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GOLDEN RATIO ALGORITHMS FOR SOLVING EQUILIBRIUM PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we devise new iterative algorithms for solving pseudomonotone equilibrium problems in real Hilbert spaces. The advantage of our algorithms is that they require only one strongly convex programming problem at each iteration. Under suitable conditions, we establish the strong and weak convergence of the proposed algorithms. The results presented in the paper extend and improve some recent results in the literature. The performances and comparisons with some existing methods are presented through numerical examples.
Keywords. Equilibrium problem; Extragradient algorithm; Golden ratio algorithm; Weak convergence; Variational inequality.

1. INTRODUCTION

Equilibrium problems unify many important problems, such as optimization problems, variational inequality problems and fixed point problems, saddle point (minimax) problems, Nash equilibria problems and complementarity problems. As far as we know, the term "equilibrium problem" was coined in 1992 by Muu and Oettli [1] and has been elaborated further by Blum and Oettli [2]. The equilibrium problem (shortly, EP) is also known as the Ky Fan inequality since Fan [3] gave the first existence result of solutions of the EP. Thanks to its wide applications, many results concerning the existence of solutions for equilibrium problems have been established and generalized by a number of authors (see, e.g., [4, 5, 6, 7] and the references therein). One of the most interesting and important problems in the equilibrium problem theory is the study of efficient iterative algorithms for finding approximate solutions, and the convergence analysis of algorithms. Several methods have been proposed to solve equilibrium problems in finite and infinite dimensional spaces (see, e.g., [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]

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and the references theirein). In [8, 15, 16], some general iterative schemes based on the proximal method, the viscosity approximation method and the hybrid method were introduced for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the equilibrium problem. But in the proximal method, we must solve an regularized equilibrium problem at each iteration of the method. This task is not easy. To overcome this difficulty, Antipin [18] and Quoc et al. [12] replaced the regularized equilibrium problem by two strongly convex optimizations, which seem computationally easier than solving the regularized equilibrium problem in the proximal method. Their method is known under the name of the extragradient method. The reason is that when the problem (EP) is a variational inequality problem, this method reduces to the classical extragradient method introduced by Korpelevich [19]. In 2008, Quoc et al. [12] extended the extragradient algorithm for the Bregman distance case and proved some important results as the foundation for later studies. It was proved that if the bifunction associated with the (EP) is pseudomonotone and satisfies a Lipschitz-type condition, then the extragradient method is weakly convergent in the framework of Hilbert spaces. Since then, many variants of the extragradient algorithm were developed to improve the efficiency of the method; see, e.g., [13, 20, 21, 22, 23] for a survey. In most algorithms, at each iteration, it must either solve two strongly convex programming problems or solve one strongly convex programming problem with one additional projection onto the feasible set. There is even an algorithm that solve three strongly convex programming problems at each iteration. Therefore, the evaluation of the subprogram involved in such algorithms is in general very expensive if the bifunctions and the feasible sets have complicated structures. For more details, we refer to, for instance, [21, 23, 24, 25].

Note that the extragradient algorithm must solve two strongly convex programming problems at each iteration. Therefore, their computations are expensive if the bifunctions and the feasible sets have complicated structures. These observations lead us to the following question.

Question. *Can we improve the extragradient algorithm such that we use one strongly convex programming problem only at each iteration?*

In this paper, we give a positive answer to this question. Motivated and inspired by the algorithms in [12, 14, 18, 26], we will introduce some new algorithms for solving the EP. The advantage of our methods is that it only requires solving one strongly convex optimization problem or computing one projection onto the feasible set. Besides, the assumptions on f can be relaxed and the convergence is still guaranteed. Numerical examples are presented to describe the efficiency of the proposed approach.

The rest of the paper is organized as follows. After collecting some definitions and basic results in Section 2, we prove, in Section 3, the weak convergence of the proposed algorithm. In Section 4, we deal with strong convergence by using the strong pseudomonotonicity. The particular case when the equilibrium problem reduces to the variational inequality problem is given in Section 5. Finally, in Section 6, we provide some numerical results to illustrate the convergence of our algorithm, and compare it with the previous algorithms.

2. PRELIMINARIES

From now on, we will assume that C is a nonempty closed convex subset of a real Hilbert space H, and $f : H \times H \to \mathbb{R} \cup \{+\infty\}$ a bifunction such that $C \times C$ is contained in the domain of f. Consider the following problem which is known as an equilibrium problem (see Muu and

Oettli [1] and Blum and Oettli [2]):

$$\text{Find } \bar{x} \in C \text{ such that } f(\bar{x}, y) \geq 0, \ \ \forall y \in C. \tag{2.1}$$

The set of solutions of EP (2.1) will be denoted by $\mathrm{Sol}(C, f)$, i.e.,

$$\mathrm{Sol}(C, f) := \{x \in C : f(x, y) \geq 0, \ \ \forall y \in C\}.$$

In 2015, Dong et al. [24] introduced and analyzed the following General Extragradient Algorithm (GEA) for solving equilibrium problem (2.1):

$$\begin{cases} x^0 \in C, \\ \bar{x}^k = \operatorname*{argmin}_{y \in C} \left\{ \alpha_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 \right\}, \\ \tilde{x}^k = \operatorname*{argmin}_{y \in C} \left\{ \beta_k f(\bar{x}^k, y) + \frac{1}{2}\|y - \bar{x}^k\|^2 \right\}, \\ x^{k+1} = \operatorname*{argmin}_{y \in C} \left\{ \beta_k f(\tilde{x}^k, y) + \frac{1}{2}\|y - \tilde{x}^k\|^2 \right\}, \end{cases} \tag{2.2}$$

where $\alpha_k \geq 0$ and $\beta_k > 0$.

It is easy to see that when $\alpha_k = 0$ for all k, Algorithm GEA reduces to the classical extragradient algorithm [12, 18]. In 2017, Hieu [20] introduced an extragradient algorithm for a class of strongly pseudomonotone equilibrium problems as follows

$$\begin{cases} x^0 \in C, \\ y^n = \operatorname*{argmin}_{y \in C} \left\{ \lambda_n f(x^n, y) + \dfrac{1}{2}\|y - x^n\|^2 \right\}, \\ x^{n+1} = \operatorname*{argmin}_{y \in C} \left\{ \lambda_n f(y^n, y) + \dfrac{1}{2}\|y - x^n\|^2 \right\}, \end{cases} \tag{2.3}$$

where $\{\lambda_n\}$ is a non-summable and diminishing sequence, i.e.,

$$\lim_{n \to \infty} \lambda_n = 0, \ \ \sum_{n=0}^{\infty} \lambda_n = +\infty. \tag{2.4}$$

In 2018, Hieu [27] proposed a Popov type algorithm for strongly pseudomonotone equilibrium problems below

$$\begin{cases} x^0, y^0 \in C, \\ x^{n+1} = \operatorname*{argmin}_{y \in C} \left\{ \lambda_n f(y^n, y) + \dfrac{1}{2}\|y - x^n\|^2 \right\}, \\ y^{n+1} = \operatorname*{argmin}_{y \in C} \left\{ \lambda_n f(y^n, y) + \dfrac{1}{2}\|y - x^{n+1}\|^2 \right\}, \end{cases} \tag{2.5}$$

where $\{\lambda_n\}$ is a nonincreasing sequence satisfying condition (2.4).

Targeting an improvement of the above algorithms, we will introduce the so-called golden ratio algorithm for equilibrium problems in Section 3.

Now let us begin with some concepts and auxiliary results needed in the sequel. Let H be a real Hilbert space endowed with inner product $\langle .,. \rangle$ and associated norm $\|.\|$. It is easy to see that

$$\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \tag{2.6}$$

for all $x,y \in H$, and for all $t \in \mathbb{R}$.

When $\{x^k\}$ is a sequence in H, we denote the strong convergence of $\{x^k\}$ to $x \in H$ by $x^k \to x$, and the weak convergence by $x^k \rightharpoonup x$. Let C be a nonempty closed convex subset of H. For every element $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, that is

$$||x-P_C x|| = \min\{||x-y|| : \ y \in C\}.$$

The operator P_C is called the *metric projection* of H onto C and some of its properties are summarized in the next lemma, see, e.g., [28].

Lemma 2.1. *Let $C \subseteq H$ be a closed convex set. Then P_C has the following propositions*

(1) $\langle x-P_C x, y-P_C x\rangle \le 0$ *for all $x \in H$ and $y \in C$;*

(2) $\|P_C x - y\|^2 \le \|x-y\|^2 - \|x-P_C x\|^2$ *for all $x \in H$, $y \in C$.*

For a proper, convex and lower semicontinuous function $g : H \to (-\infty,\infty]$ and $\gamma > 0$, the Moreau envelope of g of parameter γ is the convex function

$$^{\gamma}g(x) = \inf_{y\in H}\left\{g(y)+\frac{1}{2\gamma}\|y-x\|^2\right\}, \ \ \forall x \in H.$$

For all $x \in H$, the function

$$y \mapsto g(y)+\frac{1}{2\gamma}\|y-x\|^2$$

is proper, strongly convex and lower semicontinuous, thus the infimum is attained, i.e., $^{\gamma}g : H \to \mathbb{R}$.

The unique minimum of

$$y \mapsto g(y)+\frac{1}{2}\|y-x\|^2 \tag{2.7}$$

is called proximal point of g at x and it is denoted by $\mathrm{prox}_g(x)$. The operator

$$\begin{aligned}\mathrm{prox}_g(x) &: H \to H\\ x &\mapsto \operatorname*{argmin}_{y\in H}\left\{g(y)+\frac{1}{2\gamma}\|y-x\|^2\right\}\end{aligned}$$

is well-defined and is said to be the proximity operator of g. When $g = \iota_C$ (the indicator function of the convex set C), one has

$$\mathrm{prox}_{\iota_C}(x) = P_C(x)$$

for all $x \in H$.

