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ON THE INERTIAL RELAXED CQ ALGORITHM IN HILBERT SPACES

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Abstract. In this paper, we study the inertial relaxed CQ algorithm for solving a split feasibility problem in Hilbert spaces. For this algorithm, we establish two convergence theorems under two different conditions. The first condition is weaker than the existing condition, and the second condition is completely different from the existing one. Moreover the preliminary numerical experiment indicates that our proposed algorithms converge faster than the existing algorithms.
Keywords. Hilbert space; Relaxed CQ algorithm; Split feasibility problem; Strong convergence.

1. INTRODUCTION

The split feasibility problem (SFP), introduced by Censor and Elfving [1], has received much attention due to its various applications in signal processing and image reconstruction [2, 3]. The SFP requires to find a point $x \in H_1$ satisfying

$$x \in C \text{ and } Ax \in Q, \tag{1.1}$$

where C and Q are nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively, and A is a linear bounded operator from H_1 into H_2.

In what follows, we define $f : H_1 \to (-\infty, +\infty]$ by

$$f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2,$$

where I is the identity operator and P_Q denotes the orthogonal projection onto Q. Then the convex function $f(x)$ is differentiable and its gradient is given by

$$\nabla f(x) = A^*(I - P_Q)Ax,$$

where A^* is the adjoint operator of the linear operator A. The CQ algorithm introduced by Byrne [4] is a very successful method for solving the SFP. For any initial guess x^0, the CQ algorithm is defined as:

$$x^{k+1} = P_C(x^k - \tau_k \nabla f(x^k)), \tag{1.2}$$

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where P_C is the orthogonal projections onto C. If the step-size τ_k is chosen so that $\tau_k \equiv \tau \in (0, 2\|A\|^{-2})$, then the CQ algorithm is weakly convergent to a solution of the SFP.

We now consider a specific SFP whenever C and Q are level subsets of given convex functions. More specifically, it requires to find a point $x \in H_1$ satisfying

$$x \in C = \{x \in H_1 : c(x) \leq 0\}, \tag{1.3}$$

and

$$Ax \in Q = \{y \in H_2 : q(y) \leq 0\}, \tag{1.4}$$

where $c : H_1 \to (-\infty, +\infty]$ and $q : H_2 \to (-\infty, +\infty]$ are two proper convex functions. A standard assumption of this particular SFP is as follows:

(a1) the solution set S of the SFP is nonempty;
(a2) for any $x \in H_1$, at least one subgradient $\xi \in \partial c(x)$ can be calculated;
(a3) for any $y \in H_2$, at least one subgradient $\zeta \in \partial q(y)$ can be calculated;
(a4) both $\partial c(x)$ and $\partial q(y)$ are bounded on bounded sets.

In this case, the CQ algorithm does not work since the associated projections P_C and P_Q do not have closed-form expressions. To overcome this difficulty, Yang [5] presented the relaxed CQ algorithm:

$$x^{k+1} = P_{\tilde{C}_k}(x^k - \tau_k \nabla f_k(x^k)), \tag{1.5}$$

where

$$\tilde{C}_k = \left\{x \in H_1 \mid c(x^k) + \langle \xi^k, x - x^k \rangle \leq 0\right\},$$

with $\xi^k \in \partial c(x^k)$, and

$$\tilde{Q}_k = \left\{y \in H_2 \mid q(Ax^k) + \langle \zeta^k, y - Ax^k \rangle \leq 0\right\},$$

with $\zeta^k \in \partial q(Ax^k)$. Here, for each $k \geq 0$, we define

$$f_k(x) = \frac{1}{2}\|(I - P_{\tilde{Q}_k})Ax\|^2, \nabla f_k(x) = A^*(I - P_{\tilde{Q}_k})Ax.$$

Since $\tilde{C}_k$ and $\tilde{Q}_k$ above are both half spaces, the projections $P_{\tilde{C}_k}$ and $P_{\tilde{Q}_k}$ have closed-form expressions. Consequently, the relaxed CQ algorithm can be easily implemented. There are several variants of the relaxed CQ algorithm [6, 7, 8, 9, 10, 11, 12].

