot,y)$ is upper semicontinuous, for all $x \in C$,*
(c_5) *the set*

$$\left\{x \in C : \inf_{y \in D_\varphi(x)} f(x,y) \ge \alpha\right\}$$

is closed.

Then problem (1.2) has a solution.

Clearly, if $\varphi(x,y) = \delta(y, K(x))$, then $\alpha = 0$ and Theorem 2.3 comes down to Theorem 2.1.

Example 2.3. Let us consider the quasiequilibrium problem (1.1) with $C = [0,2]$, $K : C \rightrightarrows C$ defined by

$$K(x) = \{y \in C : 0 \le y \le -x^2 + 2x\}$$

and $f : C \times C \to \mathbb{R}$ defined by $f(x,y) = (y-x)(5 - 4\mathbf{1}_{\mathbb{Q}}(y-x)) + 1$, where $\mathbf{1}_{\mathbb{Q}}$ is the Dirichlet function. The fixed point set of K is $[0,1]$. Clearly $f(x,\cdot)$ is not convex and $f(\cdot,y)$ is not upper semicontinuous. Therefore, the assumptions (a_3) and (a_4) of Theorem 2.1 fail. Now we decompose f as the sum of $g_1(x,y) = 5(y-x) + 1$ and $g_2(x,y) = -4\mathbf{1}_{\mathbb{Q}}(y-x)$ and we transform the problem considered into (1.2):

$$\text{find } x \in C \text{ such that } 5(y-x) + 1 + (-4\mathbf{1}_{\mathbb{Q}}(y-x) + \delta(y, K(x))) \ge \delta(x, K(x)) \text{ for all } y \in C.$$

Taking $\varphi(x,y) = -4\mathbf{1}_{\mathbb{Q}}(y-x) + \delta(y, K(x))$, we have

$$\alpha = \sup\{4\mathbf{1}_{\mathbb{Q}}(y-x) : x, y \in K(x)\} = 4\sup\{-x^2 + x : x \in [0,1]\} = 1.$$

Since $g_1(x,x) = 1$ for every $x \in C$, all the assumptions of Theorem 2.3 hold and the existence of solutions is guaranteed. The solution set is $[0,1/5]$.

References

[1] K. Fan, A minimax inequality and applications, In: Shisha, O. (ed.) Inequalities III, pp. 103-113, Academic Press, New York, 1972.

[2] G. Bigi, M. Castellani, M. Pappalardo, M. Passacantando, Nonlinear programming techniques for equilibria, Springer, Berlin, 2019.

[3] J.-L. Joly, U. Mosco, A propos de l'existence et de la régularité des solutions de certaines inéquations quasi-variationnelles, J. Funct. Anal. 34 (1979), 107-137.

[4] J.-P. Aubin, I. Ekeland, Applied nonlinear analysis, John Wiley & Sons, New York, 1984.

[5] F. Flores-Bazán, Existence theorems for generalized noncoercive equilibrium problems: the quasi-convex case, SIAM J. Optim. 11 (2020), 675-690.