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APPLICATIONS OF CERTAIN BASIC (OR q-) DERIVATIVES TO SUBCLASSES OF MULTIVALENT JANOWSKI TYPE q-STARLIKE FUNCTIONS INVOLVING CONIC DOMAINS

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Abstract. In this paper, with the help of certain higher-order q-derivatives, we first introduce some new subclasses of the class of multivalent q-starlike functions which are associated with the Janowski functions and involve certain conic domains. For these newly-defined function classes, we then investigate several interesting properties including (for example) distortion theorems and radius problems. A number of coefficient inequalities and a sufficient condition are also discussed. Our results are shown to be connected with several earlier works related to the field of our present investigation. Finally, in the concluding section, we have chosen to reiterate the well-demonstrated fact that any attempt to produce the rather straightforward $(\mathfrak{p}, q)$-variations of the results, which we have presented in this paper, will be a rather trivial and inconsequential exercise, simply because the additional parameter $\mathfrak{p}$ is obviously redundant.

Keywords. Conic domains; Distortion theorems; Janowski functions; Multivalent functions; Radius problems.

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1. Introduction, Definitions and Motivation

The basic (or q-) calculus is referred to as quantum calculus in which the symbol q apparently stands for *quantum*. Owing to its demonstrated applications in various fields of the mathematical and physical sciences, the study of the q-calculus has attracted many researchers and users of mathematics.

In Geometric Function Theory of Complex Analysis, the q-derivative (or the q-difference) operator D_q provides an important tool that has been used in order to investigate the various subclasses of analytic functions (see, for example, [1, 2, 3, 4, 5, 6]). A q-extension of the class of starlike functions was first introduced in [7] by means of the q-difference operator D_q. A firm footing of the usage of the q-calculus in the context of Geometric Function Theory of Complex Analysis was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory of Complex Analysis in a book chapter by Srivastava (see, for details, [8]), which was published in 1989.

More recently, in a survey-cum-expository review article by Srivastava [4], the state-of-the-art survey and applications of the q-calculus, the fractional q-calculus, the q-derivative operator and the fractional q-derivative operators in Geometric Function Theory of Complex Analysis were investigated and, at the same time, the obvious triviality of the so-called rather inconsequential $(\mathfrak{p}, q)$-calculus involving a redundant parameter $\mathfrak{p}$ was exposed.

In the advancement of Geometric Function Theory of Complex Analysis, aforementioned works inspired a number of researchers to contribute significantly. Several convolution and fractional q-operators, which have already been defined, were surveyed in the above-cited work by Srivastava [4]. For example, for some subclasses of q-starlike functions, certain inclusion properties, several coefficient inequalities, and a sufficient condition were studied by Wongsaijai and Sukantamala (see [9]). Subsequently, Srivastava *et al.* [10] systematically generalized the work of Wongsaijai and Sukantamala [9]. They used the q-calculus and the Janowski functions, and defined three new subclasses of the q-starlike function class (see [10] and [11]). On the other hand, the close-to-convexity of q-Mittag-Leffler functions was also studied [12]. Srivastava *et al*. [13] also studied the class of q-starlike functions in the conic region, while the upper bound of the third Hankel determinant for the class of q-starlike functions was investigated in [14] (see also [15, 16]). Moreover, several authors in their recent works (see, for example, [10, 11, 17]) concentrated upon the studies of the classes of q-starlike functions related with the Janowski or other functions from several different viewpoints. For some more recent investigations about the q-calculus, we may refer the interested reader to [1, 2, 3, 15, 18, 19, 20, 21].

Our present investigation is motivated by the well-established potential for the usages of the basic (or q-) calculus and the fractional basic (or q-) calculus in Geometric Function Theory of Complex Analysis, just as it was described in a recently-published survey-cum-exposition review article by Srivastava [4], by the investigations by Wongsaijai and Sukantamala [9], Khan *et al.* [22], Srivastava *et al.* (see [10] and [11]), and also by other related works which we cited above. In this paper, we shall consider three new subfamilies of multivalent q-starlike functions with respect to higher-order q-derivatives involving the Janowski functions and associated with certain conic domains. Several such properties and characteristics as, for example, inclusion results, sufficient conditions, distortion theorems and radius problems, shall be discussed here. We shall also point out some relevant connections of our results with the existing results.

The class of multivalent (or p-valent) functions in the open unit disk $\mathbb{U}$, given by

$$\mathbb{U}=\{z: z\in\mathbb{C} \quad \text{and} \quad |z|<1\},$$

and with the following series representation:

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{n+p}z^{n+p} \quad (p\in\mathbb{N}=\{1,2,3,\cdots\}), \tag{1.1}$$

is denoted, as usual, by $\mathscr{A}(p)$. We note that

$$\mathscr{A}(1):=\mathscr{A}.$$

Moreover, the class of multivalent (or p-valent) functions is denoted by $\mathscr{S}^*(p)$, which consists of functions $f\in\mathscr{A}(p)$ that satisfy the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0 \qquad (\forall z\in\mathbb{U}).$$

One can easily see that

$$\mathscr{S}^*(1)=\mathscr{S}^*,$$

where $\mathscr{S}^*$ is the relatively more familiar class of starlike functions in $\mathbb{U}$.

Next, for $-1\leqq B<A\leqq 1$, Janowski [23] generalized the class $\mathscr{S}^*$ of starlike functions as follows.