Recently, inertial techniques were used to improve the performance of various optimization methods; see e.g., [13, 14, 15, 16, 17] and the references therein. Dang *et al.* [18] recently proposed the inertial relaxed CQ algorithm, which is defined as

$$\left[\begin{array}{l} w^k = x^k + \alpha_k(x^k - x^{k-1}), \\ x^{k+1} = P_{C_k}\left(w^k - \tau_k \nabla f_k(w^k)\right), \end{array}\right. \tag{1.6}$$

where $0 \leq \alpha_k < \theta < 1, 0 < \tau_k \equiv \tau < \frac{2}{\|A\|^2}$, and

$$C_k = \left\{x \in H_1 \mid c(w^k) + \langle \xi^k, x - w^k \rangle \leq 0\right\}, \tag{1.7}$$

with $\xi^k \in \partial c(w^k)$, and

$$Q_k = \left\{y \in H_2 \mid q(Aw^k) + \langle \zeta^k, y - Aw^k \rangle \leq 0\right\}, \tag{1.8}$$

with $\zeta^k \in \partial q(Aw^k)$. Here, for each $k \geq 0$, we define

$$f_k(x) = \frac{1}{2}\|(I - P_{Q_k})Ax\|^2, \nabla f_k(x) = A^*(I - P_{Q_k})Ax.$$

However, as shown in [18], the inertial relaxed CQ algorithm has only weak convergence. To attain the norm convergence, Gibali *et al.* [19] modified algorithm (1.6) as

$$\left[\begin{array}{l} w^k = x^k + \alpha_k(x^k - x^{k-1}), \\ v^k = P_{C_k}(w^k - \lambda_k \nabla f_k(w^k)), \\ x^{k+1} = (1 - \beta_k - \gamma_k)w^k + \beta_k v^k, \end{array}\right. \tag{1.9}$$

where $0 \leq \beta_k \leq 1, 0 \leq \gamma_k \leq 1, 0 \leq \alpha_k \leq \bar{\alpha}_k$, and

$$\lambda_k = \frac{\rho_k f_k(w^k)}{\|\nabla f_k(w^k)\|^2 + \|\nabla g_k(w^k)\|^2}, \tag{1.10}$$

$$\bar{\alpha}_k = \begin{cases} \min\left(\theta, \dfrac{\varepsilon_k}{\|x^k - x^{k-1}\|}\right), & x^k \neq x^{k-1}, \\ \theta, & x^k = x^{k-1} \end{cases} \tag{1.11}$$

with $0 < \rho_k < 4$ and $0 \leq \theta < 1$. Here, for each $k \geq 0$, we define

$$g_k(x) = \frac{1}{2}\|x - P_{C_k}x\|^2, \nabla g_k(x) = x - P_{C_k}x.$$

It was shown that the above algorithm converges strongly provided that

(b1) $\lim_k(\varepsilon_k/\gamma_k) = 0$;
(b2) $\inf_{k\geq 0} \beta_k(1 - \beta_k - \gamma_k) > 0$;
(b3) $\inf_{k\geq 0} \rho_k(4 - \rho_k) > 0$;
(b4) $\lim_k \gamma_k = 0$ and $\sum_{k=0}^{\infty} \gamma_k = \infty$.

In this paper, we continue to study algorithm (1.9) and establish two convergence theorems. In the first theorem, we only need $\lim_k(\varepsilon_k/\gamma_k) = 0$ and $\lim_k \gamma_k/(\beta_k(1 - \beta_k - \gamma_k)) = 0$, which is clearly weaker than conditions (b1) and (b2). In the second theorem, we assume $\sum_{k=0}^{\infty} \varepsilon_k < \infty$ and $\lim_k \gamma_k/(\beta_k(1 - \beta_k - \gamma_k)) = 0$, which is completely different from conditions (b1) and (b2).

2. Preliminary

In this section, we assume that "$\rightharpoonup$" stands for weak convergence, H is a Hilbert space, and D is a nonempty closed convex subset in H.

For any $x \in H$, the orthogonal projection onto D is defined as

$$P_D x = \operatorname{argmin}\{\|y - x\| \mid y \in D\}.$$

The projection has the following well-known properties.

