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SELF-ADAPTIVE ITERATIVE ALGORITHMS FOR THE SPLIT COMMON FIXED POINT PROBLEM WITH DEMICONTRACTIVE OPERATORS

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Abstract. In this paper, we use the dual variable to propose two self-adaptive iterative algorithms for solving the split common fixed point problem with demicontractive operators. Under suitable conditions, the weak and strong convergence results of the algorithms are obtained. Primary numerical experiments illustrate the performance and advantage of the proposed algorithms.

Keywords. Dual variable; Demicontractive operators; Split common fixed point problem; Viscosity method; Weak and strong convergence.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k)=\{x: \exists x_{k_j}\rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_k\}$. Let T be a mapping. Recall that T is said to be demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx=0$. Further, recall the following definitions. T is said to be nonexpansive if $\|Tx-Ty\|\le\|x-y\|$ for all $x,y\in H$. T is said to be quasi-nonexpansive if $F(T)\neq\emptyset$ and $\|Tx-q\|\le\|x-q\|$ for all $x\in H$ and $q\in F(T)$. T is said to be firmly nonexpansive if $\|Tx-Ty\|^2\le\|x-y\|^2-\|(x-y)-(Tx-Ty)\|^2$ for all $x,y\in H$. T is said to be firmly quasi-nonexpansive (also said to be directed) if $F(T)\neq\emptyset$ and $\|Tx-q\|^2\le\|x-q\|^2-\|x-Tx\|^2$ for all $x\in H$ and $q\in F(T)$. T is said to be k-strictly pseudocontractive if there exists a constant $k\in(0,1)$ such that $\|Tx-Ty\|^2\le\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2$ for all $x,y\in H$. T is said to be β-demicontractive if $F(T)\neq\emptyset$ and there exists a constant $\beta\in(0,1)$ such that $\|Tx-q\|^2\le\|x-q\|^2+\beta\|x-Tx\|^2$ for all $x\in H$ and $q\in F(T)$.

For a β-demicontractive operator T, we also have that the following equivalent expressions [1]:

$$\langle x-Tx,x-q\rangle\ge\frac{1-\beta}{2}\|x-Tx\|^2,\quad q\in F(T),x\in H$$

and

$$\langle x-Tx,q-Tx\rangle\le\frac{1+\beta}{2}\|x-Tx\|^2,\quad q\in F(T),x\in H.$$

Setting $T_\alpha=(1-\alpha)I+\alpha T$, for $\alpha\in(0,1)$. Then we have

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$$\|x-T_\alpha x\|^2 \le \frac{2\alpha}{1-\beta}\langle x-q, x-T_\alpha x\rangle. \tag{1.1}$$

Indeed,

$$\langle x-q, x-T_\alpha x\rangle = \alpha\langle x-q, x-Tx\rangle \ge \frac{\alpha(1-\beta)}{2}\|x-Tx\|^2 = \frac{1-\beta}{2\alpha}\|x-T_\alpha x\|^2.$$

Recall that a mapping $T : H \to H$ is said to be an averaged if it can be written as the average of the identity I and a nonexpansive mapping, that is, $T = (1-\alpha)I + \alpha S$, where α is a number in $(0,1)$ and $S : H \to H$ is nonexpansive.

Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two mappings. The split common fixed point problem (SCFP) studied in this paper consists of finding

$$x^* \in F(U) \text{ such that } Ax^* \in F(T), \tag{1.2}$$

where $F(U)$ and $F(T)$ stand for the fixed point sets of U and T, respectively. The SCFP finds a number of real applications in medical imaging, signal processing, etc; see, e.g., [2, 3, 4, 5, 6, 7] and the references therein. From [2], one knows that the SCFP has received much attention due to its applications in the verse problem of intensity-modulated radiation therapy and in the dynamic emission tomographic image reconstruction.

It is known that the SCFP was first proposed by Censor and Segal [8]. They introduced the original algorithm for directed operators U and T as follow:

$$x_{k+1} = U(x_k - \rho A^*(I-T)Ax_k), \quad \forall k \ge 0,$$

where the step size satisfies $0 < \rho < \frac{2}{\|A\|^2}$ and they obtained that $\{x_k\}$ weakly converges to a solution of the SCFP if the solution set of the SCFP is nonempty. It is obvious that the choice of the step size ρ depends on the norm of operator A, which is the disadvantage of this algorithm.

Recently, many authors studied and extended the SCFP in order to overcome this disadvantage. In 2014, Cui and Wang [9] proposed the following algorithm:

$$x_{k+1} = U_\alpha(x_k - \rho_k A^*(I-T)Ax_k), \quad \forall k \ge 0, \tag{1.3}$$

where T is a τ-demicontractive operator on H_2, $U_\alpha = (1-\alpha)I + \alpha U$ (where U is a κ-demicontractive operator on H_1 and $\alpha \in (0, 1-\tau)$, the step size ρ_k is chosen by

$$\rho_k = \begin{cases} \frac{(1-\tau)\|(I-T)Ax_k\|^2}{2\|A^*(I-T)Ax_k\|^2}, & Ax_k \ne T(Ax_k), \\ 0, & otherwise. \end{cases}$$

They proved that the sequence $\{x_k\}$ converges weakly to a solution of the SCFP of demicontractive operators. The advantage of this algorithm is that the step size ρ_k is searched automatically and does not depend on the norm of the operator A.

Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. If U and T are projection operators in the SCFP, then the SCFP is reduced to the following split feasibility problem (SFP) [10, 11], which can mathematically be formulated as the problem of finding a point $x^* \in C$ with the property

$$Ax^* \in Q. \tag{1.4}$$

Recently, many algorithms have been introduced to solve the SFP; see, e.g., [12, 13, 14, 15, 16, 17, 18] and the references therein. Note that the SFP can be written as a convex separable minimization problem [4, 19, 20]. In [4], Chen, Huang and Zhang considered finding a minimizer of the sum of two proper lower semi-continuous convex functions, i.e.,

$$x^* = \arg\min_{x\in R^n} f_1(x) + f_2(x),$$

where $f_1, f_2 \in \Gamma_0(\mathbb{R}^n)$ (all proper lower semi-continuous convex functions from $\mathbb{R}^n$ to $(-\infty,+\infty]$) and f_2 is differentiable on R^n with $1/\beta$-Lipschitz continuous gradient for some $\beta \in (0,+\infty)$. To solve the convex separable problem, they obtained the following fixed point formulation: the point x^* is a solution of the convex separable problem if and only if there exists $v^* \in \mathbb{R}^n$ such that

$$\begin{cases} v^* = (I - prox_{\frac{\gamma}{\lambda}f_1})(x^* - \gamma\nabla f_2(x^*) + (1-\lambda)v^*), \\ x^* = x^* - \gamma\nabla f_2(x^*) - \lambda v^*, \end{cases}$$

where λ and γ are two positive numbers. They also introduced the following iterative sequence:

$$\begin{cases} v_{k+1} = (I - prox_{\frac{\gamma}{\lambda}f_1})(x_k - \gamma\nabla f_2(x_k) + (1-\lambda)v_k), \\ x_{k+1} = x_k - \gamma\nabla f_2(x_k) - \lambda v_{k+1}. \end{cases}$$