Definition 1.1. (see [23]) A function $f\in\mathscr{A}$ is said to belong to the class $\mathscr{S}^*[A,B]$ if and only if

$$\Re\left(\frac{(B-1)\left(\frac{zf'(z)}{f(z)}\right)-(A-1)}{(B+1)\left(\frac{zf'(z)}{f(z)}\right)-(A+1)}\right)\geqq 0.$$

Historically speaking, Kanas *et al.* (see [24] and [25]; see also [26]) were the first to define the conic domain Ω_λ $(\lambda\geqq 0)$ as follows:

$$\Omega_\lambda=\left\{u+iv: u>\lambda\sqrt{(u-1)^2+v^2}\right\}. \tag{1.2}$$

Moreover, for fixed λ, Ω_λ represents the conic region bounded successively by the imaginary axis $(\lambda=0)$. For $\lambda=1$, it is a parabola and, for $0<\lambda<1$, it is the right-hand branch of hyperbola and, for $\lambda>1$, it is an ellipse.

For these conic regions, the following functions play the role of extremal functions:

$$p_\lambda(z)=\begin{cases}\chi_1(\lambda,z) & (\lambda=0)\\[2ex] \chi_2(\lambda,z) & (\lambda=1)\\[2ex] \chi_3(\lambda,z) & (0\leqq\lambda<1)\\[2ex] \chi_4(\lambda,z) & (\lambda>1),\end{cases} \tag{1.3}$$

where

$$\chi_1(\lambda,z)=\frac{1+z}{1-z}=1+2z+2z^2+\cdots,$$

$$\chi_2(\lambda, z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,$$

$$\chi_3(\lambda, z) = 1 + \frac{2}{1-\lambda^2} \sinh^2\left\{\left(\frac{2}{\pi}\arccos\lambda\right)\operatorname{arctanh}\left(\sqrt{z}\right)\right\},$$

$$\chi_4(\lambda, z) = 1 + \frac{1}{\lambda^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) + \frac{1}{\lambda^2-1},$$

and

$$u(z) = \frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa}z} \qquad (\forall\, z \in \mathbb{U})$$

and $\kappa \in (0,1)$ is chosen such that

$$\lambda = \cosh\left(\frac{\pi K'(\kappa)}{4K(\kappa)}\right).$$

Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and

$$K'(\kappa) = K\left(\sqrt{1-\kappa^2}\right),$$

that is, $K'(\kappa)$ is the complementary integral of $K(\kappa)$ (see, for details, [27, p. 326]).

Noor *et al.* [28] combined the concepts of the Janowski functions and the conic regions and gave the following definition.

Definition 1.2. A function $f \in \mathscr{A}$ is said to belong to the class λ-$\mathscr{S}^*[A,B]$ if and only if

$$\Re\left(\frac{(B-1)\left(\frac{zf'(z)}{f(z)}\right)-(A-1)}{(B+1)\left(\frac{zf'(z)}{f(z)}\right)-(A+1)}\right) \geqq \lambda\left|\frac{(B-1)\left(\frac{zf'(z)}{f(z)}\right)-(A-1)}{(B+1)\left(\frac{zf'(z)}{f(z)}\right)-(A+1)}-1\right|.$$

We first give some basic definitions and concept details of the basic (or q-) calculus. Throughout this article, we assume that

$$0 < q < 1 \quad \text{and} \quad p \in \mathbb{N} = \{1, 2, 3, \cdots\}.$$

Definition 1.3. (see [29] and [30]) Let $0 < q < 1$ and define the q-number $[j]_q$ by

$$[j]_q = \sum_{k=0}^{j-1} q^k = 1 + q + q^2 + \cdots + q^{j-1} \;\; (j \in \mathbb{N}) \quad \text{and} \quad [0]_q = 0.$$

More generally, for $\delta \in \mathbb{C}$, we define

$$[\delta]_q = \frac{1-q^\delta}{1-q}.$$

Definition 1.4. (see [29] and [30]) The q-factorial $[j]_q!$ is defined as follows:

$$[j]_q! = \frac{(q;q)_j}{(1-q)^j} = \begin{cases} 1 & (j=0) \\ \prod_{k=1}^{j} [k]_q & (j \in \mathbb{N}), \end{cases}$$

where

$$(\omega;q)_j = \prod_{k=0}^{j-1}\left(1-\omega q^k\right) = (1-\omega)(1-\omega q)\left(1-\omega q^2\right)\cdots\left(1-\omega q^{j-1}\right)$$

for $\omega \in \mathbb{C}$.

It is easy to see from Defintion 1.3 and Definition 1.4 that

$$\lim_{q\to 1-}[\delta]_q = \delta \quad \text{and} \quad \lim_{q\to 1-}[j]_q! = j!.$$

Definition 1.5. For a given function f, the q-derivative (or q-difference) operator D_q in a given subset of $\mathbb{C}$ is defined by (see [29] and [30])

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z} & (z \neq 0) \\ \\ f'(0) & (z=0), \end{cases} \tag{1.4}$$

provided that $f'(0)$ exists.