For all $x, y \in H$ and $z \in D$, we have

(i) $\langle x - P_D x, z - P_D x\rangle \leq 0$;
(ii) $\langle (I - P_D)x, x - z\rangle \geq \|(I - P_D)x\|^2$.

Recall that a function $f : H \to (-\infty, +\infty]$ is said to be proper if

$$\{x \in H \mid f(x) < +\infty\} \neq \emptyset.$$

A proper function f is said to be convex if, for each $t \in (0,1)$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \forall x, y \in H.$$

Let $f : H \to (-\infty, +\infty]$ be a proper function. A vector $u \in H$ is said to be a subgradient of f at a point x if

$$f(y) \geq f(x) + \langle u, y - x \rangle, \quad \forall y \in H.$$

The set of all subgradients of f at x, denoted by $\partial f(x)$, is called the subdifferential of f.

Let $f : H \to (-\infty, +\infty]$ be a proper function. Recall the following definitions.

(i) f is said to be lower semi-continuous at x if $x_n \to x$ implies

$$f(x) \leq \liminf_{n \to \infty} f(x_n).$$

(ii) f is said to be weakly lower semi-continuous at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n \to \infty} f(x_n).$$

(iii) f is said to be lower semi-continuous on H if it is lower semi-continuous at every point $x \in H$. f is said to be weakly lower semi-continuous on H if it is weakly lower semi-continuous at every point $x \in H$.

Lemma 2.1. [20] *Let $f : H \to (-\infty, +\infty]$ be a proper convex function. Then f is semi-continuous if and only if it is weakly semi-continuous.*

The following two lemmas are quite helpful when we prove the boundedness and strong convergence of sequences in H.

Lemma 2.2 ([21]). *Let $\{s_k\}, \{c_k\} \subset \mathbb{R}^+$, $\{\gamma_k\} \subset (0,1)$ and $\{b_k\} \subset \mathbb{R}$ be sequences such that*

$$s_{k+1} \leq (1 - \gamma_k) s_k + b_k + c_k, \text{ for all } k \geq 0.$$

If $\sum_{k=0}^{\infty} \gamma_k = \infty$, $\sum_{k=0}^{\infty} c_k < \infty$, and $\limsup_k (b_k/\gamma_k) \leq 0$, then $\lim_k s_k = 0$.

Lemma 2.3. [20] *Let $x, y \in H$ and $t, s \in \mathbb{R}$. Then*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$;
(ii) $\|tx + sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x - y\|^2$.

3. Convergence analysis

It follows from Lemma 2.1 that both c and q are weakly lower semi-continuous by condition (a2) and condition (a3). Moreover, from (1.11), one sees that

$$\alpha_k \|x^k - x^{k-1}\| \leq \bar{\alpha}_k \|x^k - x^{k-1}\| \leq \varepsilon_k. \tag{3.1}$$

The following lemmas play an important role in our subsequent analysis.

Lemma 3.1. *Let $\{x^k\}$ and $\{w^k\}$ be the sequences generated by (1.9). If $\{x^k\}$ is bounded, then, for any $z \in S$, there exists $M > 0$ such that*

$$\|w^k - z\|^2 \leq \|x^k - z\|^2 + M\varepsilon_k. \tag{3.2}$$

Proof. Since $\{x^k\}$ is bounded, we may assume that there is $M > 0$ such that

$$2\|x^k - z\| + 2\|x^k - x^{k-1}\| \leq M \text{ for all } k \geq 0.$$

It follows from Lemma 2.3 that

$$\begin{aligned}\|w^k - z\|^2 &= \|(1+\alpha_k)(x^k - z) - \alpha_k(x^{k-1} - z)\|^2 \\ &= \|x^k - z\|^2 + \alpha_k(\|x^k - z\|^2 - \|x^{k-1} - z\|^2) + \alpha_k(1+\alpha_k)\|x^k - x^{k-1}\|^2 \\ &\le \|x^k - z\|^2 + \alpha_k\|x^k - x^{k-1}\|(\|x^k - z\| + \|x^{k-1} - z\|) + 2\alpha_k\|x^k - x^{k-1}\|^2 \\ &= \|x^k - z\|^2 + \alpha_k\|x^k - x^{k-1}\|(\|x^k - z\| + \|x^{k-1} - z\| + 2\|x^k - x^{k-1}\|) \\ &\le \|x^k - z\|^2 + M\alpha_k\|x^k - x^{k-1}\|.\end{aligned}$$

