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EMBEDDING DERIVATIVES OF FOCK SPACES AND GENERALIZED WEIGHTED COMPOSITION OPERATORS

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Abstract. In this paper, we give complete characterizations of those positive Borel measures such that the differentiation operator $D^{(n)}f = f^{(n)}$ is bounded or compact from Fock spaces $F^p(\mathbb{C})$ into a class weighted Lebesgue spaces on complex plane $\mathbb{C}$. As an application, we characterize the boundedness and compactness of generalized weighted composition operators acting on Fock spaces. We also study the essential norm of these operators on Fock spaces.

Keywords. Fock space; Generalized weighted composition operator; Carleson measure.

1. INTRODUCTION

Let $H(\mathbb{C})$ be the space of entire functions on $\mathbb{C}$. For $0 < p < \infty$, the Fock space $F^p(\mathbb{C})$ is the set of entire functions on complex plane $\mathbb{C}$ with

$$\|f\|_p^p = \frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}}\, dA(z) < \infty,$$

where dA is the Lebesgue measure on $\mathbb{C}$. For $p = \infty$, the space $F^\infty(\mathbb{C})$ contains entire functions on $\mathbb{C}$ with

$$\|f\|_\infty = \operatorname*{ess\,sup}_{z\in\mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty.$$

It is well known that when $1 \leqslant p \leqslant \infty$ the space $F^p(\mathbb{C})$ is a Banach space. When $0 < p < 1$, $F^p(\mathbb{C})$ is a complete metric space with the distance $d(f,g) = \|f-g\|_p^p$. For any $w \in \mathbb{C}$, let

$$k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}, \quad z \in \mathbb{C}.$$

It is clear that each k_w is a unit vector in $F^p(\mathbb{C})$ for $0 < p \leqslant \infty$, and k_w converges to 0 as $|w| \to \infty$. For more details about Fock spaces, we refer the reader to [1]. For the study of Toeplitz operators and weighted composition operators on Fock spaces, we refer to [2, 3, 4, 5].

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In 1962, Carleson [6] characterized the positive Borel measure μ on the unit disk $\mathbb{D}$ such that the identity operator $I_d : H^p \to L^p(d\mu)$ is bounded. Indeed, he proved that, for each $f \in H^p$ $(1 < p < \infty)$,

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq \|f\|_{H^p}^p$$

holds if and only if there is a positive constant C such that, for all arcs $I \subseteq \partial\mathbb{D}$, $\mu(S(I)) \leq C|I|$, which was called the Carleson measure. Here $S(I) = \left\{z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I\right\}$. In [7], Duren generalized Carleson's result and characterized the positive Borel measure μ on the unit disk $\mathbb{D}$ such that the identity operator $I_d : H^p \to L^q(d\mu)$ is bounded when $0 < p \leq q < \infty$. Characterizations of when the differentiation operator $D^{(n)} f = f^{(n)}$ is bounded or compact from H^p into $L^p(d\mu)$ were given in [8, 9, 10]. For recent related results, we refer to [2, 3, 8, 9, 10, 11, 12, 13] and the references therein.

Let $0 < q < \infty$ and μ a positive Borel measure. Let $L^q_W(\mathbb{C}, \mu)$ denote the space of all measure functions f on complex plane $\mathbb{C}$ such that

$$\int_{\mathbb{C}} \left| f(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) < \infty.$$

Let $0 < p \leqslant \infty$, $0 < q < \infty$, n be a nonnegative integer, and μ be a positive Borel measure on complex plane $\mathbb{C}$. One of the aims of this paper is to characterize that the differentiation operator $D^{(n)} f = f^{(n)}$ is bounded or compact from Fock spaces $F^p(\mathbb{C})$ into $L^q_W(\mathbb{C}, \mu)$. We say that $D^{(n)} : F^p(\mathbb{C}) \to L^q_W(\mathbb{C}, \mu)$ is compact if

$$\lim_{j \to \infty} \int_{\mathbb{C}} \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) = 0,$$

whenever $\{f_j\}$ is a bounded sequence in $F^p(\mathbb{C})$ that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $j \to \infty$. We remark here that Rozenblum and Vasilevski studied the boundedness and compactness of $D^{(n)}$ when $p = q = 2$ in [14], i.e., they showed that the equivalence of (1) and (3) in Theorem 3.1 in this paper when $p = q = 2$. For the simplicity, we say that μ is a (p, q, n) Fock-Carleson measure if $D^{(n)} : F^p(\mathbb{C}) \to L^q_W(\mathbb{C}, \mu)$ is bounded, and μ is a vanishing (p, q, n) Fock-Carleson measure if $D^{(n)} : F^p(\mathbb{C}) \to L^q_W(\mathbb{C}, \mu)$ is compact.

