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# $L^{\infty}$ ERROR ESTIMATE FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS OF QUASI-VARIATIONAL INEQUALITIES

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**Abstract.** This paper deals with the standard finite element approximation of a class of semilinear elliptic systems of quasi-variational inequalities (QVIs) arising in stochastic control theory. Under a realistic assumption on the nonlinearity and by means of the concept of subsolutions for linear systems of QVIs, we derive sharp $L^{\infty}$ error estimate of the approximation.

**Keywords.** Semilinear system; Quasi-variational inequalities; Subsolutions; Finite element; $L^{\infty}$ error estimate.

## 1. Introduction

This paper deals with the $L^{\infty}$-convergence of the standard finite element approximation for the following class of semilinear systems of quasi-variational inequalities (QVIs): Find $U = (u_1,...,u_M) \in \left(H^1(\Omega)\right)^M$ such that

$$\begin{cases} a_i(u_i, v-u_i) \geq (f_i(u_i), v-u_i), \ \forall v \in H^1(\Omega), \\ u_i \leq k+u_{i+1}, \ v \leq k+u_{i+1}, \\ u_{M+1} = u_1, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded convex polyhedral domain of $\mathbb{R}^N$, $N \geq 1$, $f_i(.)$, $i = 1,2,...,M$, are nondecreasing, nonnegative, and Lipschitz nonlinearities, $(.,.)$ is the inner product in $L^2(\Omega)$, $k$ is a positive constant, and $a_i(.,.)$ are $M$ continuous bilinear forms.

The finite element approximation in the $L^{\infty}$-norm of systems of QVIs with linear source terms was extensively studied in the last two decades (see, e.g., [1, 2, 3, 4, 5, 6, 7]) and convergence orders were derived by means of various methods. However, when it comes to nonlinear systems, this remains to be explored. Indeed, let $\mathbb{V}_h$ denote the finite elements space consisting of continuous piecewise linear functions, $h$ be the meshsize of the triangulation, and $r_h$ be the usual Lagrange interpolation operator. The discrete counterpart of (1.1) consists of seeking $U_h = (u_{1h},...,u_{Mh}) \in (\mathbb{V}_h)^M$ such that

$$\begin{cases} a_i(u_{ih}, v-u_{ih}) \geq (f_i(u_{ih}), v-u_{ih}), \ \forall v \in \mathbb{V}_h, \\ u_{ih} \leq k+u_{i+1h}, \ v \leq k+u_{i+1h}, \\ u_{M+1h} = u_{1h}. \end{cases} \tag{1.2}$$

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In this paper, our primary aim is to show that system (1.1) can be properly approximated by a finite element method and sharp $L^\infty$-error estimate can be derived. For this, we exploit, in both the continuous and discrete situations, some qualitative properties inherited by the solution of the associated linear problem such as its Lipschitz dependency upon the right-hand side and its characterization as the least upper bound of the set of subsolutions. We give here after a brief description of the method: Given $U^0 = (u_1^0, ..., u_M^0)$ a smooth initial guess, we define the sequence $(U^n)$ such that $U^n = (u_1^n, ..., u_M^n)$, $\forall n \geq 1$, solves the continuous linear system of QVIs

$$\begin{cases} a_i(u_i^n, v - u_i^n) \geq (f_i(u_i^{n-1}), v - u_i^n)\ , \forall v \in H^1(\Omega), \\ u_i^n \leq k + u_{i+1}^n,\ v \leq k + u_{i+1}^n, \\ u_{M+1}^n = u_1^n. \end{cases} \tag{1.3}$$

Likewise, given $U_h^0 = (r_h u_1^0, ..., r_h u_M^0)$, we define the sequence $\left(U_h^n\right)$ such that $U_h^n = (u_{1h}^n, ..., u_{Mh}^n)$, $\forall n \geq 1$, solves the discrete linear system of QVIs

$$\begin{cases} a_i(u_{ih}^n, v - u_{ih}^n) \geq (f_i(u_{ih}^{n-1}), v - u_{ih}^n),\ \forall v \in H^1(\Omega) \\ u_{ih}^n \leq k + u_{i+1h}^n,\ v \leq k + u_{i+1h}^n \\ u_{M+1h}^n = u_{1h}^n \end{cases} \tag{1.4}$$

The crucial part in this paper is to build a sequence of continuous subsolutions $(\beta^n)$ such that

$$\beta^n \leq U^n, \forall n \geq 1 \tag{1.5}$$

and

$$\|\beta^n - U_h^n\|_\infty \leq Ch^2 \left|\log h\right|^2, \tag{1.6}$$

and a sequence of discrete subsolutions $(\alpha_h^n)$ such that

$$\alpha_h^n \leq u_h^n,\ \forall n \geq 1$$

and

$$\|U^n - \alpha_h^n\|_\infty \leq Ch^2 \left|\log h\right|^2. \tag{1.7}$$

In this situation, we first establish a sharp error estimate for the iterative scheme:

$$\|U^n - U_h^n\|_\infty \leq Ch^2 \left|\log h\right|^2. \tag{1.8}$$

Making use of a geometrical convergence of the sequences $(U^n)$ and $\left(U_h^n\right)$ to the solutions $U$ and $U_h$, respectively, we derive the same convergence order for the system of QVIs (1.1):

$$\|U - U_h\|_\infty \leq Ch^2 \left|\log h\right|^2, \tag{1.9}$$

where

$$\|W\|_\infty = \max_{1 \leq i \leq M} \|w_i\|_{L^\infty(\Omega)}, W = (w_1, ..., w_M)$$

and in both (1.8) and ( 1.9), $C$ is a constant independent of both $h$ and $n$.