Thus, from (3.1), one obtains (3.2) immediately. □

Lemma 3.2. *Let $\{v^k\}$ and $\{w^k\}$ be the sequences generated by (1.9). Then, for any $z \in S$, it follows that*

$$\|v^k - z\|^2 \le \|w^k - z\|^2 - \tau_k, \tag{3.3}$$

where

$$\tau_k := \frac{\rho_k(4-\rho_k)f_k^2(w^k)}{\|\nabla f_k(w^k)\|^2 + \|\nabla g_k(w^k)\|^2}. \tag{3.4}$$

Proof. Note that

$$\begin{aligned}\|v^k - z\|^2 &= \|P_{C_k}(w^k - \lambda_k \nabla f_k(w^k)) - P_{C_k} z\|^2 \\ &\le \|(w^k - z) - \lambda_k\nabla f_k(w^k)\|^2 - \|(w^k - v^k) - \lambda_k \nabla f_k(w^k)\|^2 \\ &= \|w^k - z\|^2 - \|w^k - v^k\|^2 - 2\lambda_k\langle \nabla f_k(w^k), w^k - z\rangle + 2\lambda_k\langle\nabla f_k(w^k), w^k - v^k\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}2\lambda_k\langle\nabla f_k(w^k), w^k - z\rangle &= 2\lambda_k\langle(I - P_{Q_k})Aw^k, Aw^k - Az\rangle \\ &\ge 2\lambda_k\|(I - P_{Q_k})Aw^k\|^2 = 4\lambda_k f_k(w^k).\end{aligned}$$

On the other hand, by using Cauchy-Schwartz inequality, one concludes that

$$2\lambda_k|\langle\nabla f_k(w^k), w^k - v^k\rangle| \le \|w^k - v^k\|^2 + \lambda_k^2\|\nabla f_k(w^k)\|^2.$$

This implies

$$\begin{aligned}&\|v^k - z\|^2 \\ &\le \|w^k - z\|^2 - 4\lambda_k f_k(w^k) + \lambda_k^2\|\nabla f_k(w^k)\|^2 \\ &\le \|w^k - z\|^2 - \frac{4\rho_k f_k^2(w^k)}{\|\nabla f_k(w^k)\|^2 + \|\nabla g_k(w^k)\|^2} + \frac{\rho_k^2 f_k^2(w^k)}{\|\nabla f_k(w^k)\|^2 + \|\nabla g_k(w^k)\|^2}.\end{aligned}$$

Hence, the desired inequality (3.3) follows. □

Lemma 3.3. *Let $\{v^k\}$ and $\{w^k\}$ be the sequences generated by (1.9). Assume that $\{v^k\}$ and $\{w^k\}$ are bounded such that*

$$\lim_{k\to\infty}\|v^k - w^k\| = \lim_{k\to\infty}\tau_k = 0, \tag{3.5}$$

where τ_k is defined as in (3.4). If $\liminf_k \rho_k(4-\rho_k) > 0$, then each weak cluster point of $\{w^k\}$ belongs to S.

Proof. Let $z \in S$. Since ∂c and ∂q are bounded on bounded sets, we may assume that, for all $k \geq 0$, there is $M > 0$ such that

$$\|\xi^k\| + \|\zeta^k\| \leq M, \xi^k \in \partial c(w^k), \zeta^k \in \partial q(Aw^k).$$

Let $\bar{w}$ be any weak cluster point of $\{w^k\}$. Thus, there exists a subsequence $\{w^{k_i}\}$ of $\{w^k\}$ such that $\{w^{k_i}\}$ is weakly convergent to $\bar{w}$. Since A is linear, this yields that $\{Aw^{k_i}\}$ weakly converges to $A\bar{w}$.