We can see from (1.4) that

$$\lim_{q\to 1-}\left(D_q f\right)(z) = \lim_{q\to 1-}\frac{f(z)-f(qz)}{(1-q)z} = f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. Furthermore, from (1.1) and (1.4), we see for $f \in \mathscr{A}(p)$ that

$$\left(D_q^{(1)} f\right)(z) = [p]_q z^{p-1} + \sum_{n=1}^{\infty}[n+p]_q a_{n+p} z^{n+p-1} \tag{1.5}$$

and

$$\left(D_q^{(2)} f\right)(z) = [p]_q [p-1]_q z^{p-2} + \sum_{n=1}^{\infty}[n+p]_q [n+p-1]_q a_{n+p} z^{n+p-2}, \tag{1.6}$$

and, in general, that

$$\left(D_q^{(p)} f\right)(z) = \frac{(q;q)_p}{(1-q)^p} + \sum_{n=1}^{\infty}\frac{(q;q)_{n+p}}{(1-q)^n (q;q)_p} a_{n+p} z^n, \tag{1.7}$$

where $\left(D_q^{(p)} f\right)(z)$ denotes the q-derivative of $f(z)$ of order $p \in \mathbb{N}$.

Now, for each $f \in \mathscr{A}(p)$, it is easily seen by applying the operator D_q on both sides of (1.1) s times with respect to z that

$$\left(D_q^{(s)} f\right)(z) = \frac{(q;q)_p}{(1-q)^s (q;q)_{p-s}} z^{p-s} + \sum_{n=1}^{\infty}\frac{(q;q)_{n+p}}{(1-q)^s (q;q)_{n+p-s}} a_{n+p} z^{n+p-s}$$

for $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Definition 1.6. (see [7]) A function $f \in \mathscr{A}$ is said to be in the function class $\mathscr{S}_q^*$ if

$$f(0) = f'(0) - 1 = 0 \tag{1.8}$$

and

$$\left|\frac{z}{f(z)}\left(D_q f\right)(z) - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}. \tag{1.9}$$

In view of (1.9), it is obvious in the limit case when $q \to 1-$ that

$$\left| w - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}.$$

The above closed disk is merely the right-half plane and the class $\mathscr{S}_q^*$ of q-starlike functions turns into the familiar class $\mathscr{S}^*$.

We now introduce three (presumably new) subclasses of the class $\mathscr{S}_q^*$ of q-starlike functions associated with the Janowski functions and conic domains.

Definition 1.7. A function $f \in \mathscr{A}(p)$ is said to belong to the class $\mathscr{S}^*[\lambda, q, 1, v, s, A, B, p]$ if and only if

$$\Re\left(\frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} \right) \geqq \lambda \left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - 1 \right|.$$

We call $\mathscr{S}^*[\lambda, q, 1, v, s, A, B, p]$ the class of λ-uniformly multivalent q-starlike functions of Type 1 associated with the Janowski functions.

Definition 1.8. A function $f \in \mathscr{A}(p)$ is said to belong to the class $\mathscr{S}^*[\lambda, q, 2, v, s, A, B, p]$ if and only if

$$\left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - \frac{1+\lambda-\lambda q}{(1-q)(\lambda+1)} \right| < \frac{1}{(1-q)(\lambda+1)}.$$

We call $\mathscr{S}^*[\lambda, q, 2, v, s, A, B, p]$ the class of λ-uniformly multivalent q-starlike functions of Type 2 associated with the Janowski functions.

Definition 1.9. A function $f \in \mathscr{A}(p)$ is said to belong to the class $\mathscr{S}^*[\lambda, q, 3, v, s, A, B, p]$ if and only if

$$\left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - 1 \right| < \frac{1}{\lambda+1}.$$

We call $\mathscr{S}^*[\lambda, q, 3, v, s, A, B, p]$ the class of λ-uniformly mutlivalent q-starlike functions of Type 3 associated with the Janowski functions.

Remark 1.1. First of all, it is easy to see for $(0 \leqq \eta < 1)$ that

$$\mathscr{S}^*[0, q, 1, 1, 0, (1-2\eta), -1, 1] = \mathscr{S}^*[q, 1, \eta],$$
$$\mathscr{S}^*[0, q, 2, 1, 0, (1-2\eta), -1, 1] = \mathscr{S}^*[q, 2, \eta]$$

and

$$\mathscr{S}^*[0, q, 3, 1, 0, (1-2\eta), -1, 1] = \mathscr{S}^*[q, 3, \eta],$$

where

$$\mathscr{S}^*[q, 1, \eta],\ \mathscr{S}^*[q, 2, \eta] \quad \text{and} \quad \mathscr{S}^*[q, 3, \eta]$$

are the function classes introduced and studied by Wongsaijai and Sukantamala (see [9]). Secondly, we have

$$\mathscr{S}^*[0,q,1,1,0,A,B,1] = \mathscr{S}^*[q,1,A,B],$$
$$\mathscr{S}^*[0,q,2,1,0,A,B,1] = \mathscr{S}^*[q,2,A,B],$$

and

$$\mathscr{S}^*[0,q,3,1,0,A,B,1] = \mathscr{S}^*[q,3,A,B],$$

where

$$\mathscr{S}^*[q,1,A,B], \ \ \mathscr{S}^*[q,2,A,B] \quad \text{and} \quad \mathscr{S}^*[q,3,A,B]$$

are the function classes introduced and studied by Srivastava *et al.* (see [10]). Thirdly, for $(0 \leqq \eta < 1)$, we have

$$\mathscr{S}^*[0,q,1,v,s,(1-2\eta),-1,p] = \mathscr{S}^*[q,1,v,s,\eta,p],$$
$$\mathscr{S}^*[0,q,2,v,s,(1-2\eta),-1,p] = \mathscr{S}^*[q,2,v,s,\eta,p],$$