We also recall that the subdifferential of $g : H \to (-\infty,\infty]$ at $x \in \mathrm{dom} g$ is defined as the set of all subgradient of g at x:

$$\partial g(x) := \{w \in H : g(y)-g(x) \ge \langle w, y-x\rangle, \ \ \forall y \in H\}.$$

The function g is said to be subdifferentiable at x if $\partial g(x) \neq \emptyset$, g is said to be subdifferentiable on a subset $C \subset H$ if it is subdifferentiable at each point $x \in C$, and it is said to be subdifferentiable if it is subdifferentiable at each point $x \in H$, i.e., if $D(\partial g) = H$. The normal cone of C at $x \in C$ is defined by

$$N_C(x) := \{q \in H : \langle q, y-x\rangle \le 0, \ \forall y \in C\}.$$

We now recall classical concepts of monotonicity for nonlinear operators.

Definition 2.1. (see [29]) An operator $A : C \to H$ is said to be

(1) monotone on C if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \ \forall x, y \in C.$$

(2) pseudomonotone on C if

$$\langle Ax, y - x\rangle \geq 0 \Longrightarrow \langle Ay, x - y\rangle \leq 0, \ \ \forall x, y \in C.$$

(3) strongly pseudomonotone on C with modulus $\gamma > 0$ if there exists $\gamma > 0$ such that, for any $x, y \in C$,

$$\langle Ax, y - x\rangle \geq 0 \Longrightarrow \langle Ay, x - y\rangle \leq -\gamma\|x - y\|^2, \ \ \forall x, y \in C. \tag{2.8}$$

Analogous to Definition 2.1, we have the following concepts for equilibrium problems.

Definition 2.2. (see [30]) The bifunction $f : H \times H \to \mathbb{R} \cup \{+\infty\}$ is said to be

(1) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \ \ \forall x, y \in C.$$

(2) pseudomonotone on C if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \ \ \forall x, y \in C.$$

(3) strongly pseudomonotone on C with modulus $\gamma > 0$ if there exists $\gamma > 0$ such that, for any $x, y \in C$,

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq -\gamma\|x - y\|^2.$$

Remark 2.1. It is obvious that if $A : C \to H$ is monotone (pseudomonotone) on C in the sense of Definition 2.1, then the corresponding bifunction defined by $f(x, y) = \langle Ax, y - x\rangle$ is monotone (pseudomonotone) on C in the sense of Definition 2.2.

Example 2.1. Suppose that $H = L^2([0, 1])$ with inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt, \ \forall x, y \in H$$

and induced norm

$$\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}, \ \forall x \in H.$$

Let us set

$$C = \{x \in H : \|x\| \leq 1\}, \ \ f(x, y) = \left\langle \frac{2x + y}{1 + \|x\|^2}, y - x\right\rangle.$$

We now show that f is strongly pseudomonotone on C. Indeed, let $x,y \in C$ be such that $f(x,y) = \left\langle \frac{2x+y}{1+\|x\|^2}, y-x \right\rangle \geq 0$. This implies that $\langle 2x+y, y-x\rangle \geq 0$. Consequently,

$$\begin{aligned} f(y,x) &= \left\langle \frac{2y+x}{1+\|y\|^2}, x-y \right\rangle \\ &\leq \frac{1}{1+\|y\|^2}(\langle 2y+x, x-y\rangle - \langle 2x+y, x-y\rangle) \\ &\leq -\frac{1}{2}\|x-y\|^2 \\ &= -\gamma\|x-y\|^2, \end{aligned}$$

where $\gamma := \frac{1}{2} > 0$.

On the other hand, f is neither strongly monotone nor monotone on C. To see this, we take $x = \frac{\sqrt{3t}}{2}$, $y = \sqrt{2t}$ and observe that

$$\begin{aligned} f(x,y)+f(y,x) &= \left\langle \frac{2y+x}{1+\|y\|^2} - \frac{2x+y}{1+\|x\|^2}, x-y \right\rangle \\ &= \left\langle \frac{1}{4}(4\sqrt{2t}+\sqrt{3t}) - \frac{8}{11}(\sqrt{3t}+\sqrt{2t}), \frac{\sqrt{3t}}{2} - \sqrt{2t} \right\rangle \\ &= \frac{1}{2}\left[\frac{1}{4}(4\sqrt{2}+\sqrt{3}) - \frac{8}{11}(\sqrt{3}+\sqrt{2})\right]\left(\frac{\sqrt{3}}{2}-\sqrt{2}\right) > 0. \end{aligned}$$

Before concluding this section, we recall the following lemmas, which will be useful for proving the convergence results of this paper.

Lemma 2.2. (Bauschke and Combettes [31, Proposition 26.5]) *Let $g : H \to \mathbb{R}\cup\{+\infty\}$ be proper, lower semicontinuous, and convex,and let C be a nonempty closed and convex subset of H. Assume either that g is continuous at some point of C, or that there is an interior point of C where g is finite. Then, $x^* \in C$ is a solution of the convex optimization problem*

$$\min\{g(x) : x \in C\}$$

if and only if

$$0 \in \partial g(x^*) + N_C(x^*).$$

Lemma 2.3. *(See [32]) Assume that $\{a_k\}$ and $\{b_k\}$ are two sequences of non-negative numbers such that*

$$a_{k+1} \leq a_k + b_k,\ \forall k \in \mathbb{N}.$$

If $\sum_{k=1}^{\infty} b_k < \infty$, then $\lim_{k\to\infty} a_k$ exists.

Lemma 2.4. (Bauschke and Combettes [31, Lemma 2.39]) *Let $\{x^k\}$ be a sequence in a real Hilbert space, and let C be a nonempty subset of H. Suppose that*

(1) for every $z \in S$, $\lim\limits_{k\to\infty} \|x^k - z\|$ exists;

(2) any weak cluster point of $\{x^k\}$ belongs to S.

Then, there exists $\bar{x} \in S$ such that $\{x^k\}$ converges weakly to $\bar{x}$.

3. The Golden Ratio Algorithm for Equilibrium Problems

3.1. **The algorithm.** In what follows, the following usual conditions will be used:

(A1) $f(x,x)=0$ for all $x\in C$;
(A2) f is pseudomonotone on C;
(A3) for any arbitrary sequence $\{z^k\}$ such that $z^k \rightharpoonup z$, if $\limsup_{k\to\infty} f(z^k,y)\geq 0$ for all $y\in C$, then $z\in \mathrm{Sol}(C,f)$;
(A4) $f(x,\cdot)$ is lower semicontinuous convex and subdifferentiable on C for every $x\in C$;
(A5) there exist positive numbers c_1 and c_2 such that the Lipschitz-type condition

$$f(x,y)+f(y,z)\geq f(x,z)-c_1\|x-y\|^2-c_2\|y-z\|^2 \tag{3.1}$$

holds for all $x,y,z\in C$;
(A6) $f(x,.)$ is continuous at some point of C, or there is an interior point of C where $f(x,.)$ is finite for every $x\in C$.
(A7) the solution set $\mathrm{Sol}(C,f)\neq\emptyset$;
(A8) f is strongly pseudomonotone on C with modulus $\gamma>0$.

Remark 3.1. Conditions (A2) and (A5) imply (A1). Indeed, taking $x=y=z\in C$ in (3.1), we obtain $f(x,x)\geq 0$. By (A2), we also have $f(x,x)\leq 0$. So, $f(x,x)=0$ for all $x\in C$.

Remark 3.2. If $f(x,y)=\langle Ax,y-x\rangle$, where $A:C\to H$ is Lipschitz continuous with constant $L>0$, then f satisfies the inequality (3.1) with constants $c_1=c_2=\frac{L}{2}$. Indeed, for each $x,y,z\in C$, we have

$$\begin{aligned}
f(x,y)+f(y,z)-f(x,z) &= \langle Ax,y-x\rangle+\langle Ay,z-y\rangle-\langle Ax,z-x\rangle\\
&= -\langle Ay-Ax,y-z\rangle\\
&\geq -\|Ax-Ay\|\|y-z\|\\
&\geq -L\|x-y\|\|y-z\|\\
&\geq -\frac{L}{2}\|x-y\|^2-\frac{L}{2}\|y-z\|^2\\
&= -c_1\|x-y\|^2-c_2\|y-z\|^2.
\end{aligned}$$

Thus f satisfies the inequality (3.1).

We now provide a concrete bifunction f satisfying (A5), (A7), and (A9).

Example 3.1. (see also [9, 33]) Let $H=L^2([0,1])$ and C be as in Example 2.1.
Let us take

$$f(x,y)=\big\langle \big(4-\|x\|\big)x,y-x\big\rangle+\|y\|-\|x\|.$$

(1) We will show that f is strongly pseudomonotone on C. Indeed, assume that $x,y\in C$ are such that

$$f(x,y)=\big\langle \big(4-\|x\|\big)x,y-x\big\rangle+\|y\|-\|x\|\geq 0.$$

Note that

$$\begin{aligned} f(y,x) &= \big\langle \big(4-\|y\|\big)y, x-y\big\rangle + \|x\| - \|y\| \\ &\le \big(4-\|y\|\big)\big(\langle y, x-y\rangle - \langle x, x-y\rangle\big) + \big\langle \big(4-\|x\|\big)x, y-x\big\rangle \\ &= -4\|x-y\|^2 + \langle -y\|y\| + \|x\|x, x-y\rangle \\ &= -4\|x-y\|^2 + \langle (x-y)\|y\| + (\|x\|-\|y\|)x, x-y\rangle \\ &\le -4\|x-y\|^2 + \big(\|x-y\|\|y\| + \big|\|x\|-\|y\|\big|\|x\|\big)\|x-y\| \\ &\le -2\|x-y\|^2. \end{aligned}$$

So f is strongly pseudomonotone on C, where $\gamma = 2$.

