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COERCIVITY CONDITIONS, ZEROS OF MAXIMAL MONOTONE OPERATORS, AND MONOTONE EQUILIBRIUM PROBLEMS

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Abstract. In this paper, some coercivity conditions, which guarantee existence and boundedness of zeros of a maximal monotone operator, are discussed in reflexive Banach spaces. Some results are also obtained for equilibrium problems with monotone bifunctions. Our results extend and improve the results by Zhang, He and Jiang [Existence and boundedness of solutions to maximal monotone inclusion problem, Optim. Lett. 11 (2017), 1565-1570] from Hilbert to Banach spaces, and from monotone operators to monotone equilibrium problems.
Keywords. Coercivity conditions; Monotone equilibrium problems; Zeros of maximal monotone operators.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$ and dual X^*. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x, x^* \rangle$. Let $X^{**} = (X^*)^*$ denote the second dual of X. The canonical map $i : X \underset{x \mapsto \hat{x}}{\to} X^{**}$, which $\hat{x}(x^*) = \langle x, x^* \rangle$, $x^* \in X^*$ gives a linear isomorphism (embedding) from X into X^{**}. X is said to be reflexive if $i : X \to X^{**}$ is surjective (see [1]). A Banach space X is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$, for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. $J : X \longrightarrow 2^{X^*}$ defined by

$$J(x) := \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\|\}$$

is called the duality mapping. A Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(X) := \{x \in X,\ \|x\| = 1\}$. In smooth Banach spaces, the duality mapping $J : X \to 2^{X^*}$ is single-valued (for geometric properties of Banach spaces and duality mappings, we refer to [2, 3]). Let X be a smooth Banach space. The real function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2.$$

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If X is also strictly convex, then

$$\phi(x,y)=0 \Leftrightarrow x=y$$

(see [4] and [5]). A multi-valued operator $A: X \longrightarrow 2^{X^*}$ with domain $D(A)=\{x\in X : A(x)\neq\emptyset\}$ and range $R(A)=\bigcup\{A(x): x\in D(A)\}$ is said to be monotone if

$$\langle x_1-x_2, y_1-y_2\rangle \geqslant 0$$

for each $x_i\in D(A)$, and $y_i\in A(x_i)$, $i=1,2$. A monotone operator A is said to be maximal if its graph $G(A)=\{(x,y):\ x\in D(A),\ y\in A(x)\}$ is not properly contained in the graph of any other monotone operator. In other words, A is monotone, and for any $y\in X$, $\langle \xi - x^*, y-x\rangle \geq 0$ for all $x\in D(A)$, and for all $x^*\in A(x)$ implies $y\in D(A)$ and $\xi\in A(y)$. If X is strictly convex and reflexive, and $A: X\longrightarrow 2^{X^*}$ is maximal monotone, then, for all $x_0\in X$ and for all $\lambda>0$, the inclusion $0\in J(x-x_0)+\lambda Ax$ has a unique answer (see [6], p. 324). We denote the unique point by $\mathscr{J}^A_\lambda x_0$, which is called the resolvent of A of order λ at the point x_0. If X is smooth, strictly convex, and reflexive Banach space, then $A: X\to 2^{X^*}$ is maximal if and only if $R(J+\lambda A)=X^*$ (see [7, 8, 9, 10, 11]). Another concept of the resolvent was defined by Matsushita and Takahashi in [12] as follows

$$\mathfrak{J}^A_\lambda x := (J+\lambda A)^{-1}Jx.$$

For the two kinds of resolvents, we have $A^{-1}(0)=\mathrm{Fix}(\mathscr{J}^A_\lambda)=\mathrm{Fix}(\mathfrak{J}^A_\lambda)$, where $\mathrm{Fix}(T)$ denotes the fixed point set of the mapping T. Two recent concepts of resolvents are coincident in Hilbert spaces. But the second one is firmly quasi-nonexpansive with respect to the function ϕ, i.e.,

$$\phi(p,\mathfrak{J}^A_\lambda x)\leq \phi(p,x)-\phi(\mathfrak{J}^A_\lambda x, x),$$

for all $p\in\mathrm{Fix}(\mathfrak{J}^A_\lambda)$ and $x\in X$ (see also [13]).

Let $A: X\longrightarrow 2^{X^*}$ be a maximal monotone set-valued mapping. Consider the monotone inclusion problem, which is to find $x\in X$ such that

$$0\in A(x). \tag{1.1}$$

The problem of existence of solutions for (1.1) (or non-emptiness of $A^{-1}(0)$) has an essential role in convex optimization, variational inequalities and differential equations. We present some examples in the following.