and

$$\mathscr{S}^*[0,q,3,v,s,(1-2\eta),-1,p] = \mathscr{S}^*[q,3,v,s,\eta,p],$$

where

$$\mathscr{S}^*[q,1,v,s,\eta,p], \ \ \mathscr{S}^*[q,2,v,s,\eta,p] \quad \text{and} \quad \mathscr{S}^*[q,3,v,s,\eta,p]$$

are the function classes introduced and studied by Khan *et al.* (see [22]). Finally, we have

$$\mathscr{S}^*[\lambda,q,1,1,0,A,B,1] = \mathscr{S}^*[q,1,A,B],$$
$$\mathscr{S}^*[\lambda,q,2,1,0,A,B,1] = \mathscr{S}^*[q,2,A,B],$$

and

$$\mathscr{S}^*[\lambda,q,3,1,0,A,B,1] = \mathscr{S}^*[q,3,A,B],$$

where

$$\mathscr{S}^*[q,1,A,B], \ \ \mathscr{S}^*[q,2,A,B] \quad \text{and} \quad \mathscr{S}^*[q,3,A,B]$$

are the function classes introduced and studied by Srivastava *et al.* (see [11]).

Geometrically, for $f \in \mathscr{S}^*[\lambda,q,r,v,s,A,B,p] \ \ (r = 1,2,3)$, the following quotient:

$$\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)}$$

lies in the domains $\Omega_j \ \ (j = 1,2,3)$ given by

$$\Omega_1 = \left\{w : w \in \mathbb{C} \quad \text{and} \quad \Re(w) > \frac{A-2\lambda-1}{B-2\lambda-1}\right\},$$

$$\Omega_2 = \left\{w : w \in \mathbb{C} \quad \text{and} \quad \left|w - \frac{2(\lambda+1)+q(A-2\lambda-1)}{(B-2\lambda-1)q+(B+2\lambda+3)}\right| \right.$$
$$\left. < \frac{A+1}{(B-2\lambda-1)q+(B+2\lambda+3)}\right\},$$

and

$$\Omega_3 = \left\{ w : w \in \mathbb{C} \qquad \text{and} \qquad \left| w - \frac{2(\lambda+1)}{(B+2\lambda+3)} \right| < \frac{A+1}{(B+2\lambda+3)} \right\},$$

respectively.

In this paper, many properties and characteristics, for example, sufficient conditions, inclusion results, distortion theorems and radius problems are investigated. We also indicate relevant connections of our results with those in a number of other related works on this subject.

2. A Set of Main Results

We first derive the inclusion results for the generalized higher-order multivalent q-starlike function classes

$$\mathscr{S}^*[\lambda, q, r, v, s, A, B, p] \quad (r = 1, 2, 3),$$

which are associated with the Janowski functions.

Theorem 2.1. *Suppose that* $\lambda \geqq 0$ *and* $-1 \leqq B < A \leqq 1$. *Then*

$$\mathscr{S}^*[\lambda, q, 3, v, s, A, B, p] \subset \mathscr{S}^*[\lambda, q, 2, v, s, A, B, p] \subset \mathscr{S}^*[\lambda, q, 1, v, s, A, B, p].$$

Proof. First of all, we suppose that $f \in \mathscr{S}^*[\lambda, q, 3, v, s, A, B, p]$. Then, it follows from Definition 1.9 that

$$\left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - 1 \right| < \frac{1}{\lambda+1},$$

that is,

$$\left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - 1 \right| + \frac{q}{1-q} < \frac{1}{\lambda+1} + \frac{q}{1-q}. \tag{2.1}$$

By using the triangle inequality in (2.1), we have

$$\left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - \frac{1+\lambda(1-q)}{(1-q)} \right| < \frac{1}{(1-q)(1+\lambda)}. \tag{2.2}$$

The last expression in (2.2) now implies that $f \in \mathscr{S}^*[\lambda, q, 2, v, s, A, B, p]$, that is,

$$\mathscr{S}^*[\lambda, q, 3, v, s, A, B, p] \subset \mathscr{S}^*[\lambda, q, 2, v, s, A, B, p].$$

Next we let $f \in \mathscr{S}^*[\lambda, q, 2, v, s, A, B, p]$, so that

$$\left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - \frac{\lambda(1-q)+1}{(1-q)} \right| < \frac{1}{(1-q)(1+\lambda)},$$

that is, that

$$\left| -\frac{(B-1)\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)}-(A-1)}{(B+1)\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)}-(A+1)}+\frac{\lambda(1-q)+1}{(1-q)}\right| < \frac{1}{(1-q)(1+\lambda)}.$$

Now, using the fact that $\Re(w) < |w|$, together with some elementary concepts, and after some straightforward simplifications, we have

$$\Re\left(\frac{(B-1)\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)}-(A-1)}{(B+1)\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)}-(A+1)}\right) > \frac{\lambda}{\lambda+1}.$$

This last inequality now shows that $f \in \mathcal{S}^*[\lambda,q,1,v,s,A,B,p]$, that is, that

$$\mathcal{S}^*[\lambda,q,2,v,s,A,B,p] \subset \mathcal{S}^*[\lambda,q,1,v,s,A,B,p].$$

This completes the proof Theorem 2.1. □

Remark 2.1. If we put $\lambda = 0$ and $v = s+1 = p = 1$ in Theorem 2.1, then we get the result which was proved earlier by Srivastava *et al.* [10].