Let φ be an analytic self-map of $\mathbb{C}$, ψ an entire function on $\mathbb{C}$, and n a nonnegative integer. The generalized weighted composition operator is defined as follows:

$$W^{(n)}_{\psi,\varphi}(f) = \psi(f^{(n)} \circ \varphi),$$

where $f^{(n)}$ is n-th derivative of f. If $\psi \equiv 1$, $n = 0$, then the generalized weighted composition operator reduces to the well-known composition operator:

$$C_\varphi(f) = f \circ \varphi.$$

The generalized weighted composition operator acting on several analytic function spaces on the unit disk has been studied in, for example, [15, 16, 17, 18, 19, 20]. Weighted composition operators and generalized weighted composition operators acting on Fock spaces have been studied in, for example, [4, 21, 22, 23, 24].

In this paper, we give complete characterizations of the boundedness and compactness of the operator $D^{(n)} : F^p(\mathbb{C}) \to L^q_w(\mathbb{C}, \mu)$. By using these results, we give complete characterizations of boundedness and compactness for generalized weighted composition operator on Fock spaces. Also, we investigate the essential norm of generalized weighted composition operators on $F^p(\mathbb{C})$.

The structure of our paper is as follows. In Section 2, we collect some basic facts about Fock spaces. In particular, we introduce the Berezin type transforms, averaging functions, and averaging sequences in our setting. Several lemmas will be given, which will be used frequently in later sections. In Section 3, we give complete characterizations by using Berezin type transforms, averaging functions and averaging sequences. In Section 4, the last section, we study the boundedness, compactness and essential norm for generalized weighted composition operators acting on Fock spaces.

Finally, throughout this paper, for $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means that there exists a positive number C, which is independent of a and b, such that $a \leq Cb$ ($a \geq Cb$, respectively). Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then we say $a \simeq b$.

2. Preliminaries

In this section, we give some auxiliary results, which will be used frequently in the following sections.

Lemma 2.1. *([1, Theorem 2.10]) If $0 < p < q < \infty$, then $F^p(\mathbb{C}) \subset F^q(\mathbb{C})$, and the inclusion is proper and continuous. Moreover,*

$$\|f\|_q \leqslant \left(\frac{q}{p}\right)^{\frac{1}{q}} \|f\|_p$$

for any $f \in F^p(\mathbb{C})$.

The following two lemmas were proved by Rozenblum and Vasilevski in [14, Propositions 5.1 and 5.2] for $p = 2$.

Lemma 2.2. *Let $0 < p \leqslant \infty$ and let n be a positive integer. If $f \in F^p(\mathbb{C})$, then*

$$|f^{(n)}(z)| \leqslant e^2 n! (1 + |z|)^n e^{\frac{|z|^2}{2}} \|f\|_p.$$

Proof. For $f \in F^p(\mathbb{C})$, from [1, Corollary 2.8], we have

$$|f(z)| \leqslant e^{\frac{|z|^2}{2}} \|f\|_p, \quad z \in \mathbb{C}. \tag{2.1}$$

For $|z| \leqslant 1$, Cauchy integral formula gives

$$\begin{aligned} |f^{(n)}(z)| &\leqslant \frac{n!}{2\pi} \int_{|z-\zeta|=1} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| \\ &\leqslant n! \max_{|z-\zeta|=1} |f(\zeta)| \\ &\leqslant n! \|f\|_p \max_{|z-\zeta|=1} e^{\frac{|\zeta|^2}{2}} \leqslant n! e^2 \|f\|_p. \end{aligned} \tag{2.2}$$

For $|z| > 1$, using Cauchy integral formula again, we obtain

$$\begin{aligned}
|f^{(n)}(z)| &\leqslant \frac{n!}{2\pi} \int_{|z-\zeta|=|z|^{-1}} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| \\
&\leqslant n!|z|^n \max_{|z-\zeta|=|z|^{-1}} |f(\zeta)| \\
&\leqslant n!|z|^n \|f\|_p \max_{|\zeta|\leqslant|z|+|z|^{-1}} e^{\frac{|\zeta|^2}{2}} \\
&= n!|z|^n e^{\frac{(|z|+|z|^{-1})^2}{2}} \|f\|_p \\
&\leqslant n! e^2 (1+|z|)^n e^{\frac{|z|^2}{2}} \|f\|_p. \qquad (2.3)
\end{aligned}$$

Combining (2.2) with (2.3), the desired estimate follows immediately. □

Lemma 2.3. *Let $0 < p < \infty$ and let n be a positive integer. If f is an entire function, then, for any $r > 1$,*

$$\left| f^{(n)}(z) e^{-\frac{|z|^2}{2}} \right| \leqslant C n! (1+|z|)^n \left(\int_{B(z,r)} \left| f(\zeta) e^{-\frac{|\zeta|^2}{2}} \right|^p dA(\zeta) \right)^{\frac{1}{p}},$$

where C is a constant depending only on r, and $B(z,r)$ denotes the disk with centre z and radius r.