It was shown [4] that, under appropriate conditions, the sequence $\{x_k\}$ converges to a solution of the convex separable problem. Since x is the primal variable related to the convex separable problem, it is very natural to ask which role the variable v plays in above algorithm. They showed that v is actually the dual variable of the primal-dual form related to the convex separable problem.

Inspired and motivated by the results mentioned above, for solving the SCFP (1.2) with demicontractive operators, we use the dual variable to propose a weak convergence algorithm where the stepsize does not depend on the operator norm $\|A\|$. We also use the viscosity approximation method to modify the proposed algorithm and prove a strong convergence result. The contents of this paper are as follows. Some useful lemmas are presented for the convergence analysis of the iterative algorithms in Section 2. A weak convergence theorem of the proposed algorithm with the dual variable is established in Section 3. A strong convergence theorem with the aid of viscosity approximation method is obtained in Section 4. Finally, in Section 5, we give some numerical experiments to illustrate the efficiency of the proposed iterative algorithms.

2. PRELIMINARIES

Lemma 2.1. [16] *Let H be a real Hilbert space. Let $T : H \to H$ be β-demicontractive operator with $F(T) \neq \emptyset$ and let $T_\alpha = (1-\alpha)I + \alpha T$. Then T_α is quasi-nonexpansive provided that $\alpha \in [0, 1-\beta]$ and*

$$\|T_\alpha x - q\|^2 \le \|x-q\|^2 - \alpha(1-\beta-\alpha)\|Tx - x\|^2, \quad \forall x \in H, q \in F(T).$$

Lemma 2.2. [21] *Let H be a real Hilbert space. Let K be a nonempty closed convex subset of H and let $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demiclosed at origin.*

Lemma 2.3. [22] *Let H be a real Hilbert space. Let K be a nonempty closed convex subset of H and let $\{x_k\}$ be a bounded sequence which satisfies the following properties:*

(a) *every weak limit point of $\{x_k\}$ lies in K;*

(b) $\lim_{k\to\infty}\|x_k - x\|$ *exists for every* $x \in K$.

Then $\{x_k\}$ *weakly converges to a point in* K.

Lemma 2.4. [23] *Assume that* $\{s_k\}$ *is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{k+1} \le (1-\lambda_k)s_k + \lambda_k\delta_k, \\ s_{k+1} \le s_k - \eta_k + \mu_k, \end{cases}$$

for each $k \ge 0$, *where* $\{\lambda_k\}$ *is a sequence in* $(0,1)$, $\{\eta_k\}$ *is a sequence of nonnegative real numbers and* $\{\delta_k\}$ *and* $\{\mu_k\}$ *are two sequences in* $\mathbb{R}$ *such that*

(a) $\sum_{k=1}^{\infty}\lambda_k = \infty$;

(b) $\lim_{k\to\infty}\mu_k = 0$;

(c) $\lim_{l\to\infty}\eta_{k_l} = 0$ *implies* $\limsup_{l\to\infty}\delta_{k_l} \le 0$ *for any subsequence* $\{k_l\} \subset \{k\}$.

Then $\lim_{k\to\infty}s_k = 0$.

3. The Weak Convergence Theorem

In this paper, we make use of the following assumptions:

(A1) $A : H_1 \to H_2$ is a bounded linear operator such that $A \ne 0$;

(A2) Γ denotes the solution set of the SCFP (1.2) and Γ is nonempty.

Now, we use the dual variable to introduce a new self-adaptive iterative algorithm for solving the SCFP (1.2) with demicontractive operators.

Algorithm 3.1. Let H_1 and H_2 be two real Hilbert spaces. Let x_0 and v_0 be two arbitrary vectors in H_1. Let U be a β_1-demicontractive operator on H_1 and let T be a β_2-demicontractive operator on H_2. Define

$$\begin{cases} y_k = x_k - \gamma_k A^*(I-T_\alpha)Ax_k, \\ v_{k+1} = (I-U_\alpha)(y_k + (1-\lambda)v_k), \\ x_{k+1} = y_k - \lambda v_{k+1}, \end{cases}$$

where $0 < \lambda \le 1$, $0 < \alpha < \frac{1-\beta_1}{2}$, the stepsize γ_k is chosen in such a way that

$$\gamma_k := \begin{cases} \frac{\rho_k\|(I-T_\alpha)Ax_k\|^2}{\|A^*(I-T_\alpha)Ax_k\|^2}, & \|(I-T_\alpha)Ax_k\| \ne 0, \\ \gamma, & \|(I-T_\alpha)Ax_k\| = 0 \end{cases} \tag{3.1}$$

with $0 < \rho_k < \frac{1-\beta_2}{\alpha}$ and $\gamma > 0$.

We remark here that the choice of the stepsize γ_k is independent of $\|A\|$ in Algorithm 3.1. Next, we will see that γ_k is well-defined.