We first prove $\bar{w} \in C$. From the definition of C_k (1.7) and the fact that $v^k \in C_k$, we have

$$c(w^k) \leq \langle \xi^k, w^k - v^k \rangle \leq M\|w^k - v^k\| \to 0,$$

where $\xi^k \in \partial c(w^k)$. Since c is weakly lower semi-continuous, one has

$$c(\bar{w}) \leq \liminf_{i\to\infty} c(w^{k_i}) \leq 0.$$

Consequently, $\bar{w} \in C$.

We next show $A\bar{w} \in Q$. Indeed, by (3.5), we have

$$\lim_{k\to\infty} \|(I - P_{Q_k})Aw^k\| = 0.$$

From (1.8) and the fact that $P_{Q_k}(Aw^k) \in Q_k$, we see that

$$q(Aw^k) \leq \langle \zeta^k, (I - P_{Q_k})Aw^k \rangle \leq M\|(I - P_{Q_k})Aw^k\| \to 0,$$

where $\zeta^k \in \partial q(Aw^k)$. Since q is clearly weakly lower semi-continuous, we have

$$q(A\bar{w}) \leq \liminf_{i\to\infty} q(Aw^{k_i}) \leq 0.$$

It turns out that $A\bar{w} \in Q$. This yields that $\bar{w}$ is a solution of the SFP. □

We now present our main result. To proceed, let us first define

$$\bar{z} = P_S(0) = \arg\min\{\|z\| \mid z \in S\}. \tag{3.6}$$

Theorem 3.1. *Suppose that there hold the conditions:*

(c1) $\lim_k(\varepsilon_k/\gamma_k) = 0$;
(c2) $\lim_k \gamma_k/(\beta_k(1 - \beta_k - \gamma_k)) = 0$;
(c3) $\inf_{k\geq 0} \rho_k(4 - \rho_k) > 0$;
(c4) $\lim_k \gamma_k = 0$ *and* $\sum_{k=0}^{\infty} \gamma_k = \infty$.

Then the sequence $\{x^k\}$ *generated by* (1.9) *converges strongly to the solution* $\bar{z}$.

Proof. We first show that $\{x^k\}$ is bounded. To see this, we may assume without loss of generality that $\varepsilon_k \le \gamma_k$ by (c1). It then follows from (3.3) and (3.1) that

$$\begin{aligned}
\|x^{k+1}-\bar{z}\| &= \|(1-\beta_k-\gamma_k)(w^k-\bar{z})+\beta_k(v^k-\bar{z})-\gamma_k\bar{z}\| \\
&\le (1-\beta_k-\gamma_k)\|w^k-\bar{z}\|+\beta_k\|w^k-\bar{z}\|+\gamma_k\|\bar{z}\| \\
&= (1-\gamma_k)\|x^k-\bar{z}+\alpha_k(x^k-x^{k-1})\|+\gamma_k\|\bar{z}\| \\
&\le (1-\gamma_k)\|x^k-\bar{z}\|+\alpha_k\|x^k-x^{k-1}\|+\gamma_k\|\bar{z}\| \\
&\le (1-\gamma_k)\|x^k-\bar{z}\|+\gamma_k\left(\frac{\varepsilon_k}{\gamma_k}+\|\bar{z}\|\right) \\
&\le (1-\gamma_k)\|x^k-\bar{z}\|+\gamma_k(1+\|\bar{z}\|) \\
&\le \max\{\|x^k-\bar{z}\|,1+\|\bar{z}\|\}.
\end{aligned}$$

By induction, we get

$$\|x^k-\bar{z}\| \le \max\{\|x^0-\bar{z}\|,1+\|\bar{z}\|\}$$

for all $k \ge 0$. Hence, $\{x^k\}$ is bounded, so are $\{w^k\}$ and $\{v^k\}$.