To the best of our knowledge, this paper contains the first $L^\infty$- finite element error estimate for nonlinear systems of QVIs.

An outline of the paper is as follows. In Section 2, we give some notations and assumptions. In Section 3, we recall some qualitative results related to linear system of QVIs and establish the existence and uniqueness for the semilinear system (1.1). In Section 3, we reproduce a discrete analog study to that in Section 2. Finally, in Section 4, we give the finite element error analysis and prove the main results.

## 2. Preliminaries

We give the bilinear forms: $\forall u, v \in H^1(\Omega)$,

$$a_i(u,v) = \int_\Omega \left( \sum_{j,k=1}^{N} a^i_{jk}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} b^i_k(x) \frac{\partial u}{\partial x_k} v + a^i_0(x) uv \right) dx, \tag{2.1}$$

such that $a^i_{jk}(x), b^i_k(x), a^i_0(x))$, $(j,k = 1,...,N)$, are given sufficiently smooth functions satisfying

$$\sum_{1 \leq\ j,k \leq\ N} a^i_{jk}(x) \xi_j \xi_k \geqq \alpha |\xi|^2,\ \forall \xi \in \mathbb{R}^N,\ \alpha > 0, \forall x \in \Omega, \tag{2.2}$$

$$a^i_0(x) \geqq \beta > 0, \forall x \in \Omega,\ \forall\ i = 1,...,M. \tag{2.3}$$

We assume that the bilinear forms are coercive, i.e.,

$$a_i(v,v) \geq \delta \|v\|^2_{H^1(\Omega)},\ \forall v \in H^1(\Omega),\ \delta > 0,\ \forall\ i = 1,...,\ M, \tag{2.4}$$

and the nonlinearities $f_i(.)$ are Lipschitz:

$$|f_i(x) - f_i(y)| \leq c\,|x-y|, \forall i = 1,...,M, \tag{2.5}$$

such that

$$\frac{c}{\beta} < 1 \tag{2.6}$$

and

$$|f_i(x)| \leq C, \forall x \in \mathbb{R}, \forall i = 1,2..,,M \tag{2.7}$$

## 3. The Continuous Problem

Next, we shall give some qualitative properties enjoyed by linear systems of QVIs.

3.1. **Continuous linear system of QVIs.** Let $(g_{1,...,}g_M) \in \left(L^\infty_+(\Omega)\right)^M$ and $\zeta = (\zeta_{1,...,}\zeta_M) \in \left(H^1(\Omega)\right)^M$ be the solution of the following linear system of QVIs:

$$\begin{cases} a_i(\zeta_i, v - \zeta_i) \geq (g_i, v - \zeta_i),\ \forall v \in H^1(\Omega), \\ \zeta_i \leq k + \zeta_{i+1},\ v \leq k + \zeta_{i+1}, \\ \zeta_{M+1} = \zeta_1. \end{cases} \tag{3.1}$$

System (3.1) is linear because the source terms $g_i$, $\forall i = 1,2...,M$ are independent of the solution. Moreover, it has a unique solution $\zeta = (\zeta_{1,...,}\zeta_M)$, which belongs to $\left(W^{2,p}(\Omega)\right)^M$, $2 \leq p < \infty$ ([8, 9]).

Let $g = (g_{1,...,}g_M)$ , $\tilde{g} = (\tilde{g}_{1,...,}\tilde{g}_M) \in \left(L^\infty_+(\Omega)\right)^M$, and $\zeta = (\zeta_{1,...,}\zeta_M)$ , $\tilde{\zeta} = \left(\tilde{\zeta}_{1,...,}\tilde{\zeta}_M\right)$ be the corresponding solutions to system (3.1).

**Lemma 3.1.** [1] *(Monotony) If $g \geq \tilde{g}$, then $\zeta \geq \tilde{\zeta}$.*

**Proposition 3.1.** [1] *(Lipschitz property) If Lemma 3.1 holds, then,*

$$\left\|\zeta-\tilde{\zeta}\right\|_{\infty}\leq(\frac{1}{\beta})\left\|g-\tilde{g}\right\|_{\infty} \tag{3.2}$$

**Definition 3.1.** (Subsolution) $(\omega_{1,\ldots,}\omega_{M})\in\left(H^{1}(\Omega)\right)^{M}$ is said to be a subsolution to the system of QVIs (3.1) if

$$\begin{cases} a_i(\omega_i,v)\leq(g_i,v-\omega_i),\ \forall v\in H^1(\Omega),\\ \omega_i\leq k+\omega_{i+1},\ v\leq k+\omega_{i+1},\\ \omega_{M+1}=\omega_1. \end{cases} \tag{3.3}$$

**Theorem 3.1.** [1] *The solution of system (3.1) is the least upper bound of the set of subsolutions.*

3.2. **Existence and uniqueness for system (1.1).** Consider the mapping

$$\begin{aligned} T:(L^{\infty}(\Omega))^{M}&\longrightarrow\left(L_{+}^{\infty}(\Omega)\right)^{M}\\ W=(w_1,\ldots,w_M)&\rightarrow TW=\zeta=(\zeta_1,\ldots,\zeta_M) \end{aligned} \tag{3.4}$$

such that $\zeta=(\zeta_1,\ldots,\zeta_M)\in\left(H^1(\Omega)\right)^M$ solves the linear system

$$\begin{cases} a_i(\zeta_i,v-\zeta_i)\geq(f_i(w_i),v-\zeta_i),\ \forall v\in H^1(\Omega),\\ \zeta_i\leq k+\zeta_{i+1},\ v\leq k+\zeta_{i+1},\\ \zeta_{M+1}=\zeta_1. \end{cases} \tag{3.5}$$

Note that $f_i(w_i)$ plays the role of $g_i$ in system (3.1). Also, we clearly have

$$U^n=TU^{n-1},\ \forall n\geq1,$$

where $U^n$ is defined in (1.3).