(2) We now prove that f satisfies the condition (A7). Firstly, we need to show that $A : C \to H$ defined by $A(x) = \big(4-\|x\|\big)x$ is Lipschitz continuous on C. We have

$$\begin{aligned} \|Ax - Ay\| &= \|(4-\|x\|)x - (4-\|y\|)y\| \\ &= \|4(x-y) - \|x\|(x-y) - (\|x\|-\|y\|)y\| \\ &\le 4\|x-y\| + \|x\|\|x-y\| + \|x-y\|\|y\| \\ &\le 6\|x-y\|. \end{aligned}$$

Next, for all bounded sequences $\{x^k\}, \{y^k\} \subset C$ such that $\|x^k - y^k\| \to 0$, we have

$$\begin{aligned} f(x^k,y^k) = \langle Ax^k, y^k - x^k\rangle + \|y^k\| - \|x^k\| &\ge -\|Ax^k\|\|x^k - y^k\| - \|x^k - y^k\| \\ &\ge -(1+M_0)\|x^k - y^k\|, \end{aligned}$$

where M_0 is a positive constant such that $\|Ax^k\| \le M_0$. Hence, we infer that

$$\limsup_{k\to\infty} f(x^k,y^k) \ge -\lim_{k\to\infty}(1+M_0)\|x^k - y^k\| = 0.$$

(3) It remains to prove that f satisfies Lipschitz-type condition (A5). Indeed, for all $x,y,z \in C$, we have

$$\begin{aligned} f(x,y) + f(y,z) &= f(x,z) + \big\langle \big(4-\|x\|\big)x - \big(4-\|y\|\big)y, y-z\big\rangle \\ &= f(x,z) + 4\langle x-y, y-z\rangle - \langle \|x\|x - y\|y\|, y-z\rangle \\ &= f(x,z) + 2[\|x-z\|^2 - \|x-y\|^2 - \|y-z\|^2] - \langle \|x\|x - y\|y\|, y-z\rangle. \end{aligned} \tag{3.2}$$

On the other hand, we have

$$\begin{aligned} \langle \|x\|x - y\|y\|, y-z\rangle &= \langle \|x\|(x-y) + y(\|x\|-\|y\|), y-z\rangle \\ &\le 2\big\|x-y\|\big\|\|y-z\big\| \\ &\le \|x-y\|^2 + \|y-z\|^2. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$f(x,y) + f(y,z) \ge f(x,z) - 3\|x-y\|^2 - 3\|y-z\|^2.$$

Hence, the Lipschitz type inequality (A5) is satisfied with $c_1 = c_2 = 3$.

Remark 3.3. It is easy to see that if $f(.,y)$ is weakly upper semicontinuous for all $y \in C$, then f satisfies the condition (A3), which was first introduced by Khatibzadeh and Mohebbi in [34]. However, the converse is not true in general. To see this, we consider the following counterexample in [11, 34].

Example 3.2. Let $H = \ell^2$, $C = \{\xi = (\xi_1, \xi_2, \dots) \in \ell^2 : \xi_i \geq 0 \ \ \forall i = 1,2,\dots\}$, and

$$f(x,y) = (y_1 - x_1)\sum_{i=1}^{\infty}(x_i)^2.$$

Taking $x^k = (0,\dots,0,\underset{k}{1},0,\dots)$, we have $x^k \rightharpoonup x = (0,\dots,0,\dots)$,and $x \in \mathrm{Sol}(C,f)$. Obviously, there is a $y \in C$ such that

$$\limsup_{k\to\infty} f(x^k,y) > 0 = f(x,y).$$

Then $f(.,y)$ is not weakly upper semicontinuous. We now show that f satisfies condition (A3). If $z^k = (z_1^k, z_2^k, \dots) \rightharpoonup z = (z_1, z_2, \dots)$ is an arbitrary sequence, and $\limsup_{k\to\infty} f(z^k,y) \geq 0$ for all $y \in C$, then

$$\limsup_{k\to\infty}(y_1 - z_1^k)\sum_{i=1}^{\infty}(z_i^k)^2 \geq 0.$$

Since $\lim_{k\to\infty}(y_1 - z_1^k) = y_1 - z_1$, we get

$$(y_1 - z_1)\limsup_{k\to\infty}\sum_{i=1}^{\infty}(z_i^k)^2 \geq 0,$$

thus $y_1 \geq z_1$. Hence, $f(z,y) \geq 0$ for all $y \in C$, i.e., f satisfies condition (A3).

From the above observations, it is clear that our conditions (A3) and (A7) are weaker than the conditions (A4) and (A6) in [25], respectively.

We are now in a position to describe a new algorithm for pseudomonotone equilibrium problems.

Algorithm 3.1 (Golden ratio algorithm for equilibrium problems)

Initialization: Let $\varphi = \frac{\sqrt{5}+1}{2}$ be the golden ratio, i.e., $\varphi^2 = \varphi + 1$. Choose the parameter λ such that

$$0 < \lambda \leq \min\left\{\frac{\varphi}{4c_1}, \frac{\varphi}{4c_2}\right\}. \tag{3.4}$$

Select initial $x^0 \in C$, $y^1 \in C$.

Iterative Step: Given x^{k-1} and y^k $(k \geq 1)$, compute

$$x^k = \frac{(\varphi-1)y^k + x^{k-1}}{\varphi}, \tag{3.5}$$

$$y^{k+1} = \operatorname{argmin}\left\{\lambda f(y^k,y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}. \tag{3.6}$$

Stopping Criterion: If $y^{k+1} = y^k = x^k$ then stop. Otherwise, let $k := k+1$ and return to **Iterative Step**.

Remark 3.4. For comparison with algorithms (2.2), (2.3) and (2.5), Algorithm 3.1 requires, at each iteration, only one strongly convex optimization problem.

3.2. **Convergence analysis.** We first wish to validate the stoping criterion of Algorithm 3.1.

Lemma 3.1. *Under the conditions (A1), (A4), and (A6), if $y^{k+1} = y^k = x^k$, then $y^k \in \mathrm{Sol}(C, f)$.*

Proof. If $y^{k+1} = y^k = x^k$, then

$$y^k = \operatorname*{argmin}_{y \in C} \left\{ \lambda f(y^k, y) + \frac{1}{2}\|y - y^k\|^2 \right\}.$$

Therefore, from Lemma 2.2, we have

$$0 \in \partial \left[\lambda f(y^k, .) + \frac{1}{2}\|. - y^k\| \right](y^k) + N_C(y^k),$$

i.e., $0 \in \partial(\lambda f(y^k, .))(y^k) + N_C(y^k)$, which implies that

$$\langle u^k, x - y^k \rangle \geq 0, \ \ \forall x \in C,$$

where $u^k \in \partial(f(y^k, .))(y^k)$. By the assumption (A1), we get

$$f(y^k, x) = f(y^k, x) - f(y^k, y^k) \geq \langle u^k, x - y^k \rangle \geq 0, \ \ \forall x \in C.$$

This means that $y^k \in \mathrm{Sol}(C, f)$. □

The next statement plays a crucial role in the proof of the convergence result.

Lemma 3.2. *Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by Algorithm 3.1 and $z \in C$. Under the conditions (A4), (A5), and (A6), the following inequality holds*

$$(1+\varphi)\|x^{k+1} - z\|^2 + \frac{\varphi}{2}\|y^k - y^{k+1}\|^2 \leq (1+\varphi)\|x^k - z\|^2 + \frac{\varphi}{2}\|y^{k-1} - y^k\|^2 - \varphi\|x^k - y^k\|^2 + 2\lambda f(y^k, z).$$

Proof. From $y^{k+1} = \operatorname*{argmin}_{y \in C} \left\{ \lambda f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 \right\}$, and Lemma 2.2, we have

$$0 = \lambda g^k + y^{k+1} - x^k + q,$$

where $g^k \in \partial f(y^k, .)(y^{k+1})$ and $q \in N_C(y^{k+1})$. Since

$$N_C(y^{k+1}) = \{q \in H : \langle q, y - y^{k+1} \rangle \leq 0, \ \forall y \in C\},$$

we have

$$\langle y^{k+1} - x^k + \lambda g^k, z - y^{k+1} \rangle \geq 0.$$

Consequently,

$$\langle x^k - y^{k+1}, z - y^{k+1} \rangle \leq \lambda \langle g^k, z - y^{k+1} \rangle \leq \lambda (f(y^k, z) - f(y^k, y^{k+1})), \tag{3.7}$$

and

$$\langle x^{k-1} - y^k, y^{k+1} - y^k \rangle \leq \lambda (f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k)). \tag{3.8}$$

Combining (3.8) with the fact that

$$y^k - x^{k-1} = \frac{1+\varphi}{\varphi}(y^k - x^k) = \varphi(y^k - x^k), \tag{3.9}$$

we obtain

$$\langle \varphi(x^k - y^k), y^{k+1} - y^k \rangle \le \lambda \left(f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k) \right). \tag{3.10}$$