Lemma 3.1. *The stepsize* γ_k *defined in Algorithm 3.1 is well-defined.*

Proof. Fix $x^* \in \Gamma$, i.e., $x^* \in F(U)$ and $Ax^* \in F(T)$. From (1.1), we have

$$\begin{aligned} -\langle x_k - x^*, A^*(I-T_\alpha)Ax_k\rangle =& \langle Ax_k - Ax^*, T_\alpha(Ax_k) - Ax_k\rangle \\ \le& -\frac{1-\beta_2}{2\alpha}\|(T_\alpha - I)Ax_k\|^2. \end{aligned}$$

Hence,

$$\|A^*(I-T_\alpha)Ax_k\|\|x_k - x^*\| \ge \langle x_k - x^*, A^*(I-T_\alpha)Ax_k\rangle \ge \frac{1-\beta_2}{2\alpha}\|(I-T_\alpha)Ax_k\|^2.$$

Consequently, we have $\|A^*(I-T_\alpha)Ax_k\| > 0$ when $\|(I-T_\alpha)Ax_k\| \neq 0$. This leads that γ_k is well defined. □

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Let U be a β_1-demicontractive operator on H_1 and let T be a β_2-demicontractive operator on H_2 such that $I-U$ and $I-T$ are demiclosed at origin. Let $\{(v_k,x_k)\}$ be the sequence generated by Algorithm 3.1. Assume that $0<\lambda\le 1$, $0<\alpha<\frac{1-\beta_1}{2}$, and*

$$0<\liminf_{k\to\infty}\rho_k \le \limsup_{k\to\infty}\rho_k < \frac{1-\beta_2}{\alpha}.$$

Then $\{x_k\}$ weakly converges to a point $\hat{x}$, where $\hat{x}\in\Gamma$, and $\{(v_k,x_k)\}$ weakly converges to the point $(0,\hat{x})$ as $k\to\infty$.

Proof. First, we show that $\lim_{k\to\infty}\|x_k-x^*\|$ exists for any $x^*\in\Gamma$. Taking $x^*\in\Gamma$, we have $x^*\in F(U)$ and $Ax^*\in F(T)$. From (1.1), we can get

$$\begin{aligned}\|v_{k+1}\|^2 &= \|(I-U_\alpha)(y_k+(1-\lambda)v_k)\|^2\\ &\le \frac{2\alpha}{1-\beta_1}\langle y_k+(1-\lambda)v_k-x^*, v_{k+1}\rangle\end{aligned}$$

and

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &= \|y_k-\lambda v_{k+1}-x^*\|^2\\ &= \|y_k-x^*\|^2-2\lambda\langle y_k-x^*,v_{k+1}\rangle+\lambda^2\|v_{k+1}\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_{k+1}-x^*\|^2+\lambda\|v_{k+1}\|^2\\ =&\|y_k-x^*\|^2-2\lambda\langle y_k-x^*,v_{k+1}\rangle+\lambda^2\|v_{k+1}\|^2+\lambda\|v_{k+1}\|^2\\ =&\|y_k-x^*\|^2-2\lambda\langle y_k-x^*,v_{k+1}\rangle+\frac{1-\beta_1}{\alpha}\lambda\|v_{k+1}\|^2-\lambda(\frac{1-\beta_1}{\alpha}-1-\lambda)\|v_{k+1}\|^2\\ \le&\|y_k-x^*\|^2-2\lambda\langle y_k-x^*,v_{k+1}\rangle+2\lambda\langle y_k+(1-\lambda)v_k-x^*,v_{k+1}\rangle\\ &-\lambda(\frac{1-\beta_1}{\alpha}-1-\lambda)\|v_{k+1}\|^2\\ =&\|y_k-x^*\|^2+2\lambda(1-\lambda)\langle v_k,v_{k+1}\rangle-\lambda(\frac{1-\beta_1}{\alpha}-1-\lambda)\|v_{k+1}\|^2.\end{aligned}$$

Since

$$2\langle v_{k+1},v_k\rangle = \|v_{k+1}\|^2-\|v_{k+1}-v_k\|^2+\|v_k\|^2.$$

We obtain

$$\begin{aligned}&\|x_{k+1}-x^*\|^2+\lambda\|v_{k+1}\|^2\\ \le&\|y_k-x^*\|^2+\lambda(1-\lambda)\|v_k\|^2-\lambda(1-\lambda)\|v_{k+1}-v_k\|^2-\lambda(\frac{1-\beta_1}{\alpha}-2)\|v_{k+1}\|^2.\end{aligned}$$

From Lemma 3.1, we include that

$$-\langle x_k-x^*,A^*(I-T_\alpha)Ax_k\rangle \le -\frac{1-\beta_2}{2\alpha}\|(I-T_\alpha)Ax_k\|^2,$$

which in turn yields that

$$\begin{aligned}
&\|y_k-x^*\|^2\\
=&\|x_k-\gamma_k A^*(I-T_\alpha)Ax_k-x^*\|^2\\
=&\|x_k-x^*\|^2-2\gamma_k\langle x_k-x^*,A^*(I-T_\alpha)Ax_k\rangle+\gamma_k{}^2\|A^*(I-T_\alpha)Ax_k\|^2\\
\le&\|x_k-x^*\|^2-\gamma_k\frac{1-\beta_2}{\alpha}\|(I-T_\alpha)Ax_k\|^2+\gamma_k{}^2\|A^*(I-T_\alpha)Ax_k\|^2\\
=&\|x_k-x^*\|^2-\gamma_k(\frac{1-\beta_2}{\alpha}\|(I-T_\alpha)Ax_k\|^2-\gamma_k\|A^*(I-T_\alpha)Ax_k\|^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|x_{k+1}-x^*\|^2+\lambda\|v_{k+1}\|^2\\
\le&\|x_k-x^*\|^2-\gamma_k(\frac{1-\beta_2}{\alpha}\|(I-T_\alpha)Ax_k\|^2-\gamma_k\|A^*(I-T_\alpha)Ax_k)\|^2)+\lambda(1-\lambda)\|v_k\|^2\\
&-\lambda(1-\lambda)\|v_{k+1}-v_k\|^2-\lambda(\frac{1-\beta_1}{\alpha}-2)\|v_{k+1}\|^2\\
=&\|x_k-x^*\|^2+\lambda\|v_k\|^2-\gamma_k(\frac{1-\beta_2}{\alpha}\|(I-T_\alpha)Ax_k\|^2-\gamma_k\|A^*(I-T_\alpha)Ax_k\|^2)\\
&-\lambda(1-\lambda)\|v_{k+1}-v_k\|^2-\lambda^2\|v_k\|^2-\lambda(\frac{1-\beta_1}{\alpha}-2)\|v_{k+1}\|^2.
\end{aligned}\tag{3.2}$$