As a special case of Theorem 2.1, if we put

$$\lambda = 0, \quad A = 1-2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad v = s+1 = p = 1 = -B,$$

then we get the following known result due to Wongsaijai and Sukantamala [9].

Corollary 2.1. (see [9]) *For* $0 \leqq \eta < 1$, *the following inclusion relation holds true*:

$$\mathcal{S}^*[q,3,\eta] \subset \mathcal{S}^*[q,2,\eta] \subset \mathcal{S}^*[q,1,\eta].$$

Finally, in this section, we give a sufficient condition for the class $\mathcal{S}^*[\lambda,q,3,v,s,A,B,p]$ of generalized q-starlike functions of Type 3, which also indicates the corresponding sufficient conditions for the classes $\mathcal{S}^*[\lambda,q,1,v,s,A,B,p]$ and $\mathcal{S}^*[\lambda,q,2,v,s,A,B,p]$ of Type 1 and Type 2, respectively.

Theorem 2.2. *A function* $f \in \mathcal{A}(p)$ *with the form given by* (1.1) *is in class* $\mathcal{S}^*[\lambda,q,3,v,s,A,B,p]$ *if it satisfies the following coefficient inequality*:

$$\sum_{n=1}^{\infty}\left(2(\lambda+1)\Lambda_2(q;n;p)+|\Lambda_3(q;n;p)|\right)\left|a_{n+p}\right| < |\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p), \tag{2.3}$$

where

$$\Lambda_1(q;p) = \left(\frac{(q;q)_p}{(1-q)^{s+v}(q;q)_{p-s-v}} - \frac{(q;q)_p}{(1-q)^s(q;q)_{p-s}}\right), \tag{2.4}$$

$$\Lambda_2(q;n;p) = \left(\frac{(q;q)_{n+p}}{(1-q)^{s+v}(q;q)_{n+p-s-v}} - \frac{(q;q)_{n+p}}{(1-q)^s(q;q)_{n+p-s}}\right), \tag{2.5}$$

$$\Lambda_3(q;n;p) = \left(\frac{(q;q)_{n+p}(B+1)}{(1-q)^{s+v}(q;q)_{n+p-s-v}} - \frac{(q;q)_{n+p}(A-1)}{(1-q)^{s}(q;q)_{n+p-s}} \right), \tag{2.6}$$

and

$$\Lambda_4(q;p) = \left(\frac{(q;q)_{p}(B+1)}{(1-q)^{s+v}(q;q)_{p-s-v}} - \frac{(q;q)_{p}(A+1)}{(1-q)^{s}(q;q)_{p-s}} \right). \tag{2.7}$$

Proof. Assuming that (2.3) holds true, it suffices to show that

$$\left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - 1 \right| < 1.$$

We have

$$\begin{aligned}
&\left| \frac{(B-1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\dfrac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - 1 \right| \\
&\quad = 2\left| \frac{z^v\left(D_q^{(v+s)}f\right)(z) - \left(D_q^{(s)}f\right)(z)}{(B+1)\left(D_q^{(v+s)}f\right)(z) - (A+1)\left(D_q^{(s)}f\right)(z)} \right| \\
&\quad = 2\left| \frac{\Lambda_1(q;p)z^{p-s} + \sum_{n=1}^{\infty}\Lambda_2(q;n;p)a_{n+p}z^{n+p-s}}{\Lambda_4(q;p)z^{p-s} + \sum_{n=1}^{\infty}\Lambda_3(q;n;p)a_{n+p}z^{n+p-s}} \right| \\
&\quad \leqq \frac{2\Lambda_1(q;p) + 2\sum_{n=1}^{\infty}\Lambda_2(q;n;p)\left|a_{n+p}\right|}{\left|\Lambda_4(q;p)\right| - \sum_{n=1}^{\infty}\left|\Lambda_3(q;n;p)\right|\left|a_{n+p}\right|} \qquad (|z|=1),
\end{aligned} \tag{2.8}$$

where $\Lambda_1(q;p)$, $\Lambda_2(q;n;p)$, $\Lambda_3(q;n;p)$ and $\Lambda_4(q;p)$ are given by (2.4), (2.5), (2.6) and (2.7), respectively. The last expression in (2.8) is easily seen to be bounded above by $\frac{1}{\lambda+1}$ if

$$\sum_{n=1}^{\infty}\left(2(\lambda+1)\Lambda_2(q;n;p) + \left|\Lambda_3(q;n;p)\right|\right)\left|a_{n+p}\right| < \left|\Lambda_4(q;p)\right| - 2(\lambda+1)\Lambda_1(q;p),$$

which completes the proof of Theorem 2.2. □

Remark 2.2. First of all, if we put $v = s+1 = p = 1$ in Theorem 2.2, then we get the corresponding result due to Srivastava *et al.* (see [10]). Secondly, if we put

$$A = 1-2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad B = -1,$$

then Theorem 2.2 yields the corresponding result that was proved earlier by Khan *et al.* [22].