Summing up (3.7) and (3.10), we get

$$\begin{aligned}\langle x^k - y^{k+1}, z - y^{k+1} \rangle + \langle \varphi(x^k - y^k), y^{k+1} - y^k \rangle \le \lambda \big[f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k) \\ - f(y^k, y^{k+1}) \big] + \lambda f(y^k, z).\end{aligned} \tag{3.11}$$

Using the identity

$$\langle a, b \rangle = \frac{1}{2} \left[\|a\|^2 + \|b\|^2 - \|a - b\|^2 \right],$$

we have from (3.11) that

$$\begin{aligned}\|y^{k+1} - z\|^2 \le \|x^k - z\|^2 - \|x^k - y^{k+1}\|^2 - \varphi \big[\|x^k - y^k\|^2 + \|y^{k+1} - y^k\|^2 \\ - \|y^{k+1} - x^k\|^2 \big] + 2\lambda \left[f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k) - f(y^k, y^{k+1}) \right] \\ + 2\lambda f(y^k, z).\end{aligned} \tag{3.12}$$

By assumption (A5), we get

$$\begin{aligned}2\lambda [f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k) - f(y^k, y^{k+1})] &\le 2\lambda [c_1 \|y^{k-1} - y^k\|^2 + c_2 \|y^k - y^{k+1}\|^2] \\ &\le \frac{\varphi}{2} \left[\|y^{k-1} - y^k\|^2 + \|y^k - y^{k+1}\|^2 \right].\end{aligned} \tag{3.13}$$

On the other hand, it can be easily seen from (3.9) that $y^{k+1} = (1+\varphi)x^{k+1} - \varphi x^k$. Hence, we have from (2.6) that

$$\begin{aligned}\|y^{k+1} - z\|^2 &= (1+\varphi)\|x^{k+1} - z\|^2 - \varphi \|x^k - z\|^2 + \varphi(1+\varphi)\|x^{k+1} - x^k\|^2 \\ &= (1+\varphi)\|x^{k+1} - z\|^2 - \varphi \|x^k - z\|^2 + \frac{1}{\varphi}\|y^{k+1} - x^k\|^2.\end{aligned} \tag{3.14}$$

It follows from (3.12), (3.13) and (3.14) that

$$\begin{aligned}(1+\varphi)\|x^{k+1} - z\|^2 \le (1+\varphi)\|x^k - z\|^2 - \varphi \left[\|x^k - y^k\|^2 + \|y^{k+1} - y^k\|^2 \right] \\ + \frac{\varphi}{2} \left[\|y^{k-1} - y^k\|^2 + \|y^k - y^{k+1}\|^2 \right] + 2\lambda f(y^k, z)\end{aligned}$$

or equivalently,

$$\begin{aligned}(1+\varphi)\|x^{k+1} - z\|^2 + \frac{\varphi}{2}\|y^k - y^{k+1}\|^2 \le (1+\varphi)\|x^k - z\|^2 + \frac{\varphi}{2}\|y^{k-1} - y^k\|^2 \\ - \varphi \|x^k - y^k\|^2 + 2\lambda f(y^k, z).\end{aligned} \tag{3.15}$$

The proof is complete. □

At this point, we can prove the following weak convergence theorem.

Theorem 3.1. *Let $f : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying assumptions (A2)-(A8). Then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges weakly to an solution of the EP (2.1).*

Proof. We split the proof into several steps:

Step 1. We show the boundedness of sequence $\{x^k\}$. Let $z \in \mathrm{Sol}(C,f)$. It follows from the pseudomonotonicity of f that $f(y^k,z) \le 0$. Then, the inequality (3.15) of Lemma 3.2 implies

$$(1+\varphi)\|x^{k+1}-z\|^2+\frac{\varphi}{2}\|y^k-y^{k+1}\|^2 \le (1+\varphi)\|x^k-z\|^2+\frac{\varphi}{2}\|y^{k-1}-y^k\|^2 - \varphi\|x^k-y^k\|^2. \quad (3.16)$$

From this we infer that $\left\{(1+\varphi)\|x^k-z\|^2+\frac{\varphi}{2}\|y^{k-1}-y^k\|^2\right\}$ is convergent. Therefore, the sequence $\{\|x^k-z\|\}$ is bounded, so is $\{x^k\}$. Moreover,

$$\lim_{k\to\infty}\|x^k-y^k\| = 0. \quad (3.17)$$

From (3.9), we have

$$\lim_{k\to\infty}\|y^{k+1}-x^k\| = 0. \quad (3.18)$$

This together with (3.17) implies that

$$\lim_{k\to\infty}\|y^{k+1}-y^k\| = 0. \quad (3.19)$$

Step 2. Let us show that any weakly cluster point of the sequence $\{x^k\}$ belongs to the solution set $\mathrm{Sol}(C,f)$.

Indeed, let $\bar{x}$ be an arbitrary weakly cluster point of $\{x^k\}$. Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_l}\}$ of $\{x^k\}$ such that $x^{k_l} \rightharpoonup \bar{x}$. From (3.17), we have $y^{k_l} \rightharpoonup \bar{x} \in C$. It follows from

$$y^{k+1} = \operatorname*{argmin}_{y\in C}\left\{\lambda f(y^k,y)+\frac{1}{2}\|y-x^k\|^2\right\},$$

and Lemma 2.2 that there exist $w^{k+1} \in \partial f(y^k,.)(y^{k+1})$, and $q^{k+1} \in N_C(y^{k+1})$ such that

$$0 = \lambda w^{k+1}+y^{k+1}-x^k+q^{k+1}.$$

From the definition of $N_C(y^{k+1})$, we deduce that

$$\langle x^k-y^{k+1}-\lambda w^{k+1}, y-y^{k+1}\rangle \le 0, \quad \forall y \in C,$$

or

$$\langle x^k-y^{k+1}, y-y^{k+1}\rangle \le \langle \lambda w^{k+1}, y-y^{k+1}\rangle, \quad \forall y \in C.$$

On the other hand, since $w^{k+1} \in \partial f(y^k,.)(y^{k+1})$, we get

$$\langle w^{k+1}, y-y^{k+1}\rangle \le f(y^k,y)-f(y^k,y^{k+1}), \ \forall y \in C.$$

Hence, we arrive at

$$\langle x^k-y^{k+1}, y-y^{k+1}\rangle \le \lambda f(y^k,y)-\lambda f(y^k,y^{k+1}), \quad \forall y \in C. \quad (3.20)$$

Replacing z in (3.15) by y^{k+1} yields that

$$\lambda f(y^k,y^{k+1}) \ge \frac{1}{2}\Big[(1+\varphi)\|x^{k+1}-y^{k+1}\|^2+\frac{\varphi}{2}\|y^k-y^{k+1}\|^2-(1+\varphi)\|x^k-y^{k+1}\|^2 - \frac{\varphi}{2}\|y^{k-1}-y^k\|^2+\varphi\|x^k-y^k\|^2\Big]. \quad (3.21)$$

From (3.20) and (3.21) we get

$$\lambda f(y^k,y) \geq \langle x^k - y^{k+1}, y - y^{k+1}\rangle + \frac{1}{2}\Big[(1+\varphi)\|x^{k+1} - y^{k+1}\|^2 + \frac{\varphi}{2}\|y^k - y^{k+1}\|^2 - (1+\varphi)\|x^k - y^{k+1}\|^2 - \frac{\varphi}{2}\|y^{k-1} - y^k\|^2 + \varphi\|x^k - y^k\|^2\Big], \quad \forall y \in C. \tag{3.22}$$

By (3.17), (3.18), and (3.19), we have that the right-hand side of (3.22) converges to zero as $k \to \infty$. Therefore, replacing k in (3.22) by k_l and passing to the limit, we get

$$\limsup_{l\to\infty} f(y^{k_l}, y) \geq 0, \quad \forall y \in C.$$

Now under condition (A3), we obtain, $\bar{x} \in \mathrm{Sol}(C,f)$.

Step 3. We claim that $x^k \rightharpoonup \bar{x}$. Since $\bar{x}$ is an arbitrary weakly cluster point, we can conclude that the set of all weakly cluster points belongs to the solution set $\mathrm{Sol}(C,f)$. Taking into account the convergence of the sequence $\big\{(1+\varphi)\|x^k - z\|^2 + \frac{\varphi}{2}\|y^{k-1} - y^k\|^2\big\}$, and (3.18), we deduce that the sequence $\big\{\|x^k - z\|\big\}$ is convergent. Hence, it follows from Lemma 2.4 that $\{x^k\}$ weakly converges to a solution of equilibrium problem (2.1). This completes the proof. □

Remark 3.5. Theorem 3.1 extends, improves, supplements, and develops the results of [12, 18, 24, 25] in the following aspects:

(1) in comparison with [12, 18, 24, 25], Algorithm 3.1 has the advantage that our method consists of one strongly convex programming problem instead of two or three ones as the methods of [12, 24, 25];
(2) the continuity imposed on f is relaxed;
(3) the sequence $\{x^k\}$ generated by Algorithm 3.1 is not Fejér monotone. Therefore, our proof techniques are different from those in [12, 18, 24].

4. Strong Convergence of the Golden Ratio Algorithm

We will use the strong pseudomonotonicity of bifunction f to establish the strong convergence of the gold ratio algorithm.