Next, we shall prove that the mapping $T$ is a contraction.

**Theorem 3.2.** *Let (2.5),(2.6) and Proposition 3.1 hold. Then the mapping $T$ is a contraction and, therefore, its unique fixed point coincides with the solution of system (1.1). Moreover, we have*

$$\left\|U^n-U\right\|_{\infty}\leq\rho^n\left\|U^0-U\right\|_{\infty},\ \textit{where } \rho=c/\beta. \tag{3.6}$$

*Proof.* Let $W=(w_1,\ldots,w_M)$, $\tilde{W}=(\tilde{w}_1,\ldots,\tilde{w}_M)$ in $(L^{\infty}(\Omega))^M$, and $\zeta=(\zeta_1,\ldots,\zeta_M)$; $\tilde{\zeta}=\left(\tilde{\zeta}_1,\ldots,\tilde{\zeta}_M\right)$ be the corresponding solutions to system (3.1). Then, making use of Proposition 3.1 with $g_i=f_i(w_i)$ and $\tilde{g}_i=f_i(\tilde{w}_i)$, respectively, we have

$$\max_{1\leq i\leq M}\left\|\zeta_i-\tilde{\zeta}_i\right\|_{L^{\infty}(\Omega)}\leq(\frac{1}{\beta})\max_{1\leq i\leq M}\left\|f_i(w_i)-f_i(\tilde{w}_i)\right\|_{L^{\infty}(\Omega)}.$$

From (2.5) and (2.6), we get

$$\max_{1\leq i\leq M}\left\|\zeta_i-\tilde{\zeta}_i\right\|_{L^{\infty}(\Omega)}\leq(\frac{c}{\beta})\max_{1\leq i\leq M}\left\|w_i-\tilde{w}_i\right\|_{L^{\infty}(\Omega)}.$$

That is,

$$\begin{aligned} \left\|TW-T\tilde{W}\right\|_{\infty}&=\left\|\zeta-\tilde{\zeta}\right\|_{\infty}\\ &\leq\frac{c}{\beta}\left\|W-\tilde{W}\right\|_{\infty}, \end{aligned}$$

which completes the proof. The proof of (3.6) is a straightforward consequence of the contraction principle. □

## 4. The Discrete Problem

We consider a regular and quasi-uniform triangulation $\tau_h$ of $\bar{\Omega}$, consisting of $n$- simplices $K$. Denote by $h = \max_{K\in\tau_h} h_K$, the meshsize of $\tau_h$ with $h_K$ being the diameter of $K$. For each $K \in \tau_h$, denote by $P_1(K)$ the set of polynomials on $K$ with degree no more than 1. The $P_1$-conforming finite element space is given by

$$\mathbb{V}_h = \left\{v : v \in C(\bar{\Omega}),\ v_{/K} \in P_1(K),\ \forall K \in \tau_h\right\}.$$

Let $M_s$, $1 \le s \le m(h)$ denote the vertices of the triangulation $\tau_h$, and let $\varphi_s$, $1 \le s \le m(h)$, denote the corresponding basis functions of $\mathbb{V}_h$. For any $v \in C(\Omega)$, the function $r_h v(x) = \sum\limits_{s=1}^{m(h)} v(M_s)\varphi_s(x)$ represents the Lagrange interpolation of $v$ over $\tau_h$.

In order to derive the discrete analogues of Lemma 3.1, Proposition 3.1, and Theorem 3.1, the stiffness matrix needs to be an M-Matrix.

**Definition 4.1.** A real $d \times d$ matrix $C = (c_{ls})$ with $c_{ls} \le 0$, $\forall l \ne s$, $1 \le l, s \le d$, is said to be an $M$-Matrix if $C$ is non singular and $C^{-1} \ge 0$ (i.e., all entries of its inverse are nonnegative).

Let us denote by $\mathbb{A}^i = (a_i(\varphi_l, \varphi_s))$, $i = 1, \ldots, M$, the stiffness matrices of the system. Since the bilinear form $a_i(.,.)$ are coercive, the matrices

$$\mathbb{A}^i \text{ is definite positive,}$$

and

$$a_i(\varphi_l, \varphi_l) > 0, \quad \forall l.$$

Moreover, conditions on the triangulation can be found in [10] so that

$$a_i(\varphi_l, \varphi_s) \le 0, \quad \forall l \ne s.$$

In view of the above, we have the following lemma.