1. Convex optimization. Some problems of convex optimization can be modeled as the problems of finding a zero of a monotone operator; see, e.g., [14, 15, 16]. Here we state a convex constraint optimization problem, which is modeled as a monotone operator inclusion problem. Let $f:\mathbb{R}^n\to\mathbb{R}$ be convex. Consider the minimization problem for f with constraint $Ax=b$, where A is a $m\times n$ matrix, and $b\in\mathbb{R}^m$. By KKT method, this constraint convex minimization problem is converted to a monotone inclusion problem for monotone operator B defined by

$$B(x,y):=\begin{bmatrix}\partial f(x)+A^Ty\\ b-Ax\end{bmatrix}.$$

2. Variational inequalities. Let $A: X\to 2^{X^*}$ be a maximal monotone map, and let C be a subset of X. A variational inequality problem is to find $\bar{x}\in C$ such that there exists $\bar{x}^*\in A(\bar{x})$, and

$$\langle \bar{x}^*, y-\bar{x}\rangle\geq 0,\ \ \forall y\in C.$$

This problem is equivalent to

$$0 \in A(\bar{x}) + N_C(\bar{x}),$$

where $N_C(\bar{x})$ is the normal cone of C at $\bar{x}$, and $A + N_C$ is a monotone operator.

3. Monotone differential inclusions. Let $X = H$ be a real Hilbert space, and let $A : H \to 2^H$ be a maximal monotone operator. Zeros of A are stationary solutions of the differential inclusion

$$\begin{cases} -u'(t) \in Au(t), & \text{a.e. } t \geq 0, \\ u(0) = x \in \overline{D(A)}. \end{cases} \tag{1.2}$$

The existence of a zero for A ensures the (weakly) asymptotic stability of solutions to (1.2) for each initial value $x \in \overline{D(A)}$ (see [17, 18]). Also the existence of a zero for maximal monotone operator A guarantees the existence of a unique bounded solution for the incomplete second order equation

$$\begin{cases} u''(t) \in Au(t), & \text{a.e. } t \geq 0, \\ u(0) = x \in \overline{D(A)}, \end{cases}$$

which was proved by Barbu [8] (see also [17, 18]).

The paper is organized as follows. In Section 2, we introduce some coercivity conditions in Definitions 2.1, 2.2 and 2.3. Section 3 is devoted to the equivalency between the coercivity conditions in Definition 2.2, and the existence and boundedness of the solution set of problem (1.1). In Section 4, we show that the coercivity conditions in Definition 2.3 are necessary and sufficient conditions for non-emptiness and boundedness of the zero set of a maximal monotone operator in Banach spaces. We also remove additional condition assumed in [19, Theorem 4.1] (the convexity of $D(A)$) in Theorem 4.2 of this section. In Section 5, we present the coercivity conditions for monotone bifunctions. We establish some similar results as in Sections 3 and 4 for equilibrium problems. We show the relation between the coercivity conditions with the existence and boundedness of the solution set of the equilibrium problems for monotone bifunctions.

2. Coercivity Conditions

In order to study Problem (1.1), some coercivity conditions are needed. In this section, we introduce some of them. The first coercivity condition (Definition 2.1 blew) was introduced in [19, 20, 21]. Throughout this section, $A : X \to 2^{X^*}$ is assumed to be a maximal monotone operator.

Definition 2.1.

(A) There exists $\rho > 0$ such that, for every $x \in D(A)$ with $\|x\| > \rho$, there is $y \in D(A)$ with $\|y\| < \|x\|$ satisfying $\inf_{x^* \in A(x)} \langle x^*, x - y \rangle \geqslant 0$.

(B) There exists $\rho > 0$ such that, for every $x \in D(A)$ with $\|x\| > \rho$, there is $y \in D(A)$ with $\|y\| < \|x\|$ satisfying $\sup_{y^* \in A(y)} \langle y^*, x - y \rangle > 0$.

The following coercivity conditions are weaker than the coercivity conditions (A) and (B) in the above definition.

Definition 2.2.

(A) There exist $\rho > 0$ and $z \in X$ such that, for every $x \in D(A)$ with $\|x - z\| > \rho$, there is $y \in D(A)$ with $\|y - z\| < \|x - z\|$ satisfying $\inf_{x^* \in A(x)} \langle x^*, x - y \rangle \geqslant 0$.

(B) There exist $\rho > 0$ and $z \in X$ such that, for every $x \in D(A)$ with $\|x-z\| > \rho$, there is $y \in D(A)$ with $\|y-z\| < \|x-z\|$ satisfying $\sup_{y^* \in A(y)} \langle y^*, x-y\rangle > 0$.