Proof. If $|z| \leqslant 1$, for any $r_0 > 0$, (2.2) and mean value theorem imply that

$$\begin{aligned}
\left| f^{(n)}(z) e^{-\frac{|z|^2}{2}} \right| &\leqslant |f^{(n)}(z)| \\
&\leqslant n!(1+|z|)^n \max_{|\xi|\leqslant 2} |f(\xi)| \\
&\leqslant \frac{e^{\frac{(2+r_0)^2}{2}}}{(\pi r_0^2)^{\frac{1}{p}}} n!(1+|z|)^n \max_{|\xi|\leqslant 2} \left(\int_{B(\xi,r_0)} \left| f(\zeta) e^{-\frac{|\zeta|^2}{2}} \right|^p dA(\zeta) \right)^{\frac{1}{p}} \\
&\leqslant \frac{e^{\frac{(2+r_0)^2}{2}}}{(\pi r_0^2)^{\frac{1}{p}}} n!(1+|z|)^n \left(\int_{B(z,r_0+3)} \left| f(\zeta) e^{-\frac{|\zeta|^2}{2}} \right|^p dA(\zeta) \right)^{\frac{1}{p}}. \qquad (2.4)
\end{aligned}$$

If $|z| > 1$, we obtain from (2.3) that

$$\begin{aligned}
\left| f^{(n)}(z) e^{-\frac{|z|^2}{2}} \right| &\leqslant n!(1+|z|)^n e^{-\frac{|z|^2}{2}} \max_{|w-z|\leqslant|z|^{-1}} |f(w)| \\
&\leqslant e^2 n!(1+|z|)^n \max_{|w-z|\leqslant|z|^{-1}} \left(e^{-\frac{p|w|^2}{2}} |f(w)|^p \right)^{\frac{1}{p}} \\
&\lesssim e^2 n!(1+|z|)^n \max_{|w-z|\leqslant|z|^{-1}} \left(\int_{B(w,r_0)} \left| f(\zeta) e^{-\frac{|\zeta|^2}{2}} \right|^p dA(\zeta) \right)^{\frac{1}{p}} \\
&\lesssim e^2 n!(1+|z|)^n \left(\int_{B(z,r_0+1)} \left| f(\zeta) e^{-\frac{|\zeta|^2}{2}} \right|^p dA(\zeta) \right)^{\frac{1}{p}}, \qquad (2.5)
\end{aligned}$$

where the last two inequalities follows from [3, Lemma 2.1] and $B(w,r_0) \subset B(z,r_0+1)$, respectively. □

The following lemma is an analogous result of [2, Lemma 2.2].

Lemma 2.4. *Assume that μ is positive Borel measure on $\mathbb{C}$, n is a positive integer, and $0 < r, p < \infty$. Then*

$$\int_{\mathbb{C}} \left| f^{(n)}(w) e^{-\frac{|w|^2}{2}} \right|^p d\mu(w) \lesssim \int_{\mathbb{C}} \left| f(w) e^{-\frac{|w|^2}{2}} \right|^p (1+|w|)^{np} \mu(B(w,r)) dA(w).$$

Proof. By Lemma 2.3, we have

$$\left| f^{(n)}(z) e^{-\frac{|z|^2}{2}} \right| \lesssim (1+|z|)^n \left(\int_{B(z,r)} \left| f(w) e^{-\frac{|w|^2}{2}} \right|^p dA(w) \right)^{\frac{1}{p}}.$$

Combining this with

$$\frac{1+|z|}{1+|w|} \leqslant 1+|z-w|, \qquad z,w \in \mathbb{C}, \tag{2.6}$$

we obtain

$$\begin{aligned} \int_{\mathbb{C}} \left| f^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^p d\mu(z) &\lesssim \int_{\mathbb{C}} \left| f(w) e^{-\frac{|w|^2}{2}} \right|^p \int_{\mathbb{C}} \chi_{B(z,r)}(w) (1+|z|)^{np} d\mu(z) dA(w) \\ &\lesssim \int_{\mathbb{C}} \left| f(w) e^{-\frac{|w|^2}{2}} \right|^p \int_{\mathbb{C}} \chi_{B(w,r)}(z) (1+|z|)^{np} d\mu(z) dA(w) \\ &\lesssim \int_{\mathbb{C}} \left| f(w) e^{-\frac{|w|^2}{2}} \right|^p (1+|w|)^{np} \mu(B(w,r)) dA(w). \end{aligned}$$

The proof is complete. □

Now, let us recall the concept of lattices for complex plane $\mathbb{C}$. For $r > 0$, a sequence of distinct points $\{a_k\} \subset \mathbb{C}$ is an $\frac{r}{2}$-lattice if $\cup_{k=1}^{\infty} B(a_k, r) = \mathbb{C}$ and $\{B(a_k, \frac{r}{2})\}$ are pairwise disjoint. We will use such $\frac{r}{2}$-lattice $\{a_k\}$ with r fixed in the following sections.