4.1. **An algorithm without knowledge of Lipschitz-type constants.** In general, the Lipschitz-type condition (3.1) is not satisfied, and even if f satisfies (3.1), then finding the constants c_1 and c_2 is not easy. To overcome this drawback, we propose the following algorithm

Algorithm 4.1 (Golden ratio algorithm without knowledge of Lipschitz-type constants)

Initialization: Let $\varphi = \frac{\sqrt{5}+1}{2}$. Take a positive and nonincreasing sequence $\{\lambda_k\}$ satisfying

$$\lim_{k\to\infty} \lambda_k = 0, \quad \lim_{k\to\infty} \frac{\lambda_k}{\lambda_{k-1}} = 1, \quad \sum_{k=0}^{\infty} \lambda_k = +\infty. \tag{4.1}$$

Select initial $x^0 \in C$, $y^1 \in C$.

Iterative Step: Given x^{k-1} and y^k ($k \geq 1$), compute

$$x^k = \frac{(\varphi - 1)y^k + x^{k-1}}{\varphi},$$
$$y^{k+1} = \operatorname{argmin}\Big\{\lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\Big\}.$$

Stopping Criterion: If $y^{k+1} = y^k = x^k$ then stop. Otherwise, let $k := k+1$ and return to **Iterative Step**.

We now state and prove the following strong convergence result for Algorithm 4.1.

Theorem 4.1. *Under the assumptions (A4)-(A6), (A8), and (A9), the sequence $\{x^k\}$ generated by Algorithm 4.1 converges strongly to the unique solution of the EP (2.1).*

Proof. Arguing as in the proof of Lemma 3.2, we replace λ in (3.7) by λ_k to obtain that

$$\langle x^k - y^{k+1}, z - y^{k+1}\rangle \leq \lambda_k\langle g^k, z - y^{k+1}\rangle \leq \lambda_k(f(y^k, z) - f(y^k, y^{k+1})). \tag{4.2}$$

Hence

$$\langle x^{k-1} - y^k, y^{k+1} - y^k\rangle \leq \lambda_{k-1}(f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k)), \tag{4.3}$$

which is equivalent to

$$\frac{\lambda_k}{\lambda_{k-1}}\langle x^{k-1} - y^k, y^{k+1} - y^k\rangle \leq \lambda_k(f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k)). \tag{4.4}$$

Summing up (4.2) and (4.4), we find from (3.9) that

$$\begin{aligned}\langle x^k - y^{k+1}, z - y^{k+1}\rangle + \frac{\lambda_k}{\lambda_{k-1}}\langle \varphi(x^k - y^k), y^{k+1} - y^k\rangle \leq \lambda_k\big[f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k)\\ - f(y^k, y^{k+1})\big] + \lambda_k f(y^k, z).\end{aligned} \tag{4.5}$$

By the same way as (3.12), we obtain

$$\begin{aligned}\|y^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \|x^k - y^{k+1}\|^2 - \varphi\frac{\lambda_k}{\lambda_{k-1}}\big[\|x^k - y^k\|^2 + \|y^{k+1} - y^k\|^2\\ - \|y^{k+1} - x^k\|^2\big] + 2\lambda_k\big[f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k) - f(y^k, y^{k+1})\big]\\ + 2\lambda_k f(y^k, z).\end{aligned} \tag{4.6}$$

By the assumption (A5), we get

$$\begin{aligned}2\lambda_k[f(y^{k-1}, y^{k+1}) - f(y^{k-1}, y^k) - f(y^k, y^{k+1})] \leq 2\lambda_k[c_1\|y^{k-1} - y^k\|^2\\ + c_2\|y^k - y^{k+1}\|^2].\end{aligned} \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$\|y^{k+1}-z\|^2 \le \|x^k-z\|^2 - \|x^k-y^{k+1}\|^2 - \varphi\frac{\lambda_k}{\lambda_{k-1}}\left[\|x^k-y^k\|^2+\|y^{k+1}-y^k\|^2\right] + \varphi\|y^{k+1}-x^k\|^2 + 2\lambda_k[c_1\|y^{k-1}-y^k\|^2+c_2\|y^k-y^{k+1}\|^2] + 2\lambda_k f(y^k,z). \tag{4.8}$$

Using (3.14), we obtain

$$(1+\varphi)\|x^{k+1}-z\|^2 \le (1+\varphi)\|x^k-z\|^2 - \varphi\frac{\lambda_k}{\lambda_{k-1}}\left[\|x^k-y^k\|^2+\|y^{k+1}-y^k\|^2\right] + 2\lambda_k[c_1\|y^{k-1}-y^k\|^2+c_2\|y^k-y^{k+1}\|^2] + 2\lambda_k f(y^k,z).$$

Consequently,

$$(1+\varphi)\|x^{k+1}-z\|^2 + \frac{\varphi}{2}\|y^k-y^{k+1}\|^2 \le (1+\varphi)\|x^k-z\|^2 + \frac{\varphi}{2}\|y^{k-1}-y^k\|^2 - \Big(\frac{\varphi}{2}-2\lambda_k c_1\Big)\|y^{k-1}-y^k\|^2 - \Big(\varphi\frac{\lambda_k}{\lambda_{k-1}}-\frac{\varphi}{2}-2\lambda_k c_1\Big)\|y^k-y^{k+1}\|^2 - \varphi\frac{\lambda_k}{\lambda_{k-1}}\|x^k-y^k\|^2 - 2\gamma\lambda_k\|y^k-z\|^2, \tag{4.9}$$

where the last inequality is obtained from the strong pseudomonotonicity of f. Since $\lim_{k\to\infty}\lambda_k = 0$, and $\lim_{k\to\infty}\frac{\lambda_k}{\lambda_{k-1}} = 1$, there exists k_0 such that $\frac{\varphi}{2}-2\lambda_k c_1 > \frac{\varphi}{4}$, and $\varphi\frac{\lambda_k}{\lambda_{k-1}}-\frac{\varphi}{2}-2\lambda_k c_2 > \frac{\varphi}{4}$ for all $k\ge k_0$. This together with (4.9) implies that the sequence $\big\{(1+\varphi)\|x^k-z\|^2+\frac{\varphi}{2}\|y^{k-1}-y^k\|^2\big\}$ is convergent and

$$\lim_{k\to\infty}\|y^k-y^{k+1}\| = \lim_{k\to\infty}\|x^k-y^k\| = 0. \tag{4.10}$$

Therefore,

$$\lim_{k\to\infty}\|x^k-z\|^2 \in \mathbb{R}. \tag{4.11}$$

On the other hand, we have from (4.9) that

$$2\gamma\lambda_k\|y^k-z\|^2 \le \sigma_k - \sigma_{k+1},\ \forall k\ge k_0, \tag{4.12}$$

where

$$\sigma_k = (1+\varphi)\|x^k-z\|^2 + \frac{\varphi}{2}\|y^{k-1}-y^k\|^2.$$

We fix a number $N\in\mathbb{N}$ and consider the inequality (4.13) for all the numbers $k_0,\ldots,N$. Adding these inequalities, we obtain

$$2\gamma\sum_{k=k_0}^{N}\lambda_k\|y^k-z\|^2 \le \sigma_{k_0}-\sigma_{N+1} \le \sigma_{k_0}, \tag{4.13}$$

which implies

$$\sum_{k=0}^{\infty}\lambda_k\|y^k-z\|^2 < +\infty.$$

Hence, it follows from (4.1) that

$$\liminf_{k\to\infty}\|y^k-z\| = 0. \tag{4.14}$$

Combining (4.10) and (4.14), we get

$$\liminf_{k\to\infty} \|x^k - z\| = 0. \tag{4.15}$$

Finally, by (4.11) and (4.15), we conclude that $\lim_{k\to\infty} \|x^k - z\| = 0$. The proof is complete. □

4.2. **An algorithm without Lipschitz-type condition.** To avoid Lipschitz-type condition (3.1), we introduce the following self-adaptive algorithm

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Algorithm 4.2 (Golden ratio algorithm without Lipschitz-type condition)

Initialization: Let $\varphi = \frac{\sqrt{5}+1}{2}$. Take a positive sequence $\{\beta_k\}$ satisfying

$$\sum_{k=0}^{\infty} \beta_k = +\infty, \quad \sum_{k=0}^{\infty} \beta_k^2 < +\infty. \tag{4.16}$$

Select initial $x^0 \in C$, $y^1 \in C$.
Iterative Step: Given x^{k-1} and y^k $(k \geq 1)$, compute

$$x^k = \frac{(\varphi - 1)y^k + x^{k-1}}{\varphi}. \tag{4.17}$$

Take $g(y^k) \in \partial(f(y^k,.))(y^k)$ $(k \geq 1)$. Calculate

$$\eta_k = \max\{1, \|g(y^k)\|\}, \quad \lambda_k = \frac{\beta_k}{\eta_k} \tag{4.18}$$

and

$$y^{k+1} = P_C(x^k - \lambda_k g(y^k)). \tag{4.19}$$

Stopping Criterion: If $y^{k+1} = y^k = x^k$ then stop. Otherwise, let $k := k+1$ and return to **Iterative Step**.

The following lemma is quite helpful to analyze the convergence of Algorithm 4.2.

Lemma 4.1. *If $y^{k+1} = y^k = x^k$, then $y^k \in \mathrm{Sol}(C, f)$.*

Proof. If $y^{k+1} = y^k = x^k$, then we obtain from (4.19) and Lemma 2.1 (1) that

$$\langle y^k - \lambda_k g(y^k) - y^k, y - y^k \rangle \leq 0 \ \ \forall y \in C,$$

or equivalently,

$$\langle g(y^k), y - y^k \rangle \geq 0 \ \ \forall y \in C. \tag{4.20}$$

Therefore, from (4.20) and the definition of $\partial(f(y^k,.))(y^k)$, we get

$$f(y^k, y) = f(y^k, y) - f(y^k, y^k) \geq \langle g(y^k), y - y^k \rangle \geq 0, \ \ \forall y \in C.$$

Hence, $y^k \in \mathrm{Sol}(C, f)$. □

To establish the strong convergence of Algorithm 4.2, we will use the following requirement:

(A10) If $\{x^k\} \subset C$ is bounded, then the sequence $\{g^k\}$ with $g^k \in \partial(f(x^k,.))(x^k)$ is bounded.