By substituting the norm function in Definition 2.2 by the function $\phi(\cdot,\cdot)$, we may define the modified coercivity conditions as follows.

Definition 2.3.

(A) There exist $\rho > 0$ and $z \in X$ such that, for every $x \in D(A)$ with $\phi(x,z) > \rho$, there is $y \in D(A)$ with $\phi(y,z) < \phi(x,z)$ satisfying $\inf_{x^* \in A(x)} \langle x^*, x-y\rangle \geqslant 0$.

(B) There exist $\rho > 0$ and $z \in X$ such that, for every $x \in D(A)$ with $\phi(z,x) > \rho$, there is $y \in D(A)$ with $\phi(z,y) < \phi(z,x)$ satisfying $\sup_{y^* \in A(y)} \langle y^*, x-y\rangle > 0$.

Remark 2.1. Coercivity conditions of Definition 2.3 in Hilbert spaces are the same coercivity conditions of Definition 2.2. In Definition 2.2, $(B) \Rightarrow (A)$, but we do not know whether it is true for Definition 2.3 or not?

3. The Coercivity Conditions of Definition 2.2 and Problem (1.1)

The results of this section are the extensions of [19, Theorems 3.1 and 4.1] from Hilbert spaces to reflexive, strictly convex, and smooth Banach spaces by using the coercivity conditions of Definition 2.2, which is weaker than the coercivity conditions of Definition 2.1.

Theorem 3.1. *Let X be reflexive, strictly convex, and smooth Banach space, and let $A : X \longrightarrow 2^{X^*}$ be a maximal monotone operator. Then the coercivity condition (A) of Definition 2.2 holds if and only if inclusion problem (1.1) has a solution.*

Proof. We prove "only if" part. The "if" part is similar to the proof of "if" part of Theorem 1.3 of [19] in Hilbert space. Suppose that the coercivity condition (A) of Definition 2.2 holds. Let $z \in X$ and $\rho > 0$ be as in Definition 2.2. Since A is maximal monotone, for $z \in X$ and $n \in \mathbb{N}$, there is $x_n \in D(A)$ such that $\frac{-1}{n} J(x_n - z) \in Ax_n$. Now, we claim that the sequence $\{x_n\}$ is bounded. If not, by the coercivity condition (A) of Definition 2.2, for every x_n with $\|x_n - z\| > \rho$, there exists $y \in D(A)$ with $\|y-z\| < \|x_n - z\|$ such that

$$\langle \frac{-1}{n} J(x_n - z), x_n - y\rangle \geqslant 0.$$

Since

$$\langle \frac{-1}{n} J(x_n - z), x_n - y\rangle = \langle \frac{-1}{n} J(x_n - z), x_n - z\rangle + \langle \frac{-1}{n} J(x_n - z), z - y\rangle,$$

we obtain

$$\begin{aligned} \frac{1}{n}\|x_n - z\|^2 &\leqslant \langle \frac{-1}{n} J(x_n - z), z - y\rangle \\ &\leqslant \frac{1}{n}\|J(x_n - z)\|\|z-y\| \\ &= \frac{1}{n}\|x_n - z\|\|z-y\|. \end{aligned}$$

Therefore $\|x_n - z\| \leqslant \|y - z\|$, which is a contradiction. This shows that $x_n = \mathscr{J}_n^A z$ is bounded. The rest of the proof (non-emptiness of $A^{-1}(0)$) can be concluded from [22, Theorem 2] (see also [23, Theorem 4.2]). $\square$

Theorem 3.2. *Let X be a reflexive, strictly convex, and smooth Banach space, and let $A : X \longrightarrow 2^{X^*}$ be a maximal monotone operator. If $D(A)$ is a convex subset of X, then the coercivity condition (B) of Definition 2.2 holds if and only if the solution set of inclusion problem (1.1) is nonempty and bounded.*

Proof. The proof is straightforward extension of the proof in [19, Theorem 4.1] to Banach spaces. Therefore we omit it. □

4. Coercivity Conditions of Definition 2.3 and Problem (1.1)

In this section, Theorem 3.1 and Theroem 4.1 of [19] are extended to reflexive, strictly convex, and smooth Banach spaces via the coercivity conditions of Definition 2.3.