Lemma 2.5. *([25, Lemma 2.1]) Let $r > 0$ and $\{a_k\}$ be $\frac{r}{2}$-lattice for $\mathbb{C}$. Then there exists an integer $N > 0$ such that every point in $\mathbb{C}$ belongs to at most N of the disks $B(a_k, 2r)$.*

Let μ be a positive Borel measure on $\mathbb{C}$. Because the area of $B(z,r)$ is a constant for $z \in \mathbb{C}$, we denote

$$\widehat{\mu}_r(z) := \mu(B(z,r))$$

as the averaging function of μ for simplicity.

For $s \geqslant 0$, $t > 0$, we define the (s,t)-Berezin type transform of a Borel measure μ on $\mathbb{C}$ by

$$\widetilde{\mu}_{(s,t)}(z) := \int_{\mathbb{C}} (1+|w|)^s e^{-\frac{t}{2}|z-w|^2} d\mu(w).$$

Observe that if $s = 0$, $\widetilde{\mu}_{(0,t)}$ is the t-Berezin transform $\widetilde{\mu}_t$, which was defined in [2]. The role of $\widetilde{\mu}_{(s,t)}$ is similar to $\widetilde{\mu}_t = \widetilde{\mu}_{(0,t)}$ in Fock spaces. Given a function f, let $\widetilde{f}_{(s,t)}$ denote the Berezin-type transform of f, namely

$$\widetilde{f}_{(s,t)}(z) := \int_{\mathbb{C}} f(w) (1+|w|)^s e^{-\frac{t}{2}|z-w|^2} dA(w).$$

Also, we set

$$\widehat{f}_r(z) := \int_{B(z,r)} f(w) dA(w), \; f_s(z) := (1+|z|)^s f(z),$$

and

$$F_{(s,r)}(z) := (1+|z|)^s \widehat{f}_r(z).$$

Lemma 2.6. *Let $1 \leqslant p \leqslant \infty$, $r > 0$, $t > 0$ and $s \geqslant 0$. Then the operator $f_s \mapsto \widetilde{f}_{(s,t)}$ and $f_s \mapsto F_{(s,r)}$ are bounded on $L^p(dA,\mathbb{C})$, respectively.*

Proof. We will use the interpolation argument. For $p = 1$, we have

$$\begin{aligned}
\|\widetilde{f}_{(s,t)}\|_{L^1(dA,\mathbb{C})} &\leqslant \int_{\mathbb{C}}\int_{\mathbb{C}} |f(w)|(1+|w|)^s e^{-\frac{t}{2}|z-w|^2} dA(w)dA(z) \\
&= \int_{\mathbb{C}} |f_s(w)| \int_{\mathbb{C}} e^{-\frac{t}{2}|z-w|^2} dA(z)dA(w) \\
&\lesssim \int_{\mathbb{C}} |f_s(w)| dA(w) \\
&= \|f_s\|_{L^1(dA,\mathbb{C})}.
\end{aligned}$$

From (2.6), we have

$$\begin{aligned}
\|F_{(s,r)}\|_{L^1(dA,\mathbb{C})} &\leqslant \int_{\mathbb{C}} (1+|z|)^s \int_{B(z,r)} |f(w)| dA(w)dA(z) \\
&\lesssim \int_{\mathbb{C}} |f(w)|(1+|w|)^s \int_{\mathbb{C}} \chi_{B(w,r)}(z) dA(z)dA(w) \\
&\lesssim \|f_s\|_{L^1(dA,\mathbb{C})}.
\end{aligned}$$