We next show the following inequality:

$$\|x^{k+1}-\bar{z}\|^2 \le (1-\gamma_k)\|x^k-\bar{z}\|^2+\gamma_k\delta_k, \tag{3.7}$$

where

$$\delta_k := 2\langle \bar{z}-x^{k+1},\bar{z}\rangle + M\frac{\varepsilon_k}{\gamma_k}-(1-\beta_k-\gamma_k)\frac{\beta_k}{\gamma_k}(\tau_k+\|w^k-v^k\|^2).$$

Here, $M>0$ is the real number defined as in (3.2). Indeed, it follows from Lemma 2.3, Lemma 3.1 and Lemma 3.2 that

$$\begin{aligned}
&\|(1-\beta_k-\gamma_k)(w^k-\bar{z})+\beta_k(v^k-\bar{z})\|^2 \\
&= (1-\beta_k-\gamma_k)(1-\gamma_k)\|w^k-\bar{z}\|^2+\beta_k(1-\gamma_k)\|v^k-\bar{z}\|^2-(1-\beta_k-\gamma_k)\beta_k\|w^k-v^k\|^2 \\
&\le (1-\beta_k-\gamma_k)(1-\gamma_k)\|w^k-\bar{z}\|^2+\beta_k(1-\gamma_k)(\|w^k-\bar{z}\|^2-\tau_k)-(1-\beta_k-\gamma_k)\beta_k\|w^k-v^k\|^2 \\
&= (1-\gamma_k)^2\|w^k-\bar{z}\|^2-\beta_k(1-\gamma_k)\tau_k-(1-\beta_k-\gamma_k)\beta_k\|w^k-v^k\|^2 \\
&\le (1-\gamma_k)\|x^k-\bar{z}\|^2+M\varepsilon_k-\beta_k(1-\beta_k-\gamma_k)(\tau_k+\|w^k-v^k\|^2),
\end{aligned}$$

and hence

$$\begin{aligned}
\|x^{k+1}-\bar{z}\|^2 &= \|(1-\beta_k-\gamma_k)(w^k-\bar{z})+\beta_k(v^k-\bar{z})-\gamma_k\bar{z}\|^2 \\
&\le \|(1-\beta_k-\gamma_k)(w^k-\bar{z})+\beta_k(v^k-\bar{z})\|^2+2\gamma_k\langle \bar{z}-x^{k+1},\bar{z}\rangle.
\end{aligned}$$

Combining the last two inequalities yields inequality (3.7) as desired.

On the other hand, since $\{x^k\}$ is bounded, we have that $\{\delta_k\}$ is bounded from above. In fact,

$$\sup_{k\ge 0}\|\delta_k\| \le \sup_{k\ge 0}\left\{2\|\bar{z}\|(\|\bar{z}\|+\|x^k\|)+M\frac{\varepsilon_k}{\gamma_k}\right\} < \infty.$$

Furthermore, we can take a subsequence $\{k_l\}$ such that

$$\begin{aligned}\limsup_{k\to\infty}\delta_k &= \lim_{l\to\infty}\delta_{k_l}\\ &= \lim_{l\to\infty}\left(2\langle\bar{z}-x^{k_l+1},\bar{z}\rangle-(1-\beta_{k_l}-\gamma_{k_l})\frac{\beta_{k_l}}{\gamma_{k_l}}(\tau_{k_l}+\|w^{k_l}-v^{k_l}\|^2)\right).\end{aligned}\tag{3.8}$$

With no loss of generality (by selecting a further subsequence if necessary), we may also assume that there exists the limit:

$$\lim_{l\to\infty}2\langle\bar{z}-x^{k_l+1},\bar{z}\rangle.\tag{3.9}$$

Consequently, from (3.8), it turns out that there also exists the limit:

$$\lim_{l\to\infty}(1-\beta_{k_l}-\gamma_{k_l})\frac{\beta_{k_l}}{\gamma_{k_l}}(\tau_{k_l}+\|w^{k_l}-v^{k_l}\|^2).\tag{3.10}$$

Therefore, by condition (c2), we assert that

$$\lim_{l\to\infty}\tau_{k_l}+\|w^{k_l}-v^{k_l}\|^2=0.\tag{3.11}$$

Lemma 3.3 then guarantees that each weak cluster point of $\{w^{k_l}\}$ belongs to S. By the definition of x^{k_l+1}, we deduce that

$$\begin{aligned}\|x^{k_l+1}-w^{k_l}\| &= \|\beta_{k_l}(v^{k_l}-w^{k_l})-\gamma_{k_l}w^{k_l}\|\\ &\le \|v^{k_l}-w^{k_l}\|+\gamma_{k_l}\|w^{k_l}\|.\end{aligned}$$