Theorem 4.1. *Let X be reflexive, strictly convex and smooth, and let $A : X \to 2^{X^*}$ be a maximal monotone operator. Then the coercivity condition (A) in Definition 2.3 holds if and only if there exists $x_0 \in D(A)$ satisfying $0 \in A(x_0)$.*

Proof. Only if part: Suppose the coercivity condition (A) of Definition 2.3 holds. Then, there exist $\rho > 0$ and $z \in X$ such that, for every $x \in D(A)$ with $\phi(x,z) > \rho$, there is $y \in D(A)$ with $\phi(y,z) < \phi(x,z)$ satisfying: $\inf_{x^* \in A(x)} \langle x^*, x-y\rangle \geqslant 0$. From the maximal monotonicity of A, for $n \in \mathbb{N}$, we have $\frac{1}{n}(Jz - J\mathfrak{J}_n^A z) \in A\mathfrak{J}_n^A z$. Setting $\mathfrak{J}_n^A z := x_n$, we get $\frac{1}{n}(Jz - Jx_n) \in A(x_n)$. Now, we claim that the sequence $\{x_n\}$ is bounded. If not, then by the coercivity condition (A) of Definition 2.3, for any x_n with $\phi(x_n,z) > \rho$, there exists $y \in D(A)$ with $\phi(y,z) < \phi(x_n,z)$ such that $\langle \frac{1}{n}(Jz - Jx_n), x_n - y\rangle \geqslant 0$. This implies

$$\begin{aligned} \|x_n\|^2 - \langle x_n, Jz\rangle &\leq \langle y, Jx_n\rangle - \langle y, Jz\rangle \\ &\leq \|y\|\|x_n\| - \langle y, Jz\rangle \\ &\leq \frac{1}{2}\|y\|^2 + \frac{1}{2}\|x_n\|^2 - \langle y, Jz\rangle. \end{aligned}$$

Hence $\phi(x_n,z) \leqslant \phi(y,z)$, which is a contradiction, therefore $\{x_n\}$ is bounded. The reflexivity of X implies that there is a subsequence of x_n, say x_{n_j}, converging weakly to x_0. Fix $y \in D(A)$ and $y^* \in A(y)$. From the monotonicity of A, we have

$$0 \leq \langle \frac{1}{n}(Jz - Jx_n) - y^*, x_n - y\rangle.$$

It follows

$$\begin{aligned} 0 &\leq \langle \frac{1}{n}(Jz - Jx_n), x_n - y\rangle + \langle -y^*, x_n - y\rangle \\ &\leq \frac{1}{n}\|Jz - Jx_n\|\|x_n - y\| + \langle -y^*, x_n - y\rangle \end{aligned}$$

Substituting n by n_j in the last inequality, and letting $j \to \infty$, we get

$$\langle -y^*, x_0 - y\rangle \geq 0.$$

The maximal monotonicity of A implies $x_0 \in D(A)$, and $0 \in A(x_0)$.

If part: Suppose that there exists $x_0 \in D(A)$ such that $0 \in A(x_0)$. Take $z = 0$ and $\rho = \|x_0\|^2$. For all $x \in D(A)$ with $\phi(x,0) = \|x\|^2 > \rho$, take $y = x_0$. Obviously

$$\phi(y,0) = \|y\|^2 = \|x_0\|^2 < \|x\|^2 = \phi(x,0).$$

Using the monotonicity of A, we have $\langle x^*, x - x_0 \rangle \geq 0$, for all $x^* \in A(x)$. Hence, the coercivity condition (A) of Definition 2.3 holds. $\square$

In the next theorem, in addition to the extension of [19, Theorem 4.1] from Hilbert to Banach spaces, we also remove the convexity condition on $D(A)$. The proof of "if" part of Theorem is completely different and shorter than the proof of [19, Theorem 4.1]. Our approach is based on the resolvent of monotone operators, which is due to Matsushita and Takahashi [12].

Theorem 4.2. *Let X be a reflexive, strictly convex and smooth Banach space, and let $A : X \longrightarrow 2^{X^*}$ be a maximal monotone operator with $A^{-1}(0) \neq \emptyset$. Then coercivity condition (B) of Definition 2.3 holds if and only if the set $A^{-1}(0)$ is bounded.*

Proof. *Only if part*: Let condition (B) of Definition 2.3 hold. Then, there exist $\rho > 0$ and $z \in X$ such that for every $x \in D(A)$ with $\phi(z,x) > \rho$ there is $y \in D(A)$ with $\phi(z,y) < \phi(z,x)$ satisfying $\sup_{y^* \in A(y)} \langle y^*, x - y \rangle > 0$. If $A^{-1}(0)$ is not bounded, then, for any $m > 0$, there exists $x_m \in D(A)$ such that $\phi(z, x_m) > m$ and $0 \in A(x_m)$. By the monotonicity of A for all $y \in D(A)$ and $y^* \in A(y)$, we obtain $\langle y^*, x_m - y \rangle \leq 0$, which implies

$$\sup_{y^* \in A(y)} \langle y^*, x_m - y \rangle \leq 0, \quad \forall y \in D(A). \tag{4.1}$$