We are now in a position to establish the strong convergence of the sequence generated by Algorithm 4.2.

Theorem 4.2. *Under the assumptions (A1), (A4), (A8), (A9) and (A10), the sequence $\{x^k\}$ generated by Algorithm 4.2 converges strongly to the unique solution of EP (2.1).*

Proof. Write $w^k = x^k - \lambda_k g(y^k)$ and let $z \in \mathrm{Sol}(C, f)$. Using Lemma 2.1 (2), we know that

$$\begin{aligned}\|y^{k+1} - z\|^2 &\le \|w^k - z\|^2 - \|w^k - y^{k+1}\|^2 \\ &= \|x^k - \lambda_k g(y^k) - z\|^2 - \|x^k - \lambda_k g(y^k) - y^{k+1}\|^2 \\ &= \|x^k - z\|^2 - \|x^k - y^{k+1}\|^2 + 2\lambda_k \langle z - y^{k+1}, g(y^k)\rangle \\ &= \|x^k - z\|^2 - \|x^k - y^{k+1}\|^2 + 2\lambda_k \langle g(y^k), y^k - y^{k+1}\rangle \\ &\qquad - 2\lambda_k \langle g(y^k), y^k - z\rangle. \end{aligned} \tag{4.21}$$

From (4.19) and $y^{k+1} \in C$, we have

$$\begin{aligned} &\langle y^k - x^{k-1} + \lambda_{k-1} g(y^{k-1}), y^{k+1} - y^k\rangle \ge 0 \\ \Longleftrightarrow\ &\langle \varphi(y^k - x^k) + \lambda_{k-1} g(y^{k-1}), y^{k+1} - y^k\rangle \ge 0. \end{aligned} \tag{4.22}$$

It follows from (4.21) and (4.22) that

$$\begin{aligned}\|y^{k+1} - z\|^2 \le\ &\|x^k - z\|^2 - \|x^k - y^{k+1}\|^2 + 2\langle \varphi(y^k - x^k), y^{k+1} - y^k\rangle \\ &+ 2\langle \lambda_k g(y^k) - \lambda_{k-1} g(y^{k-1}), y^k - y^{k+1}\rangle - 2\lambda_k \langle g(y^k), y^k - z\rangle. \end{aligned} \tag{4.23}$$

Moreover,

$$\langle \varphi(y^k - x^k), y^{k+1} - y^k\rangle = \frac{\varphi}{2}\Big[\|y^{k+1} - x^k\|^2 - \|x^k - y^k\|^2 - \|y^{k+1} - y^k\|^2\Big].$$

This equality, together with (3.14) and (4.23), yields

$$\begin{aligned}(1+\varphi)\|x^{k+1} - z\|^2 \le\ &(1+\varphi)\|x^k - z\|^2 - \varphi\left[\|x^k - y^k\|^2 + \|y^{k+1} - y^k\|^2\right] \\ &+ 2\langle \lambda_k g(y^k) - \lambda_{k-1} g(y^{k-1}), y^k - y^{k+1}\rangle - 2\lambda_k \langle g(y^k), y^k - z\rangle. \end{aligned} \tag{4.24}$$

Using the definition of the diagonal subdifferential, and the fact that f is strongly pseudomonotone on C, we have

$$\langle g(y^k), z - y^k\rangle \le f(y^k, z) \le -\gamma \|y^k - z\|^2. \tag{4.25}$$

Setting $LS := \langle \lambda_k g(y^k) - \lambda_{k-1} g(y^{k-1}), y^k - y^{k+1}\rangle$, we find that

$$\begin{aligned} LS &\le \left\| \frac{\beta_k}{\eta_k} g(y^k) - \frac{\beta_{k-1}}{\eta_{k-1}} g(y^{k-1}) \right\| \|y^k - y^{k+1}\| \\ &\le (\beta_k + \beta_{k-1})\|y^{k+1} - y^k\| \\ &\le \frac{1}{2}\Big((\beta_k + \beta_{k-1})^2 + \|y^{k+1} - y^k\|^2\Big) \\ &\le \beta_k^2 + \beta_{k-1}^2 + \frac{1}{2}\|y^{k+1} - y^k\|^2. \end{aligned} \tag{4.26}$$

In virtue of (4.24), and (4.26), we obtain

$$\begin{aligned}(1+\varphi)\|x^{k+1} - z\|^2 \le\ &(1+\varphi)\|x^k - z\|^2 - \varphi\|x^k - y^k\|^2 - (\varphi - 1)\|y^{k+1} - y^k\|^2] \\ &- \gamma\lambda_k \|y^k - z\|^2 + 2\beta_k^2 + 2\beta_{k-1}^2. \end{aligned} \tag{4.27}$$

This yields

$$a_{k+1} \le a_k + b_k, \tag{4.28}$$

where $a_k = (1+\varphi)\|x^k - z\|^2$, $b_k = 2\beta_k^2 + 2\beta_{k-1}^2$. The use of Lemma 2.3 leads to the convergence of the sequence $\{\|x^k - z\|^2\}$. Hence, $\{x^k\}$ is bounded. From (4.16) and (4.27), we get immediately

$$\lim_{k\to\infty} \|y^k - y^{k+1}\| = \lim_{k\to\infty} \|x^k - y^k\| = 0. \tag{4.29}$$

Therefore, $\{y^k\}$ is also bounded. Using (A10), we infer that there exists $M_1 > 0$ such that $\|g(y^k)\| \le M_1$ for all $k \in \mathbb{N}$. Setting $M_2 := \max\{1, M_1\}$, we have from (4.18) that

$$\beta_k \ge \lambda_k = \frac{\beta_k}{\eta_k} \ge \frac{1}{M_2}\beta_k, \quad \forall k \in \mathbb{N},$$

which together with (4.16) yields

$$\sum_{k=0}^{\infty} \lambda_k = +\infty; \quad \lim_{k\to\infty} \lambda_k = 0. \tag{4.30}$$

The rest of the proof is similar to the proof of Theorem 4.1, so we omit the details here. The proof is complete. □

5. Application to the Variational Inequality Problem

If the equilibrium bifunction f is defined by $f(x,y) = \langle Ax, y - x\rangle$ for every $x, y \in C$, with $A : C \to H$, then the equilibrium problem (2.1) reduces to the *variational inequality problem* (VIP):

$$\text{find } x^* \in C \text{ such that } \langle Ax^*, y - x^*\rangle \ge 0, \quad \forall y \in C. \tag{5.1}$$

The set of solutions of the problem (5.1) is denoted by $\mathrm{Sol}(C,A)$. In this situation, Algorithm 3.1 reduces to the golden ratio algorithm for variational inequalities, which was recently considered by Malitsky [26].

Algorithm 5.1 (Golden ratio algorithm for variational inequalities)

Initialization: Let $\varphi = \frac{\sqrt{5}+1}{2}$ and $\lambda > 0$.
Select initial $x^0 \in C$, $y^1 \in C$.

Iterative Step: Given x^{k-1} and y^k ($k \ge 1$), compute

$$x^k = \frac{(\varphi - 1)y^k + x^{k-1}}{\varphi} \tag{5.2}$$

and

$$y^{k+1} = P_C(x^k - \lambda A y^k). \tag{5.3}$$

Stopping Criterion: If $y^{k+1} = y^k = x^k$ then stop. Otherwise, let $k := k+1$ and return to **Iterative Step**.

Remark 5.1. Algorithm 5.1 requires, at each iteration, only one projection onto the feasible set C.

We now remind the following concept for single-valued operators.

Definition 5.1. ([35]) Let X be a normed space with X^* its dual space, and let K be a closed convex subset of X. The mapping $A : K \to X^*$ is called *F-hemicontinuous* iff, for all $y \in K$, the function $x \mapsto \langle A(x), x-y\rangle$ is weakly lower semicontinuous on K (or equivalently, $x \mapsto \langle A(x), y-x\rangle$ is weakly upper semicontinuous on K).

Clearly, any weak-to-strong continuous mapping is also F-hemicontinuous, but vice-versa not, as the following example shows.

Example 5.1. ([36]) Consider the Hilbert space $\ell^2 = \{x = (x^i)_{i\in\mathbf{N}} : \sum_{i=1}^{\infty} |x^i|^2 < \infty\}$, and let A: $\ell^2 \to \ell^2$ be the identity operator. Take an arbitrary sequence $\{x_n\} \subseteq \ell^2$ converging weakly to $\bar{x}$. Since the function $x \longmapsto \|x\|^2$ is continuous and convex, it is weakly lower semicontinuous. Hence,

$$\|\bar{x}\|^2 \le \liminf_{n\to\infty} \|x_n\|^2,$$

which clearly implies

$$\langle \bar{x}, \bar{x}-y\rangle \le \liminf_{n\to\infty} \langle x_n, x_n - y\rangle,$$

for all $y \in \ell^2$, i.e., A is F-hemicontinuous.

On the other hand, we take $x_n = e_n = (0,0,\ldots,0,1,0,\ldots)$ with 1 in the n^{th} position. It is obvious that $e_n \rightharpoonup 0$, but $\{e_n\}$ does not have any strongly convergent subsequence, as $\|e_n - e_m\| = \sqrt{2}$ for $m \neq n$. Therefore, A is not weak-to-strong continuous.