For $p = \infty$, we have

$$\begin{aligned}
\|\widetilde{f}_{(s,t)}\|_{L^\infty(\mathbb{C})} &= \operatorname*{ess\,sup}_{z\in\mathbb{C}} \left| \int_{\mathbb{C}} f(w)(1+|w|)^s e^{-\frac{t}{2}|z-w|^2} dA(w) \right| \\
&\leqslant \|f_s\|_{L^\infty(\mathbb{C})} \cdot \left(\operatorname*{ess\,sup}_{z\in\mathbb{C}} \int_{\mathbb{C}} e^{-\frac{t}{2}|z-w|^2} dA(w) \right) \\
&\lesssim \|f_s\|_{L^\infty(\mathbb{C})}
\end{aligned}$$

and

$$\begin{aligned}
\|F_{(s,r)}\|_{L^\infty(\mathbb{C})} &\lesssim \operatorname*{ess\,sup}_{z\in\mathbb{C}} \int_{B(z,r)} |f(\zeta)|(1+|\zeta|)^s dA(\zeta) \\
&\simeq \|f_s\|_{L^\infty(\mathbb{C})},
\end{aligned}$$

where we used (2.6) again. So, by Riesz-Thorin interpolation theorem, we obtain the desired results. □

For $s \geqslant 0$, let

$$\mu_s(z) := (1+|z|)^s \mu(z), \qquad \mu_{(s,r,B)}(z) := (1+|z|)^s \mu(B(z,r)).$$

Lemma 2.7. *Let $1 \leqslant p \leqslant \infty$, $0 \leqslant s < \infty$ and μ be a positive Borel measure. If $\mu_{(s,r_0,B)}$ belongs to $L^p(dA,\mathbb{C})$ for some $r_0 > 0$, then so does $\mu_{(s,r,B)}$ for all $r > 0$.*

Proof. The proof is similar to that of Lemma 4.2 in [25] with the aid of Lemma 2.6 instead of Lemma 4.1 in [25]. The details are omitted here. □

Lemma 2.8. *Let $1 \leqslant p \leqslant \infty$, $0 \leqslant s < \infty$ and μ be a positive Borel measure on $\mathbb{C}$. Then the following statements are equivalent:*

(1) $\widetilde{\mu}_{(s,t)} \in L^p(dA,\mathbb{C})$ *for some (or any)* $t > 0$*;*

(2) $\mu_{(s,r,B)} \in L^p(dA,\mathbb{C})$ *for some (or any)* $r > 0$*;*

(3) $\{\mu_{(s,r,B)}(a_k)\} \in \ell^p$ *for some (or any)* $r > 0$*, where* $\{a_k\}$ *is* $\frac{r}{2}$*-lattice on* $\mathbb{C}$*.*

Furthermore,

$$\|\widetilde{\mu}_{(s,t)}\|_{L^p(dA,\mathbb{C})} \simeq \|\mu_{(s,r,B)}\|_{L^p(dA,\mathbb{C})} \simeq \|\{\mu_{(s,r,B)}(a_k)\}\|_{\ell^p}. \tag{2.7}$$

Proof. $(1) \Rightarrow (2)$. For $0 \leqslant s < \infty$, $0 < r,t < \infty$,

$$\begin{aligned}\mu_{(s,r,B)}(z) &= (1+|z|)^s \int_{B(z,r)} d\mu(w) \\ &\lesssim \int_{B(z,r)} (1+|w|)^s e^{-\frac{t}{2}|z-w|^2} d\mu(w) \\ &\lesssim \widetilde{\mu}_{(s,t)}(z).\end{aligned} \tag{2.8}$$

$(2) \Rightarrow (1)$. For any entire function f and $t,r > 0$, [3, Lemma 2.1] yields

$$\begin{aligned}(1+|z|)^s \left| f(z)e^{-\frac{|z|^2}{2}} \right|^t &\lesssim (1+|z|)^s \int_{B(z,r)} \left| f(w)e^{-\frac{|w|^2}{2}} \right|^t dA(w) \\ &\lesssim \int_{B(z,r)} (1+|w|)^s \left| f(w)e^{-\frac{|w|^2}{2}} \right|^t dA(w), \quad z \in \mathbb{C}.\end{aligned} \tag{2.9}$$

Integrating both sides of above inequality over $\mathbb{C}$ against the measure $d\mu$, we obtain

$$\begin{aligned}\int_{\mathbb{C}} (1+|w|)^s \left| f(w)e^{-\frac{|w|^2}{2}} \right|^t d\mu(w) &\lesssim \int_{\mathbb{C}} (1+|w|)^s \left| f(w)e^{-\frac{|w|^2}{2}} \right|^t \int_{\mathbb{C}} \chi_{B(w,r)}(z) d\mu(z) dA(w) \\ &= \int_{\mathbb{C}} (1+|w|)^s \left| f(w)e^{-\frac{|w|^2}{2}} \right|^t \mu(B(w,r)) dA(w).\end{aligned} \tag{2.10}$$

Letting $f(w) = k_z(w)$ in (2.10), we have

$$\begin{aligned}\widetilde{\mu}_{(s,t)}(z) &= \int_{\mathbb{C}} (1+|w|)^s e^{-\frac{t}{2}|z-w|^2} d\mu(w) \\ &\lesssim \int_{\mathbb{C}} \mu(B(w,r))(1+|w|)^s e^{-\frac{t}{2}|z-w|^2} dA(w).\end{aligned}$$