By coercivity condition (B) of Definition 2.3, we take $m > \rho$ and $x = x_m$, there exists $y \in D(A)$ with $\phi(z,y) < \phi(z,x_m)$ such that

$$\sup_{y^* \in A(y)} \langle y^*, x_m - y \rangle > 0.$$

This contradicts (4.1), therefore the set $A^{-1}(0)$ is bounded.

If part: Suppose that $A^{-1}(0)$ is nonempty and bounded, and the coercivity condition (B) of Definition 2.3 is not satisfied. Then, for any $\rho > 0$ and $z \in A^{-1}(0)$, there is $x \in D(A)$ with $\phi(z,x) > \rho$ such that, for every $y \in D(A)$ with $\phi(z,y) < \phi(z,x)$, $\sup \langle y^*, x - y \rangle \leq 0$. Take $\lambda > 0$ and $y = \mathfrak{J}_\lambda^A x \in D(A)$. If $\mathfrak{J}_\lambda^A x = x$, then $x \in A^{-1}(0)$, and this contradicts the boundedness of $A^{-1}(0)$. Otherwise, $\mathfrak{J}_\lambda^A x \neq x$ and the firm quasi-nonexpansiveness of $\mathfrak{J}_\lambda^A$ with respect to $\phi(\cdot,\cdot)$ implies $\phi(z, \mathfrak{J}_\lambda^A x) < \phi(z,x)$. Therefore

$$\langle Jx - J\mathfrak{J}_\lambda^A x, x - \mathfrak{J}_\lambda^A x \rangle \leq \lambda \sup_{y^* \in A(\mathfrak{J}_\lambda^A x)} \langle y^*, x - \mathfrak{J}_\lambda^A x \rangle \leq 0.$$

Hence, it follows from the monotonicity of J that $\langle Jx - J\mathfrak{J}_\lambda^A x, x - \mathfrak{J}_\lambda^A x \rangle = 0$. Now, the strict convexity of X implies $\mathfrak{J}_\lambda^A x = x$, which contradicts $\mathfrak{J}_\lambda^A x \neq x$. It shows that the coercivity condition (B) of Definition 2.3 holds. $\square$

5. Equilibrium Problems and Coercivity Conditions

In this section, we study the results presented in previous two sections for equilibrium problems. Let X be a Banach space, and let $C \subset X$ be a closed and convex set. An equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$ consists of finding $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{5.1}$$

x is called an equilibrium point. The set of equilibrium points satisfying (5.1) is denoted by $EP(F;C)$. F is said to be monotone if $F(x,y)+F(y,x)\le 0$, $\forall x,y\in C$ (see [24]). In [25, 26], the authors corresponded an operator to a monotone bifunction as follows.

$$A^F(x)=\begin{cases}\{x^*\in X^*:\ F(x,y)\ge\langle x^*,y-x\rangle,\ \forall y\in X\}, & \text{if } x\in C,\\ \emptyset, & \text{if } x\in X\setminus C.\end{cases}$$

The monotonicity of F implies the monotonicity of $A^F:X\to 2^{X^*}$, and $EP(F;C)=(A^F)^{-1}(0)$. As defined in [26], the monotone bifunction F is said to be maximal monotone iff the corresponding operator A^F is maximal monotone. In [26], the authors studied several conditions on the monotone bifunctions, which guarantee their maximality. For example, if the monotone bifunction F with $F(x,x)=0$ is convex and lower semicontinuous with respect to the second variable and upper hemicontinuous with respect to the first variable, then F is maximal monotone [26, Proposition 3.1].