This together with (c4) and (3.11) yields that

$$\lim_{l\to\infty}\|x^{k_l+1}-w^{k_l}\|=0,$$

which implies that any weak cluster point of $\{x^{k_l+1}\}$ also belongs to S. With no loss of generality, we assume that $\{x^{k_l+1}\}$ weakly converges to $\bar{x}$. Hence $\bar{x}\in S$. Now by (3.8) and (3.9), we infer that

$$\limsup_{k\to\infty}\delta_k\le\lim_{l\to\infty}2\langle\bar{z}-x^{k_l+1},\bar{z}\rangle=2\langle\bar{z}-\bar{x},\bar{z}\rangle\le 0,$$

where the last inequality follows form the fact that $\bar{z}=P_S(0)$.

Finally, we arrive at $\|x^k-\bar{z}\|\to 0$ by applying Lemma 2.2 to (3.7). □

Remark 3.1. Our condition (c2) is clearly weaker than condition (b2). Indeed, if we set $\gamma_k=1/k$ and $\beta_k=1/\sqrt{k}$, then γ_k and β_k satisfy condition (c2), however, they do not satisfy condition (b2).

Theorem 3.2. *Suppose that there hold the conditions:*

(d1) $\sum_{k=0}^{\infty}\varepsilon_k<\infty$;
(d2) $\lim_k\gamma_k/(\beta_k(1-\beta_k-\gamma_k))=0$;
(d3) $\inf_{k\ge 0}\rho_k(4-\rho_k)>0$;
(d4) $\lim_k\gamma_k=0$ *and* $\sum_{k=0}^{\infty}\gamma_k=\infty$.

Then the sequence $\{x^k\}$ *generated by* (1.9) *converges strongly to the solution* $\bar{z}$.

Proof. We first show that $\{x^k\}$ is bounded. As a matter of fact, it then follows from (3.3) and (3.1) that

$$\begin{aligned}\|x^{k+1}-\bar{z}\| &= \|(1-\beta_k-\gamma_k)(w^k-\bar{z})+\beta_k(v^k-\bar{z})-\gamma_k\bar{z}\| \\ &\le (1-\beta_k-\gamma_k)\|w^k-\bar{z}\|+\beta_k\|w^k-\bar{z}\|+\gamma_k\|\bar{z}\| \\ &= (1-\gamma_k)\|x^k-\bar{z}+\alpha_k(x^k-x^{k-1})\|+\gamma_k\|\bar{z}\| \\ &\le (1-\gamma_k)\|x^k-\bar{z}\|+\gamma_k\|\bar{z}\|+\varepsilon_k \\ &\le \max\{\|x^k-\bar{z}\|,\|\bar{z}\|\}+\varepsilon_k.\end{aligned}$$

By induction, we get

$$\|x^k-\bar{z}\| \le \max\{\|x^0-\bar{z}\|,\|\bar{z}\|\}+\sum_{k=0}^{\infty}\varepsilon_k$$

for all $k \ge 0$. Hence, $\{x^k\}$ is bounded.

We next deduce from (3.7) that

$$\|x^{k+1}-\bar{z}\|^2 \le (1-\gamma_k)\|x^k-\bar{z}\|^2+\gamma_k\delta_k+M\varepsilon_k, \tag{3.12}$$

where

$$\delta_k := 2\langle \bar{z}-x^{k+1},\bar{z}\rangle-(1-\beta_k-\gamma_k)\frac{\beta_k}{\gamma_k}(\tau_k+\|w^k-v^k\|^2).$$

Here, $M>0$ is the real number defined as in (3.2). In a similar way, we can show $\limsup_k \delta_k \le 0$. Thus, $\|x^k-\bar{z}\| \to 0$ by applying Lemma 2.2 to (3.12). □

Remark 3.2. Our condition (d1)-(d4) is completely different from condition (b1)-(b4). Indeed, if we set $\rho_k=2, \beta_k=1/\sqrt{k}, \varepsilon_k=1/k^2$, and

$$\gamma_k = \begin{cases} 1/k, & k \text{ is odd;} \\ 1/k^2, & k \text{ is even;} \end{cases}$$

then these parameters satisfy condition (d1)-(d2), but do not satisfy condition (b1)-(b2). On the other hand, if we set $\rho_k=2, \gamma_k=\beta_k=1/\sqrt{k}$, and $\varepsilon_k=1/k$, then these parameters satisfy condition (b1)-(b4), but do not satisfy condition (d1)-(d4).