Applying Lemma 2.6, we conclude that

$$\|\widetilde{\mu}_{(s,t)}\|_{L^p(dA,\mathbb{C})} \lesssim \|\mu_{(s,r,B)}\|_{L^p(dA,\mathbb{C})} < \infty. \tag{2.11}$$

$(2) \Rightarrow (3)$. The proof of the case $p = \infty$ follows from the relation $1+|z| \simeq 1+|a_k|$ for $z \in B(a_k,r)$ and [3, Theorem 2.3]. For the remaining cases $1 \leqslant p < \infty$, by the property of $\frac{r}{2}$-lattice in Lemma 2.5, we have

$$\sum_{k=1}^{\infty} \int_{B(a_k,r)} (\mu_{(s,2r,B)}(z))^p dA(z) \leqslant N \int_{\mathbb{C}} (\mu_{(s,2r,B)}(z))^p dA(z). \tag{2.12}$$

The triangle inequality implies that $\mu(B(a_k,r)) \leqslant \mu(B(z,2r))$ for $z \in B(a_k,r)$. Then

$$\begin{aligned}\sum_{k=1}^{\infty}\int_{B(a_k,r)}(\mu_{(s,2r,B)}(z))^p dA(z) &\gtrsim \sum_{k=1}^{\infty}\int_{B(a_k,r)}(\mu_{(s,r,B)}(a_k))^p dA(z)\\ &\simeq \sum_{k=1}^{\infty}(\mu_{(s,r,B)}(a_k))^p. \end{aligned} \tag{2.13}$$

According to Lemma 2.7, if $\mu_{(s,r,B)}(z) \in L^p(dA,\mathbb{C})$, so does $\mu_{(s,2r,B)}(z)$. Hence, combining (2.12) with (2.13), we obtain the desired result.

$(3) \Rightarrow (2)$. Using Lemma 2.7, we obtain

$$\begin{aligned}\int_{\mathbb{C}}(\mu_{(s,r,B)}(z))^p dA(z) &\leqslant \sum_{k=1}^{\infty}\int_{B(a_k,r)}(\mu_{(s,r,B)}(z))^p dA(z)\\ &\lesssim \sum_{k=1}^{\infty}\int_{B(a_k,r)}(\mu_{(s,2r,B)}(a_k))^p dA(z)\\ &\simeq \sum_{k=1}^{\infty}(\mu_{(s,2r,B)}(a_k))^p.\end{aligned}$$

The equivalence of norms in (2.7) follows from the process of the proof. □

Lemma 2.9. *([2, Lemma 2.9]) Let $1 \leqslant p \leqslant \infty$. For $c = \{c_j\} \in \ell^p$, set*

$$S(c)(z) = \sum_{j=1}^{\infty} c_j k_{a_j}(z) = \sum_{j=1}^{\infty} c_j e^{\overline{a}_j z - \frac{|a_j|^2}{2}}, \quad z \in \mathbb{C},$$

where $\{a_j\}$ is $\frac{r}{2}$-lattice in $\mathbb{C}$. Then S is a bounded operator from ℓ^p to $F^p(\mathbb{C})$.

The following result gives the higher order derivative characterizations for Fock spaces. See [2, Theorem 2.1], [26, Theorem 1] or [27, Theorem 1].

Lemma 2.10. *For $0 < p \leqslant \infty$, let f be an entire function on $\mathbb{C}$. Then $f \in F^p(\mathbb{C})$ if and only if*

$$\frac{f^{(n)}(z)}{(1+|z|)^n} e^{-\frac{|z|^2}{2}} \in L^p(dA,\mathbb{C})$$

for any positive integer n. Moreover, if $0 < p < \infty$, then

$$\left(\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z)\right)^{\frac{1}{p}} \simeq \sum_{k=0}^{n-1}|f^{(k)}(0)| + \left(\int_{\mathbb{C}}\left|\frac{f^{(n)}(z)}{(1+|z|)^n}e^{-\frac{|z|^2}{2}}\right|^p dA(z)\right)^{\frac{1}{p}}.$$

If $p = \infty$, then

$$\|f\|_\infty \simeq \sum_{k=0}^{n-1}|f^{(k)}(0)| + \sup_{z\in\mathbb{C}}\left|\frac{f^{(n)}(z)}{(1+|z|)^n}e^{-\frac{|z|^2}{2}}\right|.$$

3. Fock-Carleson Measure for Derivatives

In this section, we give the characterizations of the boundedness and compactness of $D^{(n)}: F^p(\mathbb{C}) \to L^q_w(\mathbb{C},\mu)$ for all $0 < p,q < \infty$.