Definition 5.1. Let $F:C\times C\to\mathbb{R}$ be a monotone bifunction, $z\in X$ and $\lambda>0$. The regularized bifunctions $\hat{F},\bar{F}:C\times C\to\mathbb{R}$ are defined by

$$\hat{F}(x,y):=\lambda F(x,y)+\langle J(x-z),y-x\rangle,\forall x,y\in C, \tag{5.2}$$

and

$$\bar{F}(x,y):=\lambda F(x,y)+\langle Jx-Jz,y-x\rangle,\forall x,y\in C. \tag{5.3}$$

If F is a maximal monotone bifunction, then $\hat{F}$ and $\bar{F}$ have the unique equilibrium points (for $\hat{F}$, see [26, Proposition 2.6], and for $\bar{F}$, the proof is similar). The unique equilibrium points of (5.2) and (5.3) are respectively denoted by $\mathscr{J}_\lambda^F z$ and $\mathfrak{J}_\lambda^F z$, that are the resolvents of F of order λ at z. In fact, $\mathscr{J}_\lambda^F=\mathscr{J}_\lambda^{A^F}$ and $\mathfrak{J}_\lambda^F=\mathfrak{J}_\lambda^{A^F}$ were defined in the introduction. Therefore,

$$\mathrm{Fix}(\mathscr{J}_\lambda^{\mathrm{F}})=\mathrm{Fix}(\mathfrak{J}_\lambda^{\mathrm{F}})=EP(F;C),$$

and $\mathfrak{J}_\lambda^F$ is firmly quasi-nonexpansive with respect to $\phi(\cdot,\cdot)$, i.e.,

$$\phi(p,\mathfrak{J}_\lambda^F z)\le\phi(p,z)-\phi(\mathfrak{J}_\lambda^F z,z),\ \ \forall p\in\mathrm{Fix}(\mathfrak{J}_\lambda^F).$$

In the following, we present some coercivity conditions for monotone bifunctions similar to the coercivity conditions of Definitions 2.2 and 2.3 for monotone operators.

Definition 5.2. Let $C\subset X$ and $F:C\times C\to\mathbb{R}$ be a bifunction. Coercivity conditions of Definitions 2.2 and 2.3 of Section 2 can be rewritten for monotone bifunctions as follows.

(1) There exist $\rho>0$ and $z\in X$ such that, for every $x\in C$ with $\|x-z\|>\rho$, there is $y\in C$ with $\|y-z\|<\|x-z\|$ such that $F(x,y)\leqslant 0$.
(2) There exist $\rho>0$ and $z\in X$ such that, for every $x\in C$ with $\|x-z\|>\rho$, there is $y\in C$ with $\|y-z\|<\|x-z\|$ such that $F(y,x)>0$.
(3) There exist $\rho>0$ and $z\in X$ such that, for every $x\in C$ with $\phi(x,z)>\rho$, there is $y\in C$ with $\phi(y,z)<\phi(x,z)$ such that $F(x,y)\leqslant 0$.
(4) There exist $\rho>0$ and $z\in X$ such that, for every $x\in C$ with $\phi(z,x)>\rho$, there is $y\in C$ with $\phi(z,y)<\phi(z,x)$ such that $F(y,x)>0$.

In the following theorems, we establish the relation between the coercivity conditions defined in Definition 5.2 with non-emptiness and boundedness of the set of solutions of equilibrium problem (5.1). The existence of solutions to equilibrium problems by the coercivity condition (1) of Definition 5.2 was studied in [27, 28].

Theorem 5.1. *Let X be a reflexive, strictly convex and smooth Banach space, and let $F : C \times C \to \mathbb{R}$ be a maximal monotone bifunction. Assume that one of the following conditions hold:*
(i) F is weak upper semicontinuous in the first argument,
(ii) F is convex and lower semicontinuous in the second argument and upper hemicontinuous in the first argument.
Then
(1) the coercivity condition (1) of Definition 5.2 is equivalent to the existence of a solution for equilibrium problem (5.1);
(2) the coercivity condition (3) of Definition 5.2 is equivalent to the existence of a solution for equilibrium problem (5.1).

Proof. We prove (2). The proof of (1) is similar.
Only if part: Suppose that the coercivity condition (3) of Definition 5.2 holds. Then, there exist $\rho > 0$ and $z \in X$ such that, for every $x \in C$ with $\phi(x,z) > \rho$, there is $y \in C$ with $\phi(y,z) < \phi(x,z)$ and $F(x,y) \leqslant 0$. For $x \in C$ and $n \in \mathbb{N}$, we have

$$nF(\mathfrak{J}_n^F(x),y) + \langle J\mathfrak{J}_n^F(x) - Jz, y - \mathfrak{J}_n^F(x)\rangle \geq 0, \ \ \forall y \in C. \tag{5.4}$$

Let $x_n = \mathfrak{J}_n^F(x)$. We prove $\{x_n\}$ is bounded. If not, by (5.4) and the coercivity condition (3) of Definition 5.2, for any x_n with $\phi(x_n,z) > \rho$, there exists $y \in C$ with $\phi(y,z) < \phi(x_n,z)$ such that

$$0 \geq F(x_n,y) \geq \frac{-1}{n}\langle Jx_n - Jz, y - x_n\rangle, \ \ \forall y \in C. \tag{5.5}$$