Let $s_k=\|x_k-x^*\|^2+\lambda\|v_k\|^2$. For the case of $(I-T_\alpha)Ax_k=0$, we have

$$s_{k+1}\le s_k-\lambda^2\|v_k\|^2-\lambda(1-\lambda)\|v_{k+1}-v_k\|^2-\lambda(\frac{1-\beta_1}{\alpha}-2)\|v_{k+1}\|^2,\tag{3.3}$$

otherwise,

$$\begin{aligned}
s_{k+1}\le&s_k-\rho_k(\frac{1-\beta_2}{\alpha}-\rho_k)\frac{\|(I-T_\alpha)Ax_k\|^4}{\|A^*(I-T_\alpha)Ax_k\|^2}-\lambda^2\|v_k\|^2-\lambda(1-\lambda)\|v_{k+1}-v_k\|^2\\
&-\lambda(\frac{1-\beta_1}{\alpha}-2)\|v_{k+1}\|^2.
\end{aligned}\tag{3.4}$$

From the assumption on ρ_k and λ, we get $s_{k+1}\le s_k$, which implies that $\{s_k\}$ is non-increasing and lower bounded by 0. Hence $\lim_{k\to\infty}s_k$ exists. It follows that $\{s_k\}$ is bounded, so is $\{x_k\}$. From (3.3) and (3.4), we have $\lambda^2\|v_k\|^2\le s_k-s_{k+1}$, which implies that

$$\lim_{k\to\infty}\|v_k\|=0.\tag{3.5}$$

So $\lim_{k\to\infty}\|x_k-x^*\|^2=\lim_{k\to\infty}(s_k-\lambda\|v_k\|^2)=\lim_{k\to\infty}s_k$ exists.

Next, we show that $\lim_{k\to\infty}\|(I-T_\alpha)Ax_k\|=\lim_{k\to\infty}\|x_k-y_k\|=0$. If $\|(I-T_\alpha)Ax_k\|=0$, then

$$x_k-y_k=\gamma_k A^*(I-T_\alpha)Ax_k=0.\tag{3.6}$$

Otherwise, it follows from (3.4) that

$$\rho_k(\frac{1-\beta_2}{\alpha}-\rho_k)\frac{\|(I-T_\alpha)Ax_k\|^4}{\|A^*(I-T_\alpha)Ax_k\|^2}\le s_k-s_{k+1},$$

which implies

$$\lim_{k\to\infty}\frac{\|(I-T_\alpha)Ax_k\|^4}{\|A^*(I-T_\alpha)Ax_k\|^2}=0.$$

Hence, we have

$$\lim_{k\to\infty} \frac{\|(I-T_\alpha)Ax_k\|^2}{\|A^*(I-T_\alpha)Ax_k\|} = 0, \tag{3.7}$$

which implies that

$$\begin{aligned}\frac{1}{\|A\|}\|(I-T_\alpha)Ax_k\| &= \|(I-T_\alpha)Ax_k\| \frac{\|(I-T_\alpha)Ax_k\|}{\|A\|\|(I-T_\alpha)Ax_k\|} \\ &\le \|(I-T_\alpha)Ax_k\| \frac{\|(I-T_\alpha)Ax_k\|}{\|A^*(I-T_\alpha)Ax_k\|} \\ &= \frac{\|(I-T_\alpha)Ax_k\|^2}{\|A^*(I-T_\alpha)Ax_k\|} \to 0\end{aligned}$$

as $k\to\infty$. We obtain

$$\lim_{k\to\infty} \|(I-T)Ax_k\| = \frac{1}{\alpha}\lim_{k\to\infty} \|(I-T_\alpha)Ax_k\| = 0. \tag{3.8}$$

On the other hand, we also have

$$\|x_k - y_k\| = \|\gamma_k A^*(I-T_\alpha)Ax_k\| = \rho_k \frac{\|(I-T_\alpha)Ax_k\|^2}{\|A^*(I-T_\alpha)Ax_k\|} \to 0$$

as $k\to\infty$. It follows from (3.6) that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{3.9}$$

In view of (3.5), we have

$$\begin{aligned}&\lim_{k\to\infty} \|y_k + (1-\lambda)v_k - U(y_k + (1-\lambda)v_k)\| \\ =&\lim_{k\to\infty} \frac{1}{\alpha}\|y_k + (1-\lambda)v_k - U_\alpha(y_k + (1-\lambda)v_k)\| \\ =&\frac{1}{\alpha}\lim_{k\to\infty} \|v_{k+1}\| = 0.\end{aligned} \tag{3.10}$$

Now, we prove that $\omega_w(x_k)\subset\Gamma$. Assume that $\tilde{x}\in\omega_w(x_k)$, i.e., there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j}\rightharpoonup\tilde{x}$ as $j\to\infty$. Then, we have $Ax_{k_j}\rightharpoonup A\tilde{x}$. From (3.5) and (3.9), we have

$$y_{k_j} + (1-\lambda)v_{k_j} \rightharpoonup \tilde{x}$$

as $j\to\infty$. Since $I-U$ and $I-T$ are demiclosed at origin, it follows form (3.8) and (3.10) that $\tilde{x}\in F(U_\alpha) = F(U)$ and $A\tilde{x}\in F(T_\alpha) = F(T)$, which implies $\tilde{x}\in\Gamma$. So $\omega_w(x_k)\subset\Gamma$.

Finally, it follows from Lemma 2.3 that $x_k\rightharpoonup\hat{x}$, where $\hat{x}$ is a solution of the SCFP (1.2). Thus, it follows $v_k\to 0$ that $\{(v_k,x_k)\}\rightharpoonup(0,\hat{x})$. This complete the proof. □

4. The Strong Convergence Theorem

In this section, we modify Algorithm 3.1 and prove the following strong convergence theorem.