4. Demonstration examples

In this section, we present two numerical experiments to illustrate the performance of the proposed algorithms. Our numerical experiments are coded in MATLAB R2012b running on personal computer with 3.50 GHz Intel Core i3 and 4GB RAM. For our convenience, we denote in what follows by Algorithm G the method (1.9) under conditions (b1)-(b4), Algorithm 1 the method (1.9) under conditions (c1)-(c4), and Algorithm 2 the method (1.9) under conditions (c1)-(c4).

Example 4.1. In this experiment, we condsider the following case:

$$\begin{aligned} C &= \{x\in\mathbb{R}^n : -x_1+x_2^2+\cdots+x_n^2 \le 0\}, \\ Q &= \{y\in\mathbb{R}^m : y_1+y_2^2+\cdots+y_m^2 \le 1\}, \end{aligned} \tag{4.1}$$

where A is an $m\times n$ matrix randomly generated by a standardized normal distribution.

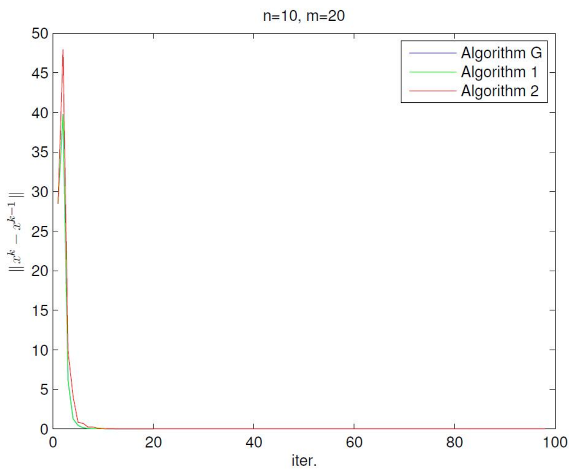

FIGURE 1.

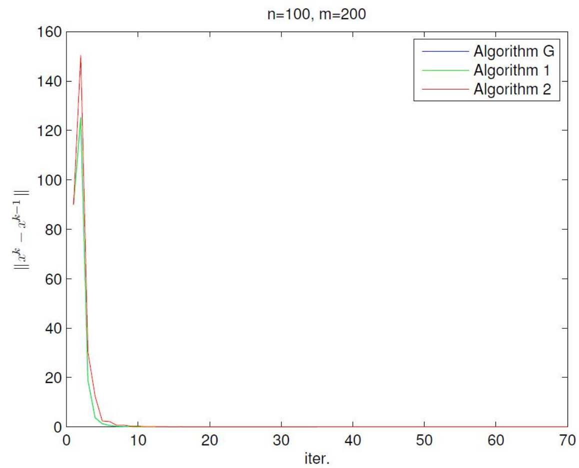

FIGURE 2.

In Algorithm G, we set $\varepsilon_k = 1/k^2, \beta_k \equiv 0.5, \rho_k \equiv 2$ and $\gamma_k = 1/k$. In Algorithm 1, $\varepsilon_k = 1/k^2, \beta_k = 1/\sqrt{k}, \rho_k \equiv 2$ and $\gamma_k = 1/k$. In Algorithm 2, $\varepsilon_k = 1/k^2, \beta_k = 1/\sqrt{k}, \rho_k \equiv 2$ and

$$\gamma_k = \begin{cases} 1/k, & k \text{ is odd}, \\ 1/k^2, & k \text{ is even}. \end{cases}$$

The stopping criteria is $\|x^{k+1} - x^k\| < 10^{-4}$, and the initial points are set respectively set as $x_0 = (0,0,\cdots,0)^T$ and $x_1 = (1,1,\cdots,1)^T$. The numerical results of the compared algorithms of the number of iterations and the time of execution in seconds are respectively reported in Fig. 3-Fig. 4 with different choices of m and n. As shown in these results, it is readily seen that our proposed algorithms converge faster, and thus have better performance compared with Algorithm G.

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