3. Analytic and p-Valent Functions with Negative Coefficients

In this section, we introduce new subclasses of multivalent q-starlike functions, which involve negative coefficients. Let $\mathscr{T}$ be a subset of $\mathscr{A}(p)$ consisting of functions with negative coefficient, that is,

$$f(z) = z^p - \sum_{n=1}^{\infty} \left|a_{n+p}\right| z^{n+p}. \tag{3.1}$$

We also let

$$\mathscr{T}\mathscr{S}^*[\lambda, q, j, v, s, A, B, p] := \mathscr{S}^*[\lambda, q, j, v, s, A, B, p] \cap \mathscr{T} \qquad (j = 1, 2, 3). \tag{3.2}$$

Theorem 3.1. *Let* $-1 \leqq B < A \leqq 1$. *Then*

$$\mathscr{T}\mathscr{S}^*[\lambda, q, 1, v, s, A, B, p] \equiv \mathscr{T}\mathscr{S}^*[\lambda, q, 2, v, s, A, B, p] \equiv \mathscr{T}\mathscr{S}^*[\lambda, q, 1, v, s, A, B, p].$$

Proof. In view of Theorem 2.1, it is sufficient here to show that

$$\mathscr{T}\mathscr{S}^*[\lambda, q, 1, v, s, A, B, p] \subseteq \mathscr{T}\mathscr{S}^*[\lambda, q, 3, v, s, A, B, p].$$

Indeed, if we assume that $f \in \mathscr{T}\mathscr{S}^*[\lambda, q, 1, v, s, A, B, p]$, then

$$\Re\left(\frac{(B-1)\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)}\right) \geqq \frac{\lambda}{\lambda+1},$$

so that

$$\Re\left(\frac{(B-1)\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A-1)}{(B+1)\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)} - (A+1)} - 1\right) \geqq -\frac{1}{\lambda+1}.$$

Thus, after a simple calculation, we find that

$$\Re\left(\frac{2\left[\left(D_q^{(s)}f\right)(z) - z^v\left(D_q^{(v+s)}f\right)(z)\right]}{(B+1)z^v\left(D_q^{(v+s)}f\right)(z) - (A+1)\left(D_q^{(s)}f\right)(z)}\right) \geqq -\frac{1}{\lambda+1},$$

that is, that

$$-2\Re\left(\frac{\Lambda_1(q;p)\,z^{p-s} - \sum\limits_{n=1}^{\infty}\Lambda_2(q;n;p)\,a_{n+p}z^{n+p-s}}{\Lambda_4(q;p)\,z^{p-s} - \sum\limits_{n=1}^{\infty}\Lambda_3(q;n;p)\,a_{n+p}z^{n+p-s}}\right) \geqq -\frac{1}{\lambda+1}.$$

If we now let z lie on the real axis in the complex z-plane, then the value of the following quotient:

$$\frac{z^v\left(D_q^{(v+s)}f\right)(z)}{\left(D_q^{(s)}f\right)(z)}$$

is real. In this case, upon letting $z \to 1-$ along the real axis, we get

$$2(\lambda+1)\Lambda_1(q;p) - \sum_{n=1}^{\infty} 2(\lambda+1)\Lambda_2(q;n;p)\left|a_{n+p}\right| \geqq -\left|\Lambda_4(q;p)\right| + \sum_{n=1}^{\infty}\left|\Lambda_3(q;n;p)\right|\left|a_{n+p}\right|, \tag{3.3}$$

where $\Lambda_1(q;p)$, $\Lambda_2(q;n;p)$, $\Lambda_3(q;n;p)$ and $\Lambda_4(q;p)$ are given by (2.4), (2.5), (2.6) and (2.7), respectively. The last expression in (3.3) is easily seen to satisfy the inequality in (2.3). Thus, by appealing to Theorem 2.2, we complete the proof of Theorem 3.1. □

Remark 3.1. First of all, if we put

$$\lambda+1 = v = s+1 = p = 1$$

in Theorem 3.1, then we get the corresponding result due to Srivastava *et al.* (see [10]). Secondly, upon setting

$$A = 1-2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad B = -1 = \lambda - 1,$$

then Theorem 3.1 yields the corresponding known result that was proven by Khan *et al.* (see [22]). Thirdly, if we put

$$v = 1 = p \quad \text{and} \quad s = 0$$

in Theorem 3.1, then we get the corresponding result due to Srivastava *et al.* (see [11]).

As a special case of Theorem 3.1, if we put

$$A = 1-2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad -B = v = s+1 = p = 1 = \lambda+1,$$

then we get the following known result due to Wongsaijai and Sukantamala [9].

Corollary 3.1. (see [9, Theorem 8]) *If* $0 \leqq \eta < 1$, *then*

$$\mathscr{T}\mathscr{S}^*[q,1,\eta] \equiv \mathscr{T}\mathscr{S}^*[q,1,\eta] \equiv \mathscr{T}\mathscr{S}^*[q,1,\eta].$$

Corollary 3.2. *Let the function f of the form (3.1) be in the class $\mathscr{T}\mathscr{S}^*[\lambda,q,j,v,s,A,B,p]$ $(j = 1,2,3)$. Then*

$$a_{n+p} \leqq \frac{\left|\Lambda_4(q;p)\right| - 2(\lambda+1)\Lambda_1(q;p)}{2(\lambda+1)\Lambda_2(q;n;p) + \left|\Lambda_3(q;n;p)\right|}. \tag{3.4}$$

The result is sharp for the function $f_t(z)$ given by

$$f_t(z) = z^p - \frac{\left|\Lambda_4(q;p)\right| - 2(\lambda+1)\Lambda_1(q;p)}{2(\lambda+1)\Lambda_2(q;n;p) + \left|\Lambda_3(q;n;p)\right|} z^{p+1}, \tag{3.5}$$

where $\Lambda_1(q;p)$, $\Lambda_2(q;n;p)$, $\Lambda_3(q;n;p)$ and $\Lambda_4(q;p)$ are given by (2.4), (2.5), (2.6) and (2.7), respectively.