Theorem 4.1. *Let H_1 and H_2 be two real Hilbert spaces. Let U be a β_1-demicontractive operator on H_1 and let T be a β_2-demicontractive operator on H_2 such that $I-U$ and $I-T$ are* *demiclosed at origin. Let* $f : H_1 \to H_1$ *be a* ρ*-contractive mapping with* $0 \le \rho \le \frac{1}{\sqrt{2}}$ *and* x_0 *and* v_0 *be two arbitrary vectors in* H_1*. Let* $\{(v_k, x_k)\}$ *be a sequence generated by*

$$\begin{cases} y_k = x_k - \gamma_k A^*(I - T_\alpha)Ax_k, \\ \bar{v}_k = (I - U_\alpha)(y_k + (1-\lambda)v_k), \\ \bar{x}_k = y_k - \lambda \bar{v}_k, \\ v_{k+1} = \theta_k f(v_k) + (1-\theta_k)\bar{v}_k, \\ x_{k+1} = \theta_k f(x_k) + (1-\theta_k)\bar{x}_k, \end{cases}$$

where $\{\theta_k\} \subset (0,1)$ *such that* $\theta_k \to 0$ *as* $k \to \infty$*,* $\sum_{k=0}^{\infty} \theta_k = \infty$*,* $0 < \alpha < \frac{1-\beta_1}{2}$*. The stepsize* γ_k *is chosen by (3.1) with* $0 < \lim_{k\to\infty} \inf \rho_k \le \lim_{k\to\infty} \sup \rho_k < \frac{1-\beta_2}{\alpha}$*. Then the sequence* $\{(v_k, x_k)\}$ *strongly converges to* $(0, x^*)$*, where* x^* *is a solution of the SCFP (1.2) and* x^* *solves the following variational inequality problem:*

$$\langle (I - f)x^*, x - x^* \rangle \ge 0, \quad \forall x \in \Gamma. \tag{4.1}$$

Proof. Let $x^* \in \Gamma$ be the solution of variational inequality (4.1). From (3.2), one has

$$\begin{aligned} &\|\bar{x}_k - x^*\|^2 + \lambda \|\bar{v}_k\|^2 \\ \le &\|x_k - x^*\|^2 + \lambda\|v_k\|^2 - \gamma_k\Big(\frac{1-\beta_2}{\alpha}\|(I - T_\alpha)Ax_k\|^2 - \gamma_k\|A^*(I - T_\alpha)Ax_k\|^2\Big) \\ &- \lambda(1-\lambda)\|\bar{v}_k - v_k\|^2 - \lambda^2\|v_k\|^2 - \lambda\Big(\frac{1-\beta_1}{\alpha} - 2\Big)\|\bar{v}_k\|^2. \end{aligned} \tag{4.2}$$

Hence,

$$\|\bar{x}_k - x^*\|^2 + \lambda\|\bar{v}_k\|^2 \le \|x_k - x^*\|^2 + \lambda\|v_k\|^2. \tag{4.3}$$

It follows that

$$\begin{aligned} &\|x_{k+1} - x^*\|^2 + \lambda\|v_{k+1}\|^2 \\ \le &\theta_k\|f(x_k) - x^*\|^2 + (1-\theta_k)\|\bar{x}_k - x^*\|^2 + \lambda(\theta_k\|f(v_k)\|^2 + (1-\theta_k)\|\bar{v}_k\|^2) \\ \le &2\theta_k(\|f(x_k) - f(x^*)\|^2 + \|f(x^*) - x^*\|^2) + (1-\theta_k)\|\bar{x}_k - x^*\|^2 \\ &\quad + \lambda[2\theta_k(\|f(v_k) - f(0)\|^2 + \|f(0)\|^2) + (1-\theta_k)\|\bar{v}_k\|^2] \\ \le &2\theta_k\rho^2\|x_k - x^*\|^2 + 2\theta_k\|f(x^*) - x^*\|^2 + (1-\theta_k)\|\bar{x}_k - x^*\|^2 \\ &\quad + 2\lambda\theta_k\rho^2\|v_k\|^2 + 2\lambda\theta_k\|f(0)\|^2 + \lambda(1-\theta_k)\|\bar{v}_k\|^2 \\ \le &(1-\theta_k)(\|x_k - x^*\|^2 + \lambda\|v_k\|^2) + 2\theta_k\rho^2(\|x_k - x^*\|^2 + \lambda\|v_k\|^2) \\ &\quad + 2\theta_k(\|f(x^*) - x^*\|^2 + \lambda\|f(0)\|^2) \\ = &[1 - \theta_k(1-2\rho^2)](\|x_k - x^*\|^2 + \lambda\|v_k\|^2) \\ &\quad + \theta_k(1-2\rho^2)\frac{2}{1-2\rho^2}(\|f(x^*) - x^*\|^2 + \lambda\|f(0)\|^2). \end{aligned}$$

Setting $s_k = \|x_k - x^*\|^2 + \lambda\|v_k\|^2$, we have

$$s_{k+1} \le [1 - \theta_k(1-2\rho^2)]s_k + \theta_k(1-2\rho^2)\frac{2}{1-2\rho^2}(\|f(x^*) - x^*\|^2 + \lambda\|f(0)\|^2).$$

It follows that

$$s_k \leq \max\{s_0, \frac{2}{1-2\rho^2}(\|f(x^*)-x^*\|^2+\lambda\|f(0)\|^2)\}$$

for each $k \geq 0$. That implies that $\{x_k\}$ and $\{v_k\}$ are bounded. Hence $\{\bar{x}_k\}$, $\{\bar{v}_k\}$, $\{f(x_k)\}$ and $\{f(v_k)\}$ are all bounded. Note that

$$\begin{aligned}
&\|x_{k+1}-x^*\|^2+\lambda\|v_{k+1}\|^2 \\
=&\theta_k^2\|f(x_k)-x^*\|^2+2\theta_k(1-\theta_k)\langle f(x_k)-x^*,\bar{x}_k-x^*\rangle+(1-\theta_k)^2\|\bar{x}_k-x^*\|^2 \\
&+\lambda(\theta_k^2\|f(v_k)\|^2+2\theta_k(1-\theta_k)\langle f(v_k),\bar{v}_k\rangle+(1-\theta_k)^2\|\bar{v}_k\|^2) \\
=&\theta_k^2\|f(x_k)-x^*\|^2+2\theta_k(1-\theta_k)(\langle f(x_k)-f(x^*),\bar{x}_k-x^*\rangle \\
&+\langle f(x^*)-x^*,\bar{x}_k-x^*\rangle)+(1-\theta_k)^2\|\bar{x}_k-x^*\|^2+\lambda\theta_k^2\|f(v_k)\|^2 \\
&+2\lambda\theta_k(1-\theta_k)(\langle f(v_k)-f(0),\bar{v}_k\rangle+\langle f(0),\bar{v}_k\rangle)+\lambda(1-\theta_k)^2\|\bar{v}_k\|^2 \\
\leq&\theta_k^2\|f(x_k)-x^*\|^2+\theta_k(1-\theta_k)(\|f(x_k)-f(x^*)\|^2+\|\bar{x}_k-x^*\|^2) \\
&+2\theta_k(1-\theta_k)\langle f(x_k)-x^*,\bar{x}_k-x^*\rangle+(1-\theta_k)^2\|\bar{x}_k-x^*\|^2+\lambda\theta_k^2\|f(v_k)\|^2 \\
&+\lambda\theta_k(1-\theta_k)(\|f(v_k)-f(0)\|^2+\|\bar{v}_k\|^2) \\
&+2\lambda\theta_k(1-\theta_k)\langle f(0),\bar{v}_k\rangle+\lambda(1-\theta_k)^2\|\bar{v}_k\|^2 \\
\leq&\theta_k^2\|f(x_k)-x^*\|^2+\theta_k(1-\theta_k)\rho^2\|x_k-x^*\|^2+(1-\theta_k)\|\bar{x}_k-x^*\|^2 \\
&+2\theta_k(1-\theta_k)(\langle f(x^*)-x^*,\bar{x}_k-x^*\rangle+\lambda\theta_k^2\|f(v_k)\|^2 \\
&+\lambda\theta_k(1-\theta_k)\rho^2\|v_k\|^2+\lambda(1-\theta_k)\|\bar{v}_k\|^2+2\lambda\theta_k(1-\theta_k)\langle f(0),\bar{v}_k\rangle.
\end{aligned} \tag{4.4}$$