**Lemma 4.1.** [10] *The matrices with generic coefficient* $(a_i(\varphi_l, \varphi_s)), 1 \le l, s \le m(h)$, $i = 1, \ldots,$ *M, are M-Matrices.*

### 4.1. Discrete linear system of QVIs.

The discrete analogue of system (3.1) consists of finding $\zeta_h = (\zeta_{1h,\ldots,}\zeta_{Mh}) \in (\mathbb{V}_h)^M$ such that

$$\begin{cases} a_i(\zeta_{ih}, v - \zeta_{ih}) \ge (g_i, v - \zeta_{ih}),\ \forall v \in \mathbb{V}_h, \\ \zeta_{ih} \le k + \zeta_{i+1h},\ v \le k + \zeta_{i+1h}, \\ \zeta_{M+1h} = \zeta_{1h}. \end{cases} \tag{4.1}$$

Under conditions of Lemma 4.1, system (4.1) has a unique solution.

Let $g = (g_{1,\ldots,}g_M)$, $\tilde{g} = (\tilde{g}_{1,\ldots,}\tilde{g}_M) \in \left(L^\infty_+(\Omega)\right)^M$, and $\zeta = \left(\zeta_{1h,\ldots,}\zeta_{Mh}\right)$, $\tilde{\zeta}_h = \left(\tilde{\zeta}_{1h,\ldots,}\tilde{\zeta}_{Mh}\right)$ be the corresponding solutions to system (4.1).

**Lemma 4.2.** [1] *(Monotony) If lemma 4.1 holds, then* $g \ge \tilde{g}$ *implies* $\zeta_h \ge \tilde{\zeta}_h$,

**Proposition 4.1.** [1] *(Lipschitz property) If Lemma 4.2 holds, then*

$$\left\|\zeta_h - \tilde{\zeta}_h\right\|_\infty \leq (\frac{1}{\beta}) \left\|g - \tilde{g}\right\|_\infty. \tag{4.2}$$

**Definition 4.2.** [1] (Subsolution) $\omega_h = (\omega_{1h,\ldots,}\omega_{Mh}) \in (\mathbb{V}_h)^M$ is said to be a subsolution to the system of QVIs (4.1) if

$$\begin{cases} a_i(\omega_{ih}, \varphi_s) \leq (g_i, \varphi_s),\ \forall \varphi_s, \\ \omega_{ih} \leq k + \omega_{i+1h}, \\ \omega_{M+1h} = \omega_{1h}. \end{cases} \tag{4.3}$$

**Theorem 4.1.** [1] *Let Lemma 4.1 hold. Then the solution of system (4.1) is the least upper bound of the set of subsolutions.*

**Theorem 4.2.** [4] *Let* $\zeta \in \left(W^{2,p}(\Omega)\right)^M$, $2 \leq p < \infty$. *Then,*

$$\left\|\zeta - \tilde{\zeta}_h\right\|_\infty \leq Ch^2 \left|\ln h\right|^2. \tag{4.4}$$

4.2. **Existence and uniqueness for system (1.2).** Consider the mapping

$$\begin{aligned} T_h : \left(L^\infty_+(\Omega)\right)^M &\longrightarrow (\mathbb{V}_h)^M \\ W \to T_h W = \zeta_h &= (\zeta_{1h}, \ldots, \zeta_{Mh}) \end{aligned} \tag{4.5}$$

such that $\zeta_h = (\zeta_{1h}, \ldots, \zeta_{Mh})$ solves the discrete linear system

$$\begin{cases} a_i(\zeta_{ih}, v - \zeta_{ih}) \geq (f_i(w_i), v - \zeta_{ih}),\ \forall v \in \mathbb{V}_h, \\ \zeta_{ih} \leq k + \zeta_{i+1h},\ v \leq k + \zeta_{i+1h}, \\ \zeta_{M+1h} = \zeta_{1h}. \end{cases} \tag{4.6}$$

We clearly have

$$U_h^n = TU_h^{n-1},\ \forall n \geq 1,$$

where $U_h^n$ is defined in (1.4).

As in the continuous case, we assert that $T_h$ is a contraction. The proof will be omitted as it is very similar to that of Theorem 3.2.

**Theorem 4.3.** *Let* (2.5) , (2.6) *and Proposition 4.1 hold. Then mapping $T_h$ is a contraction and, therefore, its unique fixed point coincides with the solution of system (1.2). Moreover,*

$$\left\|U_h^n - U_h\right\|_\infty \leq \rho^n \left\|U_h^0 - U_h\right\|_\infty. \tag{4.7}$$

## 5. $L^\infty$ - Error Analysis

This section is devoted to deriving main results of the paper. (From now on, $C$ will denote a constant independent of both $n$ and $h$).

5.1. **$L^\infty$ - error estimate for the iterative scheme.** In order to estimate the error between the continuous iterative scheme and its finite element counterpart, we introduce the following sequences of linear systems of QVIs.

5.1.1. *A sequence of continuous linear systems of QVIs.* We introduce the sequence $(\bar{U}^n)_{n\geq 1}$ such that $\bar{U}^n = \left(\bar{u}_1^n, ..., \bar{u}_M^n\right)$, $\forall n \geq 1$, solves the continuous linear system**:**

$$\begin{cases} a_i(\bar{u}_i^n, v - \bar{u}_i^n) \geq (f_i(u_{ih}^{n-1}), v - \bar{u}_i^n), \ \forall v \in H^1(\Omega), \\ \bar{u}_i^n \leq k + \bar{u}_{i+1}^n, \ v \leq k + \bar{u}_{i+1}^n, \\ \bar{u}_{M+1}^n = \bar{u}_i^n, \end{cases} \tag{5.1}$$

where $u_{ih}^n$ is defined in (1.4).