The following result is an extension of the corresponding result of Malitsky to infinite dimensional spaces.

Corollary 5.1. *Let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a pseudomonotone, F-hemicontinuous, Lipschitz continuous mapping with constant $L > 0$ such that* $\mathrm{Sol}(C,A) \neq \emptyset$. *Let $\{x^k\}$, $\{y^k\}$ be the sequences generated by Algorithm 5.1 with $\lambda \in \big(0, \frac{\varphi}{2L}\big]$. Then the sequences $\{x^k\}$ and $\{y^k\}$ converge weakly to the same point $x^* \in \mathrm{Sol}(C,A)$.*

Proof. For each pair $x, y \in C$, we define

$$f(x,y) := \begin{cases} \langle Ax, y-x\rangle, & \text{if } x,y \in C,\\ +\infty, & \text{otherwise.}\end{cases} \tag{5.4}$$

Note that the formula (3.6) of Algorithm 3.1 can be equivalently written as

$$\begin{aligned} y^{k+1} &= \operatorname*{argmin}_{y\in C}\left\{\lambda\langle Ay^k, y-y^k\rangle + \frac{1}{2}\|y-x^k\|^2\right\},\\ &= \operatorname*{argmin}_{y\in C}\left\{\frac{1}{2}\|y-(x^k-\lambda Ay^k)\|^2\right\}\\ &= P_C(x^k - \lambda Ay^k).\end{aligned}$$

From the assumptions, it is easy to check conditions (A2)-(A6) and (A8) are satisfied. By Theorem 3.1, the sequences $\{x^k\}$ and $\{y^k\}$ converge weakly to $x^* \in \mathrm{Sol}(C,f)$. It means that the sequences $\{x^k\}$ and $\{y^k\}$ converge weakly to $x^* \in \mathrm{Sol}(C,A)$. Hence, the result is true and the proof is complete. □

6. Preliminary Numerical Results

In this section, we provide numerical examples to illustrate our algorithms and compare with other existing algorithms in [20, 24, 27]. All the codes were written in Matlab (R2015a) and run on PC with Intel(R) Core(TM) i3-370M Processor 2.40 GHz. In the numerical results reported in the following tables, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

Example 6.1. Consider the equilibrium problem given in [12], where the bifunction

$$f:\mathbb{R}^5\times\mathbb{R}^5\to\mathbb{R}$$

is defined, for every $x,y\in\mathbb{R}^5$, by

$$f(x,y)=\langle Px+Qy+q,y-x\rangle,$$

where the vector $q\in\mathbb{R}^5$, and the matrices P and Q are two square matrices of order 5 such that Q is symmetric positive semidefinite, and $Q-P$ is negative semidefinite. To illustrate our algorithms, the matrices P,Q and the vector q are chosen as follows:

$$q=\begin{bmatrix}1\\-2\\-1\\2\\-1\end{bmatrix},P=\begin{bmatrix}3.1&2&0&0&0\\2&3.6&0&0&0\\0&0&3.5&2&0\\0&0&2&3.3&0\\0&0&0&0&3\end{bmatrix},Q=\begin{bmatrix}1.6&1&0&0&0\\1&1.6&0&0&0\\0&0&1.5&1&0\\0&0&1&1.5&0\\0&0&0&0&2\end{bmatrix}.$$

The feasible set is

$$C=\left\{x\in\mathbb{R}^5:\sum_{i=1}^{5}x_i\geq-1,-5\leq x_i\leq 5,\ i=1,\dots,5\right\}.$$

We now verify that the above bifunction f satisfies condition (A7). Indeed, for all bounded sequences $\{x^k\},\{y^k\}\subset C$ such that $\|x^k-y^k\|\to 0$, we see that

$$\begin{aligned}f(x^k,y^k)=\langle Px^k+Qy^k+q,y^k-x^k\rangle &\geq -\left[\|Px^k\|+\|Qy^k\|+\|q\|\right]\|x^k-y^k\|\\ &\geq -M_3\|x^k-y^k\|,\end{aligned}$$

where M_3 is a positive constant such that $\|Px^k\|+\|Qy^k\|+\|q\|\leq M_3$. Therefore, we get

$$\limsup_{k\to\infty}f(x^k,y^k)\geq-\lim_{k\to\infty}M_3\|x^k-y^k\|=0.$$

Then all the conditions (A2)-(A8) of Theorem 3.1 are satisfied. We will apply Algorithm 3.1 and algorithm (2.2) (GEA) to solve EP (2.1). In both algorithms, we will use the same starting point x^0, the same step size $\alpha_k=\beta_k=\lambda=0.27$, and the stopping rule $\|y^{k+1}-y^k\|+\|y^k-x^k\|<10^{-6}$ for Algorithm 3.1 and $\|\tilde{x}^k-\bar{x}^k\|<10^{-6}$ for GEA.

In Table 1, we have compared the performance of Algorithm 3.1 (GRA1) with the General Extragradient Algorithm (2.2) (Algorithm GEA in [24]).

	$x^0 = (-1,3,1,1,2)$		$x^0 = (1,1,1,1,1)$		$x^0 = (-1,0,0,0,0)$	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
GEA	2.8392	40	2.7768	40	2.6676	40
GRA1	2.3088	95	2.4648	96	2.2776	94

TABLE 1. Comparison of Algorithm 3.1 and Algorithm GEA in Example 6.1 with different x^0

Convergent behavior of both algorithms with different x^0 is given in Figures 1-2. In these figures, the value of errors $\|y^{k+1} - y^k\| + \|y^k - x^k\|$ (Algorithm 3.1) and $\|\tilde{x}^k - \bar{x}^k\|$ (GEA) is represented by the y-axis, number of iterations is represented by the x-axis.

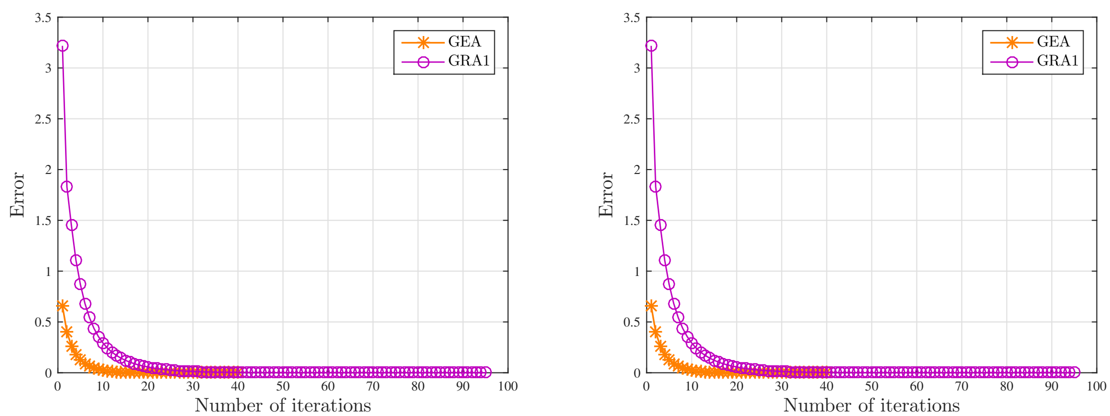

FIGURE 1. Numerical behavior of two algorithms in Example 6.1 with $x^0 = (-1,3,1,1,2)$, $x^0 = (1,1,1,1,1)$, respectively

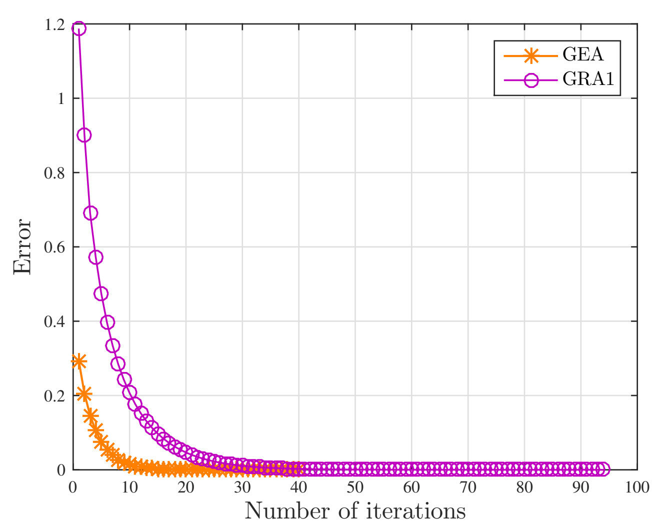

FIGURE 2. Numerical behavior of two algorithms in Example 6.1 with $x^0 = (-1,0,0,0,0)$

Let us observe that computational time of Algorithm 3.1 is smaller than that of Algorithm GEA in [24] but not much for this simple and small example.

Example 6.2. Let $H = L^2([0,1])$ and C as in Example 2.1. Let us set

$$C = \{x \in H : \|x\| \leq 1\}, \;\; f(x,y) = \Big\langle \Big(\frac{3}{2} - \|x\|\Big)x, y - x \Big\rangle.$$

In the same way as in Example 3.1, we see that

(1) f is strongly pseudomonotone on C, where $\gamma = \frac{1}{2}$. On the other hand, f is neither strongly monotone nor monotone on C. To see this, we take $x = \frac{3}{2}\sqrt{\frac{t}{2}}$, $y = \sqrt{2t}$ and observe that

$$\begin{aligned} f(x,y) + f(y,x) &= \Big\langle \Big(\frac{3}{2} - \frac{3}{4}\Big)x, y - x \Big\rangle + \Big\langle \Big(\frac{3}{2} - 1\Big)y, x - y \Big\rangle \\ &= \Big\langle \frac{9}{8}\sqrt{\frac{t}{2}} - \frac{1}{2}\sqrt{2t}, \sqrt{2t} - \frac{3}{2}\sqrt{\frac{t}{2}} \Big\rangle \\ &= \frac{1}{2}\Big(\frac{9}{8\sqrt{2}} - \frac{\sqrt{2}}{2}\Big)\Big(\sqrt{2} - \frac{3}{2\sqrt{2}}\Big) > 0. \end{aligned}$$

(2) f satisfies Lipschitz-type condition (A5) with $c_1 = c_2 = \dfrac{7}{4}$, and the other conditions of Theorem 4.1 are also satisfied.