From (4.3) and (4.4), we have

$$\begin{aligned}
s_{k+1}\leq&[1-\theta_k(1-(1-\theta_k)\rho^2)]s_k+\theta_k[\theta_k(\|f(x_k)-x^*\|^2+\lambda\|f(v_k)\|^2) \\
&+2(1-\theta_k)(\langle f(x^*)-x^*,\bar{x}_k-x^*\rangle+\lambda\langle f(0),\bar{v}_k\rangle)] \\
=&(1-\lambda_k)s_k+\lambda_k\delta_k,
\end{aligned}$$

where $\lambda_k=\theta_k(1-(1-\theta_k)\rho^2)$ and

$$\begin{aligned}
\delta_k=&\frac{2(1-\theta_k)(\langle f(x^*)-x^*,\bar{x}_k-x^*\rangle+\lambda\langle f(0),\bar{v}_k\rangle)}{1-(1-\theta_k)\rho^2} \\
&+\frac{\theta_k(\|f(x_k)-x^*\|^2+\lambda\|f(v_k)\|^2)}{1-(1-\theta_k)\rho^2}.
\end{aligned}$$

On the other hand, form (4.2) and (4.4), we see that

$$\begin{aligned} s_{k+1} \leq & [1-\theta_k(1-(1-\theta_k)\rho^2)]s_k + \theta_k^2(\|f(x_k)-x^*\|^2 + \lambda\|f(v_k)\|^2) \\ & + 2\theta_k(1-\theta_k)(\langle f(x^*)-x^*, \bar{x}_k - x^*\rangle + \lambda\langle f(0), \bar{v}_k\rangle) \\ & - (1-\theta_k)[\gamma_k(\frac{1-\beta_2}{\alpha}\|(I-T_\alpha)Ax_k\|^2 - \gamma_k\|A^*(I-T_\alpha)Ax_k\|^2) \\ & + \lambda(1-\lambda)\|\bar{v}_k - v_k\|^2 + \lambda^2\|v_k\|^2 + \lambda(\frac{1-\beta_1}{\alpha}-2)\|\bar{v}_k\|^2] \\ \leq & s_k + \theta_k^2(\|f(x_k)-x^*\|^2 + \lambda\|f(v_k)\|^2) \\ & + 2\theta_k(1-\theta_k)(\langle f(x^*)-x^*, \bar{x}_k - x^*\rangle + \lambda\langle f(0), \bar{v}_k\rangle) \\ & - (1-\theta_k)[\gamma_k(\frac{1-\beta_2}{\alpha}\|(I-T_\alpha)Ax_k\|^2 - \gamma_k\|A^*(I-T_\alpha)Ax_k\|^2) \\ & + \lambda(1-\lambda)\|\bar{v}_k - v_k\|^2 + \lambda^2\|v_k\|^2 + \lambda(\frac{1-\beta_1}{\alpha}-2)\|\bar{v}_k\|^2] \\ \leq & s_k + \theta_k^2(\|f(x_k)-x^*\|^2 + \lambda\|f(v_k)\|^2) \\ & + 2\theta_k(1-\theta_k)(\langle f(x^*)-x^*, \bar{x}_k - x^*\rangle + \lambda\langle f(0), \bar{v}_k\rangle) \\ & - (1-\theta_k)[\rho_k(\frac{1-\beta_2}{\alpha}-\rho_k)\frac{\|(I-T_\alpha)Ax_k\|^4}{\|A^*(I-T_\alpha)Ax_k\|^2} \\ & + \lambda(1-\lambda)\|\bar{v}_k - v_k\|^2 + \lambda^2\|v_k\|^2 + \lambda(\frac{1-\beta_1}{\alpha}-2)\|\bar{v}_k\|^2]. \end{aligned} \tag{4.5}$$

Set

$$\begin{aligned} \mu_k = & \theta_k^2(\|f(x_k)-x^*\|^2 + \lambda\|f(v_k)\|^2) \\ & + 2\theta_k(1-\theta_k)(\langle f(x^*)-x^*, \bar{x}_k - x^*\rangle + \lambda\langle f(0), \bar{v}_k\rangle) \end{aligned}$$

and

$$\begin{aligned} \eta_k = & (1-\theta_k)[\rho_k(\frac{1-\beta_2}{\alpha}-\rho_k)\frac{\|(I-T_\alpha)Ax_k\|^4}{\|A^*(I-T_\alpha)Ax_k\|^2} \\ & + \lambda(1-\lambda)\|\bar{v}_k - v_k\|^2 + \lambda^2\|v_k\|^2 + \lambda(\frac{1-\beta_1}{\alpha}-2)\|\bar{v}_k\|^2], \end{aligned}$$

one has $s_{k+1} \leq s_k - \eta_k + \mu_k$ for each $k \geq 0$. From the assumptions on $\{\theta_k\}$ and $\{\rho_k\}$, we have $\sum_{k=0}^{\infty}\lambda_k = \infty$, and $\lim_{k\to\infty}\mu_k = 0$. To use Lemma 2.4, it suffices to verify that, for all subsequence $\{k_l\} \subseteq \{k\}$, $\lim_{l\to\infty}\eta_{k_l} = 0$ implies