Theorem 3.1. *Let $0 < p \leqslant q < \infty$, n be a positive integer, and μ a positive Borel measure on $\mathbb{C}$. Then the following statements are equivalent:*

(1) *μ is a (p,q,n)-Fock Carleson measure;*
(2) *$\widetilde{\mu}_{(qn,t)} \in L^\infty(\mathbb{C})$ for some (or any) $t > 0$;*
(3) *$\mu_{(qn,r,B)} \in L^\infty(\mathbb{C})$ for some (or any) $r > 0$;*
(4) *$\{\mu_{(qn,r,B)}(a_k)\} \in \ell^\infty$ for some (or any) $r > 0$, where $\{a_k\}$ is $\frac{r}{2}$-lattice in $\mathbb{C}$.*

Furthermore,

$$\|\mu\|^q \simeq \|\widetilde{\mu}_{(qn,t)}\|_{L^\infty(\mathbb{C})} \simeq \|\mu_{(qn,r,B)}\|_{L^\infty(\mathbb{C})} \simeq \|\{\mu_{(qn,r,B)}(a_k)\}\|_{\ell^\infty}. \tag{3.1}$$

Proof. Lemma 2.8 for $p = \infty$ gives the equivalence of (2), (3) and (4). We next prove (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (3). If $|w| \geqslant 1$, let $f(z) = k_w(z)$. By the boundedness of $D^{(n)} : F^p(\mathbb{C}) \to L^q_W(\mathbb{C},\mu)$, we have

$$\begin{aligned}\int_{\mathbb{C}} \left| k_w^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) &= |w|^{qn} \int_{\mathbb{C}} e^{-\frac{q}{2}|z-w|^2} d\mu(z) \\ &\lesssim \|k_w\|_p^q \simeq 1.\end{aligned} \tag{3.2}$$

For $r > 0$, we obtain by (3.2) that

$$(1+|w|)^{qn} \int_{B(w,r)} e^{-\frac{q}{2}|z-w|^2} d\mu(z) \lesssim |w|^{qn} \int_{B(w,r)} e^{-\frac{q}{2}|z-w|^2} d\mu(z) \lesssim 1,$$

which implies that

$$(1+|w|)^{qn} \mu(B(w,r)) \lesssim e^{\frac{qr^2}{2}}. \tag{3.3}$$

If $|w| < 1$ and $|b| \geqslant 3$, we let $f(z) = k_{w+b}(z)$. It follows that

$$\int_{\mathbb{C}} \left| k_{w+b}^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) \lesssim \|k_{w+b}\|_p^q \simeq 1,$$

which implies that

$$|w+b|^{qn} \int_{B(w,r)} e^{-\frac{q}{2}|w+b-z|^2} d\mu(z) \lesssim 1.$$

It follows that

$$(1+|w|)^{qn} \mu(B(w,r)) \lesssim e^{\frac{q}{2}(r+|b|)^2}. \tag{3.4}$$

Combining (3.3) with (3.4), we obtain (3).

Conversely, if (3) holds, similar to the proof of Theorem 5.4 of [14], we have

$$I_{q,n}(f) := \int_{\mathbb{C}} \left| f^{(n)}(w) e^{-\frac{|w|^2}{2}} \right|^q d\mu(w) \leqslant \sum_{k=1}^{\infty} \int_{B(a_k,r)} \left| f^{(n)}(w) e^{-\frac{|w|^2}{2}} \right|^q d\mu(w), \tag{3.5}$$

where $\{a_k\}$ is $\frac{r}{2}$-lattice. For $w \in B(a_k, r)$, Lemma 2.3 gives

$$\left| f^{(n)}(w) e^{-\frac{|w|^2}{2}} \right|^q \lesssim (1+|w|)^{qn} \int_{B(a_k,2r)} |f(\xi)|^q e^{-\frac{q}{2}|\xi|^2} dA(\xi).$$

Combining this with (3.5), and using Lemmas 2.1 and 2.5, we obtain

$$I_{q,n}(f) \lesssim \|\mu_{(qn,r,B)}\|_{L^\infty(\mathbb{C})} \sum_{k=1}^{\infty} \int_{B(a_k,2r)} |f(\xi)|^q e^{-\frac{q}{2}|\xi|^2} dA(\xi)$$
$$\lesssim N\|\mu_{(qn,r,B)}\|_{L^\infty(\mathbb{C})}\|f\|_q^q \lesssim N \cdot \frac{q}{p}\|\mu_{(qn,r,B)}\|_{L^\infty(\mathbb{C})}\|f\|_p^q,$$

which implies (1). The equivalence of norms in (3.1) can be obtained from above proof and Lemma 2.8. The proof is complete. □