**Lemma 5.1.**

$$\|\bar{U}^n - U_h^n\|_\infty \leq Ch^2 \left|\ln h\right|^2 \tag{5.2}$$

*Proof.* As $U_h^n$ is the discrete counterpart of $\bar{U}^n$, and due to (2.7), $\left\|f_i(u_{ih}^{n-1})\right\|_{L^\infty(\Omega)} \leq C$, $\forall i = 1, ...,$ $M$, making use of Theorem 4.2, we get the desired error estimates immediately. □

5.1.2. *A sequence of discrete linear systems of QVIs.* We define the sequence $\left(\bar{U}_h^n\right)_{n\geq 1}$ such that $\bar{U}_h^n = \left(\bar{u}_{1h}^n, ..., \bar{u}_{Mh}^n\right)$ $(\forall n \geq 1)$ solves the discrete linear system of QVIs:

$$\begin{cases} a_i(\bar{u}_{ih}^n, v - \bar{u}_{ih}^n) \geq (f_i(u_i^{n-1}), v - \bar{u}_{ih}^n), \ \forall v \in \mathbb{V}_h, \\ \bar{u}_{ih}^n \leq k + r_h \bar{u}_{i+1h}^n, \ v \leq k + r_h \bar{u}_{i+1h}^n, \\ \bar{u}_{M+1h}^n = \bar{u}_{1h}^n, \end{cases} \tag{5.3}$$

where $u_i^n$ is defined in (1.3).

**Lemma 5.2.**

$$\|U^n - \bar{U}_h^n\|_\infty \leq Ch^2 \left|\ln h\right|^2. \tag{5.4}$$

*Proof.* As $\bar{U}_h^n$ is the discrete counterparts of $U^n$ and due to (2.7), $\left\|f_i(u_i^{n-1})\right\|_{L^\infty(\Omega)} \leq C$, $\forall i =$ $1, ..., M$ , then making use of Theorem 4.2, we get the desired error estimate immediately. □

**Theorem 5.1.** *Let $\rho = c/\beta$. Then*

$$\|U^n - U_h^n\|_\infty \leq \frac{1-\rho^{n+1}}{1-\rho} Ch^2 \left|\ln h\right|^2. \tag{5.5}$$

*Proof.* The proof consists of building a continuous sequence of functions $\beta^n = \left(\beta_1^n, ..., \beta_M^n\right)$ such that

$$\begin{cases} \beta^n \leq U^n, \\ \text{and} \\ \left\|\beta^n - U_h^n\right\|_\infty \leq Ch^2 \left|\ln h\right|^2, \end{cases}$$

and a discrete sequence of functions $\alpha^n = \left(\alpha_{1h}^n, ..., \alpha_{Mh}^n\right)$ such that

$$\begin{cases} \alpha_h^n \leq U_h^n, \\ \text{and} \\ \left\|\alpha_h^n - U^n\right\|_\infty \leq Ch^2 \left|\ln h\right|^2. \end{cases}$$

For this, we proceed by induction. Indeed, let $\theta(h) = Ch^2 |\ln h|^2$ and consider the problem (5.1) for $n = 1$, that is,

$$\begin{cases} a_i(\bar{u}_i^1, v - \bar{u}_i^1) \geq (f_i(u_{ih}^0), v - \bar{u}_i^1), \ \forall v \in H^1(\Omega), \\ \bar{u}_i^1 \leq k + \bar{u}_{i+1}^1, \ v \leq k + \bar{u}_{i+1}^1, \\ \bar{u}_{M+1}^1 = \bar{u}_i^1. \end{cases} \tag{5.6}$$

Now, as $\bar{U}^1$ is solution to a linear system of QVIs, it is also a subsolution to the same system, that is,

$$\begin{cases} a_i(\bar{u}_i^1, v) \leq (f_i(u_{ih}^0), v), \ \forall v \in H^1(\Omega), v \geq 0, \\ \bar{u}_i^1 \leq k + \bar{u}_{i+1}^1, \\ \bar{u}_{M+1}^1 = \bar{u}_i^1. \end{cases}$$

Moreover, since

$$\begin{aligned} f_i(u_{ih}^0) &\leq f_i(u_{ih}^0) - f_i(u_i^0) + f(u_i^0) \\ &\leq c \left\| u_{ih}^0 - u_i^0 \right\|_{L^\infty(\Omega)} + f_i(u_i^0) \\ &\leq c\theta(h) + f_i(u_i^0), \end{aligned}$$

we have

$$\begin{cases} a_i(\bar{u}_i^1, v) \leq (c\theta(h) + f_i(u_i^0), v) \forall v \in H^1(\Omega), v \geq 0, \\ \bar{u}^1 \leq k + \bar{u}_{i+1}^1, \\ \bar{u}_{M+1}^1 = \bar{u}_i^1. \end{cases}$$

Hence, $\bar{U}^1 = \left(\bar{u}_1^1, ..., \bar{u}_M^1\right)$ is a subsolution to the system of QVIs with the source term

$$\tilde{g} = [c\theta(h) + f_1(u_1^0), ..., c\theta(h) + f_M(u_M^0)].$$

Let $\Psi^1$ be the solution of such a system. Then, as $U^1 = \left(u_1^1, ..., u_M^1\right)$ is the solution of the system of QVIs with the source term $g = [f_1(u_1^0), ..., f_M(u_M^0)]$, we conclude from Proposition 3.1, we get

$$\left\| \Psi^1 - U^1 \right\|_\infty \leq (1/\beta) \left\| g - \tilde{g} \right\|_{L^\infty(\Omega)} \leq \rho\theta(h).$$