The boundedness of x_n is given via a similar argument of the proof of Theorem 4.1. The boundedness of x_n follows that $\frac{-1}{n}\langle Jx_n - Jz, y - x_n\rangle \longrightarrow 0$. Assume that $x_{n_j} \rightharpoonup \bar{x}$. Substitute n by n_j into (5.5), and let $j \longrightarrow \infty$. If F satisfies (i), then we conclude from (5.5) that

$$F(\bar{x},y) \geq \limsup_{j\to\infty} F(x_{n_j},y) \geq 0, \ \ \forall y \in C,$$

which implies that $\bar{x}$ is a solution for equilibrium problem (5.1). If F satisfies (ii), then the monotonicity of F and (5.5) imply

$$F(y,x_{n_j}) \leq \frac{1}{n_j}\langle Jx_{n_j} - Jz, y - x_{n_j}\rangle, \ \ \forall y \in C.$$

From the lower semicontinuity of F with respect to the second argument, we get

$$F(y,\bar{x}) \leqslant \liminf_{j\to\infty} F(y,x_{n_j}) \leqslant 0, \ \ \forall y \in C.$$

For each $z \in C$, we take $y = \lambda z + (1-\lambda)\bar{x}$. Hence

$$0 = F(y,y) = F(y,\lambda z + (1-\lambda)\bar{x}) \leq \lambda F(y,z) + (1-\lambda)F(y,\bar{x}) \leq \lambda F(y,z).$$

From upper hemicontinuity of F with respect to the first variable, we get

$$0 \leq \limsup_{\lambda\to 0} F(y,z) \leq F(\bar{x},z).$$

It shows that $\bar{x}$ is an equilibrium point for bifunction F.

If part: Suppose that x_0 is a solution for equilibrium problem (5.1). Let $\rho = \|x_0\|^2 + 1$, and $y = x_0$. Then, for each x with $\phi(x,0) = \|x\|^2 > \rho$, we obtain $\phi(y,0) = \|y\|^2 < \rho < \|x\|^2 = \phi(x,0)$. Since F is monotone, and $F(x_0,x) \geq 0$, then

$$F(x,x_0) \leqslant 0,$$

which concludes that the Condition (3) of Definition 5.2 holds. □

Theorem 5.2. *Let X be a reflexive, strictly convex and smooth Banach space, and let $F : C \times C \to \mathbb{R}$ be a monotone bifunction with $EP(F;C) \neq \emptyset$. Suppose that one of the following conditions holds:*
(i) F is upper hemicontinuous with respect to the first argument and convex and lower hemicontinuous with respect to the second argument,
(ii) F is maximal monotone.
Then
(1) the coercivity condition (2) of Definition 5.2 is equivalent to the boundedness of solutions to equilibrium problem (5.1);
(2) the coercivity condition (4) of Definition 5.2 is equivalent to the boundedness of solutions to equilibrium problem (5.1).

Proof. We prove (2). The proof of (1) is similar. Let the coercivity condition (4) of Definition 5.2 hold. Then, there exist $\rho > 0$ and $z \in X$ such that, for every $x \in C$ with $\phi(z,x) > \rho$, we have $y \in C$ such that $\phi(z,y) < \phi(z,x)$, and $F(y,x) > 0$. If $EP(F;C)$ is not bounded, then, for any $m > 0$, there exists $x_m \in C$ with $\phi(z,x_m) > m$ such that $F(x_m,y) \geqslant 0$, $\forall y \in C$. By the monotonicity of F, we get

$$F(y,x_m) \leqslant 0,\ \forall y \in C. \tag{5.6}$$

On the other hand, using the coercivity condition (4) of Definition 5.2, and setting $m > \rho$ and $x = x_m$, we see that there exists $y \in C$ with $\phi(z,y) < \phi(z,x_m)$ such that

$$F(y,x_m) > 0,$$

which contradicts (5.6). Therefore the set $EP(F;C)$ is bounded.