$$\limsup_{l\to\infty}\delta_{k_l} \leq 0. \tag{4.6}$$

In view of $\lim_{l\to\infty}\eta_{k_l} = 0$, we have

$$\begin{aligned} & \lim_{l\to\infty}\rho_{k_l}(\frac{1-\beta_2}{\alpha}-\rho_{k_l})\frac{\|(I-T_\alpha)Ax_{k_l}\|^4}{\|A^*(I-T_\alpha)Ax_{k_l}\|^2} + \lambda(1-\lambda)\|\bar{v}_{k_l} - v_{k_l}\|^2 \\ & + \lambda^2\|v_{k_l}\|^2 + \lambda(\frac{1-\beta_1}{\alpha}-2)\|\bar{v}_{k_l}\|^2 = 0. \end{aligned}$$

From the assumptions on λ and $\{\rho_k\}$, we obtain

$$\lim_{l\to\infty}\frac{\|(I-T_\alpha)Ax_{k_l}\|^4}{\|A^*(I-T_\alpha)Ax_{k_l}\|^2} = 0, \tag{4.7}$$

and

$$\lim_{l\to\infty}\|v_{k_l}\| = \lim_{l\to\infty}\|\bar{v}_{k_l}\| = 0. \tag{4.8}$$

In a similar way as in Theorem 3.1, we find that

$$\lim_{l\to\infty}\|(I-T_\alpha)Ax_{k_l}\| = \alpha\lim_{l\to\infty}\|(I-T)Ax_{k_l}\| = 0,$$

and $\omega_w(x_{k_l})\subseteq\Gamma$. In addition, from (4.7), (4.8) and

$$\begin{aligned}\|x_{k_l}-\bar{x}_{k_l}\| =&\|x_{k_l}-x_{k_l}+\gamma_{k_l}A^*(I-T_\alpha)Ax_{k_l}+\lambda\bar{v}_{k_l}\|\\ =&\|\gamma_{k_l}A^*(I-T_\alpha)Ax_{k_l}+\lambda\bar{v}_{k_l}\|\\ \le&\gamma_{k_l}\|A\|\|(I-T_\alpha)Ax_{k_l}\|+\lambda\|\bar{v}_{k_l}\|,\end{aligned}$$

we obtain

$$\lim_{l\to\infty}\|x_{k_l}-\bar{x}_{k_l}\| = 0. \tag{4.9}$$

Note that

$$\lim_{l\to\infty}(1-(1-\theta_{k_l})\rho^2) = 1-\rho^2$$

and

$$\lim_{l\to\infty}\theta_{k_l}(\|f(x_{k_l})-x^*\|^2+\lambda\|f(v_{k_l})\|^2) = 0.$$

To see (4.6), we only need to verify

$$\limsup_{l\to\infty}(\langle f(x^*)-x^*,\bar{x}_{k_l}-x^*\rangle+\lambda\langle f(0),\bar{v}_{k_l}\rangle)\le 0.$$

From (4.9), we can take subsequence $\{(v_{k_{l_j}},x_{k_{l_j}})\}$ of $\{(v_{k_l},v_{k_l})\}$ such that $x_{k_{l_j}}\rightharpoonup\widetilde{x}$ as $j\to\infty$ and

$$\begin{aligned}&\limsup_{l\to\infty}(\langle f(x^*)-x^*,\bar{x}_{k_l}-x^*\rangle+\lambda\langle f(0),\bar{v}_{k_l}\rangle)\\ =&\lim_{j\to\infty}(\langle f(x^*)-x^*,\bar{x}_{k_{l_j}}-x^*\rangle+\lambda\langle f(0),\bar{v}_{k_{l_j}}\rangle)\\ =&\lim_{j\to\infty}\langle f(x^*)-x^*,x_{k_{l_j}}-x^*\rangle\\ =&\langle f(x^*)-x^*,\widetilde{x}-x^*\rangle.\end{aligned}\tag{4.10}$$

Since $\omega_w(x_{k_l})\subset\Gamma$ and x^* is a solution to variational inequality (4.1), it follows from (4.10) that

$$\limsup_{l\to\infty}(\langle f(x^*)-x^*,\bar{x}_{k_l}-x^*\rangle+\lambda\langle f(0),\bar{v}_{k_l}\rangle)\le 0.$$

In view of Lemma 2.4, we conclude that

$$\lim_{k\to\infty}s_k = \lim_{k\to\infty}(\|x_k-x^*\|^2+\lambda\|v_k\|^2) = 0,$$

which implies that $x_k\to x^*$, $v_k\to 0$ and $(v_k,x_k)\to(0,x^*)$, where x^* is a solution of the SCFP (1.2), which solves variational inequality (4.1). This completes the proof. □

5. Numerical Experiments

In this section, we provide some numerical experiments and show the performance of the proposed self-adaptive iterative algorithms for solving SCFP (1.2). All the codes are written in Matlab and are performed on a personal Thinkpad computer with Pentinum(R) CPU@1.6GHZ and RAM 4.00GB.

In the following examples, we take the stopping criteria is:

$$Error := \|(I-U)x_k\| + \|(I-T)Ax_k\| + \leq 10^{-9}.$$

Example 5.1. Define the operators $U : R^3 \to R^3$ and $T : R^6 \to R^6$ as follows:

$$U(x) = (z_1, z_2, z_3)^T, \ \ T(y) = (w_1, w_2, w_3, w_4, w_5, w_6)^T,$$

where $x = (x_1, x_2, x_3)^T, y = (y_1, y_2, y_3, y_4, y_5, y_6)^T$,

$$z_i = \begin{cases} x_i, & x_i < 0, \\ -2x_i, & x_i \geq 0 \end{cases}$$

and

$$w_j = \begin{cases} y_j, & y_j < 0, \\ -3y_i, & y_j \geq 0 \end{cases}$$

for $1 \leq i \leq 3$ and $1 \leq j \leq 6$. Note that U is $\frac{1}{3}-$ demicontractive and T is $\frac{1}{2}-$ demicontractive. Let

$$A = \begin{pmatrix} 2 & 6 & 3 \\ 1 & 4 & 6 \\ 3 & 7 & 4 \\ 5 & 2 & 6 \\ 4 & 8 & 4 \\ 2 & 4 & 1 \end{pmatrix}.$$

We consider the following SCFP which consists of finding a point

$$x^* \in F(U) \ \ \text{such that} \ \ A^* \in F(T).$$

We apply the proposed self-adaptive iterative Algorithm 3.1 to solve this example. We take $\alpha = \frac{1}{5}$ and $\rho_k = 1.4$ in the experiment. We take different initial points $x_0 = (10, -20, 10)^T$, $v_0 = (0,0,0)^T$ and $x_0 = (5, -10, 5)^T$, $v_0 = (0,0,0)^T$ for this example. Since the parameter $\lambda \in (0,1]$ is important in the algorithm 3.1, we try different values of λ for solving this example. The numerical results are given in Table 1. Denote x_k as the kth iterative sequence.