Hence, making use of Theorem 3.1, we have $\bar{U}^1 \leq U^1 \leq \rho\theta(h)$. Putting $\beta^1 = \bar{U}^1 - \rho\theta(h)$, we get

$$\beta^1 \leq U^1. \tag{5.7}$$

Using Lemma 5.1 yields

$$\begin{aligned} \left\| \beta^1 - U_h^1 \right\|_\infty &\leq \left\| \bar{U}^1 - \rho\theta(h) - U_h^1 \right\|_\infty \\ &\leq \left\| \bar{U}^1 - U_h^1 \right\|_\infty + \rho\theta(h \\ &\leq (1+\rho)\theta(h). \end{aligned} \tag{5.8}$$

Now, let us consider the problem (5.3) with $n = 1$, that is,

$$\begin{cases} a_i(\bar{u}_{ih}^1, v - \bar{u}_h^1) \geq (f_i(u_i^0), v - \bar{u}_h^1), \ \forall v \in \mathbb{V}_h, \\ \bar{u}_{ih}^1 \leq k + r_h \bar{u}_{i+1h}^1, \ v \leq k + r_h \bar{u}_{i+1h}^1, \\ \bar{u}_{M+1h}^1 = \bar{u}_{1h}^1. \end{cases}$$

Then, as $\bar{U}_h^1$ is solution to a linear system of QVIs, it is also a subsolution to the same system, that is,

$$\begin{cases} a_i(\bar{u}_{ih}^1, \varphi_s) \leq (f_i(u_i^0), \varphi_s), \ \forall \varphi_s, \\ \bar{u}_{ih}^1 \leq k + r_h \bar{u}_{i+1h}^1, \\ \bar{u}_{M+1h}^1 = \bar{u}_{1h}^1. \end{cases}$$

Also, as

$$\begin{aligned} f_i(u_i^0) &\leq f_i(u_i^0) - f_i(u_{ih}^0) + f_i(u_{ih}^0) \\ &\leq c \left\| u_{ih}^0 - u_i^0 \right\|_{L^\infty(\Omega)} + f_i(u_{ih}^0) \\ &\leq c\theta(h) + f_i(u_{ih}^0), \end{aligned}$$

it follows that

$$\begin{cases} a_i(\bar{u}_{ih}^1, \varphi_s) \leq (c\theta(h) + f_i(u_{ih}^0), \varphi_s), \ \forall \varphi_s, \\ \bar{u}_h^1 \leq k + \bar{u}_{i+1h}^1, \\ \bar{u}_{M+1h}^1 = \bar{u}_{1h}^1. \end{cases}$$

So, $\bar{U}_h^1 = \left(\bar{u}_{1h}^1, ..., \bar{u}_{Mh}^1\right)$ is a subsolution for the linear system of QVIs with the source term

$$\tilde{g} = [c\theta(h) + f_1(u_{1h}^0), ..., c\theta(h) + f_M(u_{Mh}^0)].$$

Let $\Psi_h^1$ be the solution to such a system. Then, as $U_h^1 = \left(u_{1h}^1, ..., u_{Mh}^1\right)$ is the solution to the system of QVIs with the source term $g = [f_1(u_{1h}^0), ..., \ f_M(u_{Mh}^0)]$ , making use of Proposition 4.1, we get

$$\begin{aligned} \left\| \Psi_h^1 - U_h^1 \right\|_\infty &\leq (1/\beta) \left\| g - \tilde{g} \right\|_\infty \\ &\leq \rho\theta(h). \end{aligned}$$

Hence, applying Theorem 4.1, we get $\bar{U}_h^1 \leq U_h^1 + \rho\theta(h)$. Putting $\alpha_h^1 = \bar{U}_h^1 - \rho\theta(h)$, we get

$$\alpha_h^1 \leq U_h^1. \tag{5.9}$$

Hence, as $\bar{U}_h^1$ is the discrete analog of $U^1$, making use of Lemma 5.2, we also get

$$\begin{aligned} \left\| \alpha_h^1 - U^1 \right\|_\infty &\leq \left\| \bar{U}_h^1 - U^1 \right\|_\infty + \rho\theta(h) \\ &\leq (1+\rho)\,\theta(h). \end{aligned} \tag{5.10}$$

Thus, combining (5.7), (5.8), (5.9), and (5.10), we obtain

$$\begin{aligned} U^1 &\leq \alpha_h^1 + (1+\rho)\,\theta(h) \leq U_h^1 + (1+\rho)\,\theta(h) \\ &\leq \beta^1 + 2\,(1+\rho)\,\theta(h) \leq U^1 + 2\,(1+\rho)\,\theta(h), \end{aligned}$$

that is, $\left\| U^1 - U_h^1 \right\|_\infty \leq \frac{1-\rho^2}{1-\rho}\theta(h)$. Assume that

$$\left\| U^{n-1} - U_h^{n-1} \right\|_\infty \leq \frac{1-\rho^n}{1-\rho}\theta(h). \tag{5.11}$$