Now, we suppose that $EP(F;C)$ is nonempty and bounded, and condition (i) is satisfied. We prove that the condition (4) of Definition 5.2 holds. Let $x_0 \in EP(F;C)$ and $z = 0$. If the coercivity condition (4) Definition 5.2 does not hold, then, for $\rho > \|x_0\|^2 = \phi(0,x_0)$, there exists $x_\rho \in C$ with $\phi(0,x_\rho) = \|x_\rho\|^2 > \rho$ such that, for any $y \in C$, $\phi(0,y) = \|y\|^2 < \|x_\rho\|^2 = \phi(0,x_\rho)$, $F(y,x_\rho) \leq 0$. Since x_0 is an equilibrium point and F is monotone, we get

$$F(y,x_0) \leq 0.$$

Let $z = tx_0 + (1-t)x_\rho$ with $t \in (0,1)$ such that $\rho - 1 < \phi(0,z) = \|z\|^2 < \rho$. Since C is convex, by condition (i), we have

$$F(y,z) = F(y,tx_0 + (1-t)x_\rho) \leq tF(y,x_0) + (1-t)F(y,x_\rho) \leq 0,$$

which implies $F(y,z) \leq 0$, $\forall y \in C$ with $\|y\| < \|x_\rho\|$. For any $y_1 \in C$ with $\|y_1\|^2 \geq \|x_\rho\|^2 > \rho$, take $y_0 = \lambda z + (1-\lambda)y_1$ with $\lambda \in (0,1)$ such that $y_0 \in C \cap \overline{B_{\sqrt{\rho}}(0)}$. Then $\|y_0\|^2 \leq \rho < \|x_\rho\|^2$.

Therefore, $F(y_0,z) \leq 0$. On the other hand, we have

$$\begin{aligned} 0 = F(y_0,y_0) &= F(y_0, \lambda z + (1-\lambda)y_1) \\ &\leq \lambda F(y_0,z) + (1-\lambda)F(y_0,y_1) \\ &\leq (1-\lambda)F(y_0,y_1), \end{aligned}$$

which follows $F(y_0,y_1) \geq 0$. Hence, the monotonicity of F implies $F(y_1,y_0) \leq 0$. Moreover, condition (i) follows

$$F(y_1,z) \leq \liminf_{\lambda \to 1} F(y_1, \lambda z + (1-\lambda)y_1) \leq 0.$$

Therefore $F(y,z) \leq 0$, $\forall y \in C$. Now, set $y = \lambda z + (1-\lambda)x$, $\forall x \in C$, we have

$$0 = F(y,y) = F(y, \lambda z + (1-\lambda)x) \leq \lambda F(y,z) + (1-\lambda)F(y,x) \leq (1-\lambda)F(y,x),$$

which implies $F(y,x) \geq 0$. Therefore

$$0 \leq \limsup_{\lambda \to 1} F(\lambda z + (1-\lambda)x, x) \leq F(z,x).$$

Hence z is an equilibrium point of F, which contradicts the boundedness of the set $EP(F;C)$.

Finally, we prove that the boundedness of $EP(F;C)$ implies the condition (4) of Definition 5.2 if F is a maximal monotone bifunction. Suppose that $EP(F;C)$ is nonempty and bounded, and (4) is not satisfied. Then, for any $\rho > 0$ and $z \in EP(F;C)$, there is $x \in C$ with $\phi(z,x) > \rho$ such that, for every $y \in C$ with $\phi(z,y) < \phi(z,x)$,

$$F(y,x) \leq 0. \tag{5.7}$$

Since F is maximal monotone bifunction, we have

$$\lambda F(\mathfrak{J}_\lambda^F(x), y) + \langle J\mathfrak{J}_\lambda^F(x) - Jx, y - \mathfrak{J}_\lambda^F(x) \rangle \geq 0, \quad \forall y \in C.$$

Letting $y = x$ in the last inequality, we get

$$\lambda F(\mathfrak{J}_\lambda^F(x), x) + \langle J\mathfrak{J}_\lambda^F(x) - Jx, x - \mathfrak{J}_\lambda^F(x) \rangle \geq 0. \tag{5.8}$$

If $\mathfrak{J}_\lambda^F(x) = x$, then $x \in EP(F;C)$, which contradicts the boundedness of $EP(F;C)$. Otherwise, from the quasi-firm nonexpansiveness of $\mathfrak{J}_\lambda^F$ with respect to ϕ, we have $\phi(z, \mathfrak{J}_\lambda^F(x)) < \phi(z,x)$. Now (5.7) implies that $F(\mathfrak{J}_\lambda^F(x), x) \leq 0$. Thus by (5.8), we have

$$\langle J\mathfrak{J}_\lambda^F(x) - Jx, x - \mathfrak{J}_\lambda^F(x) \rangle \geq 0,$$

which together with the monotonicity of J implies $\langle J\mathfrak{J}_\lambda^F(x) - Jx, x - \mathfrak{J}_\lambda^F(x) \rangle = 0$. From the strict convexity of X, we get $\mathfrak{J}_\lambda^F(x) = x$, which is a contradiction. This completes the proof. □

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