We will apply Algorithm 4.1 (GRA2) to solve EP (2.1) and compare it with the Algorithm 1 of [20] (Hieu's algorithm) and Algorithm 3.1 of [27] (Popov's algorithm). To test three algorithms, we take the same parameter $\lambda_k = \frac{40}{k+1}$, $k = 0, 1, 2, \ldots$ and

(i) the stopping criterion $\|y^{k+1} - x^k\| + \|y^k - x^k\| < 10^{-3}$ for Algorithm 4.1 and Popov's algorithm; $\|x^k - y^k\| < 10^{-3}$ for Hieu's algorithm;
(ii) the same initial point x^0.

Numerical results of three algorithms are presented in Table 2 and Figure 3.

	$x^0 = \frac{1}{200}(\sin(-3t) + \cos(-10t))$		$x^0 = \frac{1}{285}(t^3 + 1)e^{5t}$	
	Sec.	Iter.	Sec.	Iter.
Hieu's algorithm	0.0096139	86	0.008652	86
Popov's algorithm	0.013996	118	0.012966	118
GRA2	0.0062632	83	0.0070049	83

TABLE 2. The first comparison in Example 6.2

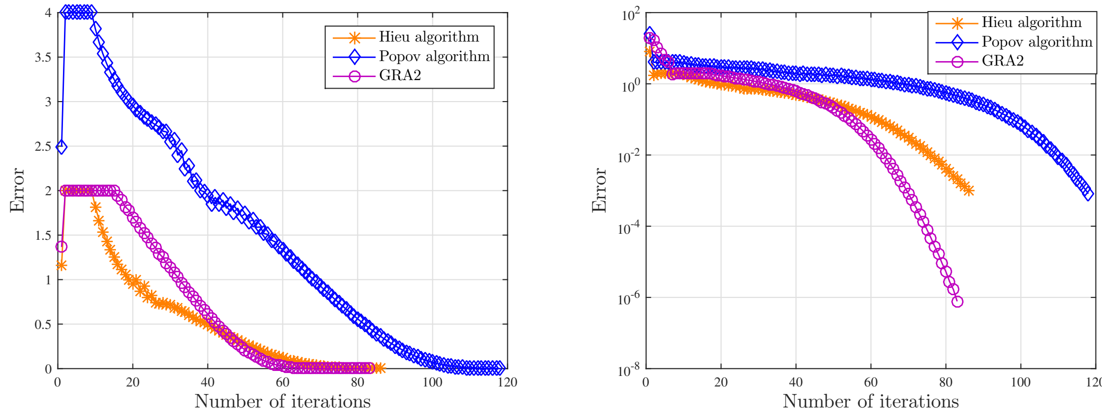

FIGURE 3. The first comparison in Example 6.2 with $x^0 = \frac{1}{200}(\sin(-3t) + \cos(-10t))$, $x^0 = \frac{1}{285}(t^3+1)e^{5t}$, respectively

We now make a comparision of our algorithms including Algorithm 3.1 (GRA1), Algorithm 4.1 (GRA2) and Algorithm 4.2 (GRA3) with the parameters $\lambda = 0.2$, $\lambda_k = \frac{40}{k+1}$ and $\beta_k = \frac{30}{k}$, respectively. Numerical performances are reported in Table 3 and Figure 4.

	$x^0 = \frac{1}{200}(\sin(-3t)+\cos(-10t))$		$x^0 = \frac{1}{185}(t^3+1)e^{5t}$	
	Sec.	Iter.	Sec.	Iter.
GRA1	0.0098299	105	0.015076	150
GRA2	0.0066354	83	0.0064861	83
GRA3	0.0074925	72	0.0078445	75

TABLE 3. The second comparison in Example 6.2

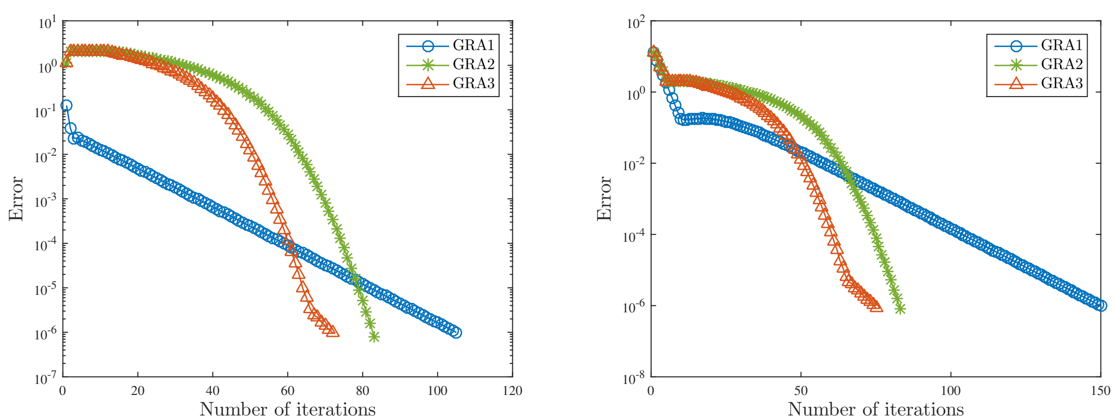

FIGURE 4. The second comparison in Example 6.2 with $x^0 = \frac{1}{200}(\sin(-3t) + \cos(-10t))$, $x^0 = \frac{1}{185}(t^3+1)e^{5t}$, respectively

Example 6.3. Let $H = L^2([0,1])$ and C, f as in Example 3.1. We will compare Algorithm 4.1 (GRA2) with Algorithm 1 of [20] (Hieu's algorithm) and Algorithm 3.1 of [27] (Popov's algorithm). To compare these algorithms, we take the same parameter $\lambda_k = \frac{20}{k+1}$, $k = 0, 1, 2, \ldots$ and stopping rules are used as in the first comparision of Example 6.2. All the strongly convex programming problems over C are solved by the function `fmincon` in Matlab (R2015a) Optimization Toolbox. Numerical results are given in Table 4 and Figure 5.

	$x^0 = \frac{1}{220}(t+1)e^{5t}$		$x^0 = \frac{1}{185}(t^3+1)e^{5t}$	
	Sec.	Iter.	Sec.	Iter.
Hieu's algorithm	13.9933	2	16.3645	2
Popov's algorithm	14.4925	3	16.1461	3
GRA2	8.0965	21	8.1745	21

TABLE 4. Comparison of three algorithms in Example 6.3

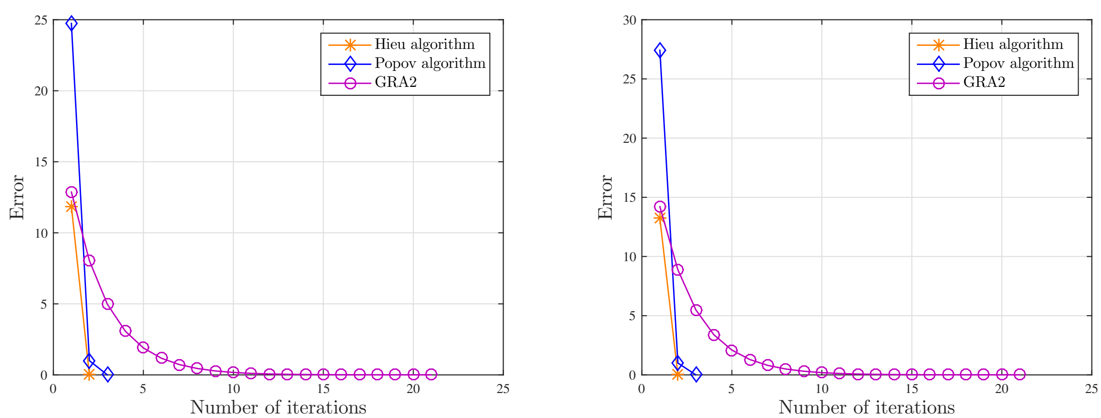

FIGURE 5. Comparison of three algorithms in Example 6.3 with $x^0 = \frac{1}{220}(t+1)e^{5t}$, $x^0 = \frac{1}{185}(t^3+1)e^{5t}$, respectively

From the above numerical results, we find that the number of iterations slightly depends on initial values but the CPU time crucially depends on both initial values and stepsizes.

7. CONCLUSIONS

This paper deals with the convergence analysis and some numerical examples of the golden ratio algorithm for pseudomonotone equilibrium problems in Hilbert spaces. The proposed algorithm is an equilibrium version of a very recent algorithm introduced by Malitsky [26] (for variational inequalities). Moreover, the proposed algorithm is convergent under a weaker condition than the joint weak lower semicontinuity of the bifunction, assumed in several papers before. Numerical results show that the algorithm performs better than some existing methods. Note that, obtaining a result for Algorithm 3.1 without using the condition (A5) seems to be more delicate, and further investigations are necessary.

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