Since $\bar{U}^n$, $\forall i = 1,\ldots, M$, is the solution to a linear system of QVIs, it is also a subsolution, that is,

$$\begin{cases} a_i(\bar{u}_i^n, v) \leq (f_i(u_{ih}^{n-1}), v),\ \forall v \in H^1(\Omega),\ v \geq 0, \\ \bar{u}^1 \leq k + \bar{u}_{i+1}^n, \\ \bar{u}_{M+1}^n = \bar{u}_1^n, \end{cases}$$

but

$$\begin{aligned} f_i(u_{ih}^{n-1}) &\leq f_i(u_{ih}^{n-1}) - f_i(u_i^{n-1}) + f_i(u_i^{n-1}) \\ &\leq c \left\| u_i^{n-1} - u_{ih}^{n-1} \right\|_{L^\infty(\Omega)} + f_i(u_i^{n-1}) \\ &\leq c\frac{1-\rho^n}{1-\rho}\theta(h) + f(u_i^{n-1}). \end{aligned}$$

It follows that

$$\begin{cases} a_i(\bar{u}_i^n, v) \leq \left(c\frac{1-\rho^n}{1-\rho}\theta(h) + f_i(u_i^{n-1}),\ v\right),\ \forall v \in H^1(\Omega), v \geq 0, \\ \bar{u}_i^n \leq k + \bar{u}_{i+1}^n, \\ \bar{u}_{M+1}^n = \bar{u}_1^n. \end{cases}$$

So, $\bar{U}^n$ is a subsolution to the linear system of QVIs with the source term

$$\tilde{g} = \left[c\frac{1-\rho^n}{1-\rho}\theta(h) + f_1(u_1^{n-1}), \ldots, c\frac{1-\rho^n}{1-\rho}\theta(h) + f_M(u_M^{n-1})\right].$$

Let $\bar{\Psi}^n$ be the solution of such a system. Then, as $U^n = \left(u_1^n, \ldots, u_M^n\right)$ is the solution of the system of QVIs with source term $g = \left[f_1(u_1^{n-1}), \ldots, f_M(u_M^{n-1})\right]$, we conclude from Proposition 3.1 that

$$\begin{aligned} \left\| U^n - \bar{\Psi}^n \right\|_\infty &\leq (1/\beta) \left\| g - \tilde{g} \right\|_\infty \\ &\leq \rho\frac{1-\rho^n}{1-\rho}\theta(h). \end{aligned}$$

Hence, applying Theorem 3.1 yields

$$\bar{U}^n \leq U^n + \rho\frac{1-\rho^n}{1-\rho}\theta(h).$$

Putting $\beta^n = \bar{U}^n - \rho\frac{1-\rho^n}{1-\rho}\theta(h)$, we get

$$\beta^n \leq U^n, \tag{5.12}$$

and

$$\begin{aligned} \left\| \beta^n - U_h^n \right\|_\infty &\leq \left\| \bar{U}^n - \rho\frac{1-\rho^n}{1-\rho}\theta(h) - U_h^n \right\|_\infty \\ &\leq \left\| \bar{U}^n - U_h^n \right\|_\infty + \rho\frac{1-\rho^n}{1-\rho}\theta(h) \\ &\leq \left(1 + \rho\frac{1-\rho^n}{1-\rho}\right)\theta(h) \\ &\leq \frac{1-\rho^{n+1}}{1-\rho}\theta(h). \end{aligned} \tag{5.13}$$

Now, we consider the system

$$\begin{cases} a(\bar{u}_{ih}^n, v-\bar{u}_{ih}^n) \geq (f_i(u_i^{n-1}), v-\bar{u}_{ih}^n),\ \forall v \in \mathbb{V}_h, \\ \bar{u}_{ih}^n \leq k+\bar{u}_{i+1h}^n,\ v \leq k+\bar{u}_{i+1h}^n, \\ \bar{u}_{M+1h}^n = \bar{u}_{1h}^n. \end{cases} \tag{5.14}$$

Then, as $\bar{U}_h^n$ is the solution of a linear system of QVIs, it is also a subsolution for the same VI, that is,

$$\begin{cases} a(\bar{u}_{ih}^n, \varphi_s) \leq (f_i(u_{ih}^{n-1}), \varphi_s),\ \forall v \in \varphi_s, \\ \bar{u}_{hh}^n \leq k+\bar{u}_{i+1h}^n,\ v \leq k+\bar{u}_{i+1h}^n, \\ \bar{u}_{M+1h}^n = \bar{u}_{1h}^n, \end{cases} \tag{5.15}$$

but,

$$\begin{aligned} f_i(u_i^{n-1}) &\leq f_i(u_i^{n-1}) - f_i(u_{ih}^{n-1}) + f_i(u_{ih}^{n-1}) \\ &\leq c\left\|u_i^{n-1} - u_{ih}^{n-1}\right\|_{L^\infty(\Omega)} + f_i(u_{ih}^{n-1}) \\ &\leq c\frac{1-\rho^n}{1-\rho}\theta(h) + f_i(u_{ih}^{n-1}). \end{aligned}$$

Hence,

$$\begin{cases} a_i(\bar{u}_{ih}^n, \varphi_s) \leq \left(c\frac{1-\rho^n}{1-\rho}\theta(h) + f_i(u_i^{n-1}),\ \varphi_s\right),\ \forall \varphi_s, \\ \bar{u}_h^n \leq k+\bar{u}_{i+1h}^n, \\ \bar{u}_{M+1h}^n = \bar{u}_{1h}^n. \end{cases}$$