By means of Theorem 3.1, it should be understood that Type 1, Type 2 and Type 3 of the multivalent q-starlike functions associated with the Janowski functions are exactly the same. Therefore, for convenience, we state and prove the following distortion theorem by using the notation $\mathscr{T}\mathscr{S}_q^*[\lambda,q,j,v,s,A,B,p]$ in which it is tacitly assumed that $j = 1,2,3$.

Theorem 3.2. *If $f \in \mathcal{T}\mathcal{S}^*[\lambda,q,j,v,s,A,B,p]$ $(j=1,2,3)$, then*

$$|f(z)| \geqq r^p - \left(\frac{|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)}{2(\lambda+1)\Lambda_2(q;1;p) + |\Lambda_3(q;1;p)|)}\right) r^{p+1} \tag{3.6}$$

$$\left(|z^p| = r^p \;\; (0<r<1)\right)$$

and

$$|f(z)| \leqq r^p + \left(\frac{|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)}{2(\lambda+1)\Lambda_2(q;1;p) + |\Lambda_3(q;1;p)|}\right) r^{p+1} \tag{3.7}$$

$$\left(|z^p| = r^p \;\; (0<r<1)\right),$$

where $\Lambda_1(q;p)$, $\Lambda_2(q;n;p)$, $\Lambda_3(q;n;p)$ and $\Lambda_4(q;p)$ are given by (2.4), (2.5), (2.6) and (2.7), respectively. The equalities in (3.6) and (3.7) are attained for the function $f(z)$ given by (3.5).

Proof. The following inequality follows from Theorem 2.2:

$$\begin{aligned}
&\left(2(\lambda+1)\Lambda_2(q;1;p) + |\Lambda_3(q;1;p)|\right)\sum_{n=1}^{\infty}|a_{n+p}| \\
&\qquad \leqq \sum_{n=1}^{\infty}\left(2(\lambda+1)\Lambda_2(q;n;p) + |\Lambda_3(q;n;p)|\right)|a_{n+p}| \\
&\qquad < |\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p),
\end{aligned}$$

which readily yields

$$\begin{aligned}
|f(z)| &\leqq r^p + \sum_{n=1}^{\infty}|a_{n+p}|\, r^{n+p} \\
&\leqq r^p + r^{p+1}\sum_{n=1}^{\infty}|a_{n+p}| \\
&\leqq r^p + \frac{|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)}{2(\lambda+1)\Lambda_2(q;1;p) + |\Lambda_3(q;1;p)|} r^{p+1}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|f(z)| &\geqq r^p - \sum_{n=1}^{\infty}|a_{n+p}|\, r^{n+p} \\
&\geqq r^p - r^{p+1}\sum_{n=1}^{\infty}|a_{n+p}| \\
&\geqq r^p - \frac{|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)}{2(\lambda+1)\Lambda_2(q;1;p) + |\Lambda_3(q;1;p)|} r^{p+1},
\end{aligned}$$

where $\Lambda_1(q;p)$, $\Lambda_2(q;n;p)$, $\Lambda_3(q;n;p)$ and $\Lambda_4(q;p)$ are given by (2.4), (2.5), (2.6) and (2.7), respectively. We have thus completed the proof of Theorem 3.2. □

Remark 3.2. First of all, if we put

$$v = 1 = p \quad \text{and} \quad \lambda = 0 = s$$

in Theorem 3.2, then we get the corresponding result due to Srivastava *et al.* (see [10]). Secondly, if we set

$$A = 1 - 2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad \lambda - 1 = B = -1,$$

then Theorem 3.2 yields the corresponding known result that was proven earlier by Khan *et al.* (see [22]). Thirdly, if we put

$$v = 1 = p \quad \text{and} \quad s = 0$$

in Theorem 3.2, then we get the corresponding result due to Srivastava *et al.* (see [11]).

In its special case when

$$A = 1 - 2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad -B = v = s + 1 = p = 1,$$

if we let $q \longrightarrow 1-$, Theorem 3.2 coincides with a simpler result (see [31]) given as follows.

Corollary 3.3. (see [31]) *If $f \in \mathscr{T}\mathscr{S}^*(\eta)$, then*

$$r - \left(\frac{1-\eta}{2-\eta}\right) r^2 \leqq |f(z)| \leqq r + \left(\frac{1-\eta}{2-\eta}\right) r^2 \qquad \big(|z| = r \ \ (0 < r < 1)\big).$$

The proof of the following result (Theorem 3.3) is smiler to the proof of Theorem 3.2, so we choose to state it here without proof.

Theorem 3.3. *If $f \in \mathscr{T}\mathscr{S}^*[\lambda, q, j, v, s, A, B, p] \ \ (j = 1, 2, 3)$, then*

$$\left|f'(z)\right| \geqq pr^{p-1} - \left(\frac{\big(|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)\big)(p+1)}{2(\lambda+1)\Lambda_2(q;1;p) + |\Lambda_3(q;1;p)|}\right) r^p$$

$$\big(|z^p| = r^p \ \ (0 < r < 1)\big)$$

and

$$\left|f'(z)\right| \leqq pr^{p-1} + \left(\frac{\big(|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)\big)(p+1)}{2(\lambda+1)\Lambda_2(q;1;p) + |\Lambda_3(q;1;p)|}\right) r^p$$

$$\big(|z^p| = r^p \ \ (0 < r < 1)\big).$$

The result is sharp for the function $f_t(z)$ given by (3.5).