In Figure 1, we compare Algorithm 3.1 for $\lambda = 0.95$ with algorithm (1.3). We take $\alpha = \frac{23}{100}, \rho_k = 1.4$, initial point $x_0 = (30, -30, 30)^T$ and $v_0 = (5,5,5)^T$. We observe that Algorithm 3.1 behaves better than algorithm (1.3) for suitable parameter $0 < \lambda < 1$.

Example 5.2. Define the operator $U : R^2 \to R^2$ and $T : R^2 \to R^2$ as follows:

$$U(x) = (\frac{1}{3}x_1, \frac{1}{3}x_2)^T, T(x) = (z_1, z_2)^T$$

where $x = (x_1, x_2)^T$ and

$$z_i = \begin{cases} x_i, & x_i < 0, \\ -x_i, & x_i \geq 0 \end{cases}$$

TABLE 1. Numerical results for solving Example 5.1 with different λ.

Initials	λ	Iterations	Error	x_k
$x_0 = (10,-20,20)^T$ $v_0 = (0,0,0)^T$	0.5	7	0	$(-0.0149,-25.3506,-0.0595)^T$
	0.6	7	0	$(-0.0140,-25.3506,-0.0562)^T$
	0.7	7	0	$(-0.0059,-25.3506,-0.0234)^T$
	0.8	8	0	$(-0.0004,-25.3506,-0.0016)^T$
	0.9	25	3.5618e-10	$(-0.0000,-25.3506,-0.0000)^T$
	1	28	4.143e-10	$(-0.0000,-25.3506,-0.0000)^T$
$x_0 = (5,-10,5)^T$ $v_0 = (0,0,0)^T$	0.5	7	0	$(-0.0159,-11.2062,-0.0111)^T$
	0.6	7	0	$(-0.0150,-11.2062,-0.0104)^T$
	0.7	7	0	$(-0.0063,-11.2062,-0.0044)^T$
	0.8	8	0	$(-0.0004,-11.2062,-0.0003)^T$
	0.9	23	9.3210e-10	$(0.0000,-11.2062,0.0000)^T$
	1	26	8.1677e-10	$(0.0000,-11.2062,0.0000)^T$

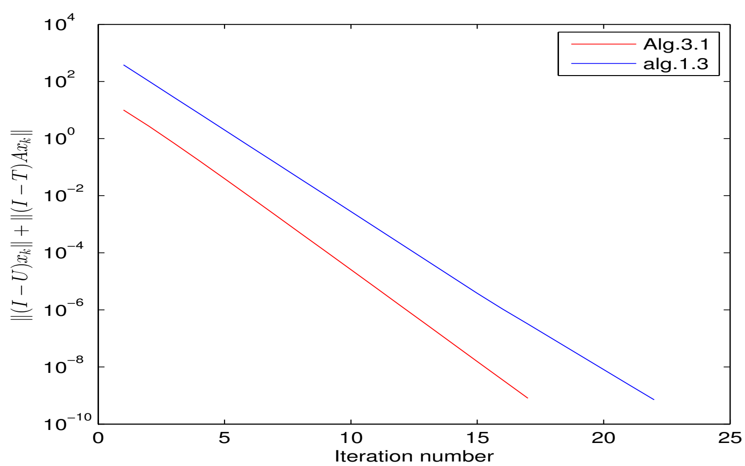

FIGURE 1. Comparison of the number of iteration of Algorithm 3.1 with algorithm (1.3).

for $1 \le i \le 2$. Obviously, U and T are 0-demicontractive and $F(U) \bigcap F(T) = \{(0,0)\}^T$. Let

$$A = \begin{pmatrix} 3 & 5 \\ -6 & 3 \end{pmatrix}.$$

We consider the following SCFP which consists of finding

$$x \in F(U) \quad \text{such that } Ax \in F(T).$$

We apply the proposed iterative Algorithm 3.1 and the algorithm (1.3) to solve the above SCFP. In Figure 2, we compare Algorithm 3.1 for $\lambda = 0.8$ with algorithm (1.3). We take $\alpha = \frac{9}{25}, \rho_k = 0.8$, the initial point $x_0 = (5,-5)^T$ and $v_0 = (0,0)^T$. We observe that Algorithm 3.1 behaves better than algorithm (1.3) for suitable parameter $0 < \lambda < 1$ in this example.

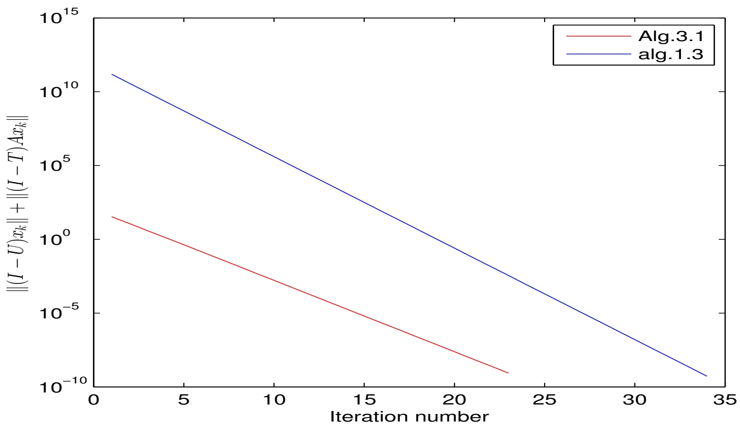

FIGURE 2. Comparison of the number of iteration of Algorithm 3.1 with algorithm (1.3).

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