So, $\bar{U}_h^n$ is a subsolution to the linear system of QVIs with the source term

$$\left[c\frac{1-\rho^n}{1-\rho}\theta(h) + f_1(u_{1h}^{n-1}),\dots,c\frac{1-\rho^n}{1-\rho}\theta(h) + f_M(u_{Mh}^{n-1})\right].$$

Let $\bar{\Psi}_h^n$ be the solution of such a system. Then, as $U_h^n$ is the solution of the discrete system of QVIs with source terms $\left[f_1(u_{1h}^{n-1}),\dots,f_M(u_{Mh}^{n-1})\right]$, by making use of Proposition 4.1, we get

$$\left\|\bar{\Psi}_h^n - U_h^n\right\|_\infty \leq (1/\beta)\left\|g-\tilde{g}\right\|_\infty \leq \rho\frac{1-\rho^n}{1-\rho}\theta(h).$$

From Theorem 4.1, we have $\bar{U}_h^n \leq U_h^n + Ch^2\left|\ln h\right|^2$. Putting

$$\alpha_h^n = \bar{U}_h^n - \rho\frac{1-\rho^n}{1-\rho}\theta(h),$$

we have

$$\alpha_h^n \leq U_h^n, \tag{5.16}$$

and

$$\begin{aligned} \left\|\alpha_h^n - U^n\right\|_\infty &\leq \left\|\bar{U}_h^n - U^n\right\|_\infty + Ch^2\left|\ln h\right|^2 \\ &\leq \left(1+\rho\frac{1-\rho^n}{1-\rho}\right)\theta(h) \\ &\leq \frac{1-\rho^{n+1}}{1-\rho}\theta(h). \end{aligned} \tag{5.17}$$

Finally, combining (5.12), (5.13), (5.16) and (5.17), we obtain

$$\begin{aligned} U^n &\le \alpha_h^n + \frac{1-\rho^{n+1}}{1-\rho}\theta(h) \le U_h^n + \frac{1-\rho^{n+1}}{1-\rho}\theta(h) \\ &\le \beta^n + 2\frac{1-\rho^{n+1}}{1-\rho}\theta(h) \le U^n + 2\frac{1-\rho^{n+1}}{1-\rho}\theta(h), \end{aligned}$$

that is,

$$\|U^n - U_h^n\|_\infty \le \frac{1-\rho^{n+1}}{1-\rho}\theta(h).$$

This completes the proof. □

5.2. **$L^\infty$- error estimate for system (1.1).**

**Theorem 5.2.**

$$\|U - U_h\|_\infty \le Ch^2 |\ln h|^2 . \tag{5.18}$$

*Proof.* Combining (3.6), (4.7), and Theorem 5.1, we have

$$\begin{aligned} \|U - U_h\|_\infty &\le \|U - U^n\|_\infty + \|U^n - U_h^n\|_\infty + \|U_h^n - U_h\|_\infty \\ &\le \rho^n \left\|U^0 - U\right\|_\infty + Ch^2 |\ln h|^2 + \rho^n \left\|U_h^0 - U_h\right\|_\infty . \end{aligned}$$

Letting $n \to \infty$, we obtain $\|U - U_h\|_\infty \le Ch^2 |\ln h|^2$. □

## References


[1] P. Cortey Dumont, J.C. Nedelec, Sur l'analyse numérique des equations de Hamilton-Jacobi-Bellman, Math. Methods Appl. Sci. 9 (1987), 198-209.

[2] M. Boulbrachene, M. Haiour, The Finite element approximation of Hamilton-Jacobi-Bellman equations, Comput. Math. Appl. 41 (2001), 993-1007.

[3] M. Boulbrachene, B. Chentouf, The finite element approximation of Hamilton-Jacobi- Bellman Equations: The noncoercive case, Appl. Math. Comput. 158 (2004), 585-592.

[4] M. Boulbrachene, P. Cortey Dumont, Optimal $L^\infty$-error estimates of a finite element method for Hamilton-Jacobi-Bellman equations, Numer. Funct. Anal. Optim. 30 (2009), 421-435.

[5] M. Boulbrachene, S. Saadi, $L^\infty$ - error analysis for a system of quasi-variational Inequalities with noncoercive operators, J. Inequal. Appl. 2006 (2006), Article ID 15704.

[6] M. Boulbrachene, Optimal $L^\infty$ - error estimate for a System of elliptic quasi variational Inequalities with Non coercive Operators. In Advances in Applied Mathematics, pp. 89-96, Springer Series in Mathematics and Statistics, 2014.

[7] M. Boulbrachene, On The Finite element approximation of a system of elliptic quasi-variational inequalities related to Hamilton-Jacobi Bellman equations, J. Appl. Numer. Math. 2 (2018), 27-41.

[8] P.L. Lions, J.L. Menaldi, Optimal control of stochastic integrals and Hamilton-Jacobi-Bellman equations (part I), SIAM. J. Control. Optim. 20 (1982), 58-81.

[9] L.C. Evans, Classical solutions of the Hamilton–Jacobi–Bellman equation for uniformly elliptic operators, Trans. Amer. Math. Soc. 275, (1983), 245-255.

[10] C. Lu, W. Huang, J. Qiu, Maximum principle in linear finite element approximations of anisotropic diffusion–convection–reaction problems, Numer. Math. 127 (2014), 515-537.