In its special case when

$$A = 1 - 2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad -B = v = s + 1 = p = 1,$$

if we let $q \longrightarrow 1-$, then Theorem 3.2 reduces to the following known result.

Corollary 3.4. (see [31]) *If $f \in \mathscr{T}\mathscr{S}^*(\eta)$, then*

$$1 - \left(\frac{2(1-\eta)}{2-\eta}\right) r \leqq \left|f'(z)\right| \leqq 1 + \left(\frac{2(1-\eta)}{2-\eta}\right) r \qquad \big(|z| = r \ \ (0 < r < 1)\big).$$

Finally, we determine the radii of close-to-convexity, starlikeness and convexity for functions in the classes $\mathscr{T}\mathscr{S}^*[\lambda, q, j, v, s, A, B, p] \ \ (j = 1, 2, 3)$.

Theorem 3.4. *Let $f \in \mathcal{TS}^*[\lambda,q,j,v,s,A,B,p]\ \ (j=1,2,3)$. Then $f(z)$ is a p-valent close-to-convex function of order ϖ $(0 \leqq \varpi < p)$ for $|z| \leqq r_0(j,p,n,A,B,\varpi)$, where*

$$r_0 = \inf_{n \geqq 1} \left(\frac{\left(2(\lambda+1)\Lambda_2(q;n;p) + |\Lambda_3(q;n;p)|\right)(p-\varpi)}{\left(|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)\right)(n+p)} \right)^{\frac{1}{n}}. \tag{3.8}$$

The result is sharp for the function $f_t(z)$ given by (3.5).

Proof. By applying Theorem 2.2 to functions $f \in \mathcal{TS}^*[\lambda,q,j,v,s,A,B,p]\ \ (j=1,2,3)$ the form (3.1), we find for $|z| < r_0$ that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p - \varpi \qquad (|z| \leqq r_0),$$

which completes the proof of Theorem 3.4. □

Remark 3.3. In its special case when

$$A = 1 - 2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad B = -1,$$

Theorem 2.1 will yield the corresponding known result that was already proven by Khan *et al.* (see [22]).

Theorem 3.5. *Let $f \in \mathcal{TS}^*[\lambda,q,j,v,s,A,B,p]\ \ (j=1,2,3)$. Then $f(z)$ is a p-valent starlike function of order ϖ $(0 \leqq \varpi < p)$ for $|z| \leqq r_1(j,p,n,A,B,\varpi)$, where*

$$r_1 = \inf_{n \geqq 1} \left(\frac{\left(2(\lambda+1)\Lambda_2(q;n;p) + |\Lambda_3(q;n;p)|\right)(p-\varpi)}{\left(|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)\right)(n+p-\varpi)} \right)^{\frac{1}{n}}. \tag{3.9}$$

The result is sharp for the function $f_t(z)$ given by (3.5).

Proof. Following the same steps as in the proof of Theorem 3.4, it is readily seen that

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \varpi \qquad (|z| \leqq r_1),$$

which evidently proves Theorem 3.5. □

Remark 3.4. Upon setting

$$A = 1 - 2\eta \quad (0 \leqq \eta < 1) \qquad \text{and} \qquad B = -1,$$

Theorem 2.1 will yield the corresponding known result that was proven by Khan *et al.* (see [22]).

Corollary 3.5. *Let $f \in \mathcal{TS}^*[\lambda,q,j,v,s,A,B,p]\ \ (j=1,2,3)$. Then $f(z)$ is a p-valent convex function of order ϖ $(0 \leqq \varpi < p)$ for $|z| \leqq r_2(j,p,n,A,B,\varpi)$, where*

$$r_2 = \inf_{n \geqq 1} \left(\frac{\left(2(\lambda+1)\Lambda_2(q;n;p) + |\Lambda_3(q;n;p)|\right)(p-\varpi)p}{\left(|\Lambda_4(q;p)| - 2(\lambda+1)\Lambda_1(q;p)\right)(n+p-\varpi)} \right)^{\frac{1}{n}}. \tag{3.10}$$

The result is sharp for the function $f_t(z)$ given by (3.5).

4. Concluding Remarks and Observations

In this paper, we are motivated by the applications of the basic (or q-) calculus and the fractional basic (or q-) calculus in Geometric Function Theory of Complex Analysis as presented in a survey-cum-expository review article by Srivastava [4]. Several subfamilies of multivalently (or, more precisely, p-valently) q-starlike functions in the open unit disk $\mathbb{U}$ were introduced and investigated here rather systematically. The importance of the results demonstrated in this paper is obvious from the fact that these results would generalize and extend various previously known results derived in many earlier works. Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials are applicable particularly in several diverse areas (see, for example, [27, pp. 350–351]; see also [6, 32, 33, 34, 35, 36]). Moreover, as we remarked above and in the introductory section, in the recently-published survey-cum-expository review article by Srivastava [4], the so-called $(\mathfrak{p},q)$-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter $\mathfrak{p}$ being redundant or superfluous (see, for details, [4, p. 340]). This observation by Srivastava [4] will indeed apply also to any attempt to produce the rather straightforward and inconsequential $(\mathfrak{p},q)$-variations of the results which we presented in this paper.

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