Theorem 3.2. *Let $0 < p \leqslant q < \infty$, n be a positive integer, and μ a positive Borel measure on $\mathbb{C}$. Then the following statements are equivalent:*
(1) *μ is a vanishing (p,q,n)-Fock Carleson measure;*
(2) *$\widetilde{\mu}_{(qn,t)}(z) \to 0$ as $|z| \to \infty$ for some (or any) $t > 0$;*
(3) *$\mu_{(qn,r,B)}(z) \to 0$ as $|z| \to \infty$ for some (or any) $r > 0$;*
(4) *$\mu_{(qn,r,B)}(a_k) \to 0$ as $k \to \infty$ for some (or any) $r > 0$, where $\{a_k\}$ is $\frac{r}{2}$-lattice in $\mathbb{C}$.*

Proof. (1) $\Rightarrow$ (2). Assume that $D^{(n)} : F^p(\mathbb{C}) \to L^q_w(\mathbb{C},\mu)$ is compact. Since k_z is an unit vector in $F^p(\mathbb{C})$ and $k_z(w) \to 0$ uniformly on any compact subset of $\mathbb{C}$ as $|z| \to \infty$, we have from Theorem 3.1 and [2, Theorem 3.1] that μ is a (p,q,n)-Fock Carleson measure if and only if $(1+|w|)^{qn}d\mu(w)$ is a (p,q)-Fock Carleson measure. Hence,

$$\widetilde{\mu}_{(qn,q)}(z) = \int_{\mathbb{C}} \left|k_z(w)e^{-\frac{|w|^2}{2}}\right|^q (1+|w|)^{qn} d\mu(w) \lesssim \|k_z\|_p^q \to 0,$$

as $|z| \to \infty$. That is, (2) holds for $t = q$.

The proof of (2) $\Rightarrow$ (3) follows from (2.8).

(3) $\Rightarrow$ (4). Applying the estimate of (2.6) in [2] and $1+|a_k| \simeq 1+|z|$ for $z \in B(a_k,r)$, we have

$$(1+|a_k|)^{qn}\mu(B(a_k,r)) \lesssim \int_{B(a_k,r)} (1+|z|)^{qn}\mu(B(z,r))dA(z).$$

Combining this with (3), we see that

$$\mu_{(qn,r,B)}(a_k) = (1+|a_k|)^{qn}\mu(B(a_k,r)) \to 0, \qquad as\ \ k \to \infty.$$

(4) $\Rightarrow$ (1). Let $\{f_j\} \subset F^p(\mathbb{C})$ be a bounded sequence and $\{f_j\}$ converge to zero uniformly on each compact subset of $\mathbb{C}$ as $j \to \infty$. On the one hand, since $\mu_{(qn,r,B)}(a_k) \to 0$ as $k \to \infty$, for any $\varepsilon > 0$, we have that there exists a positive integer N_0 such that $\mu_{(qn,r,B)}(a_k) < \varepsilon$ whenever $k > N_0$. Let $K := \cup_{k=1}^{N_0}\overline{B(a_k,2r)}$. Then there exists a positive integer j_0 such that when $j > j_0$

$$\sum_{k=1}^{N_0} \mu_{(qn,r,B)}(a_k)\left(\int_{B(a_k,2r)} \left|f_j(z)e^{-\frac{|z|^2}{2}}\right|^p dA(z)\right)^{\frac{q}{p}} \lesssim \varepsilon. \tag{3.6}$$

In view of $\frac{q}{p} \geqslant 1$, we have

$$\sum_{k=N_0+1}^{\infty} \mu_{(qn,r,B)}(a_k)\left(\int_{B(a_k,2r)} \left|f_j(z)e^{-\frac{|z|^2}{2}}\right|^p dA(z)\right)^{\frac{q}{p}}$$
$$\leqslant \varepsilon\left(\sum_{k=N_0+1}^{\infty} \int_{B(a_k,2r)} \left|f_j(z)e^{-\frac{|z|^2}{2}}\right|^p dA(z)\right)^{\frac{q}{p}} \lesssim \varepsilon. \tag{3.7}$$

From Lemma 2.3, (3.6) and (3.7), for any $\varepsilon > 0$, we have that there exists a positive integer j_0 such that, when $j > j_0$,

$$\begin{aligned}
&\int_{\mathbb{C}} \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) \qquad (3.8)\\
&\leqslant \sum_{k=1}^{\infty} \int_{B(a_k,r)} \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z)\\
&\leqslant \sum_{k=1}^{\infty} \mu(B(a_k,r)) \left(\sup_{z\in B(a_k,r)} \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^p \right)^{\frac{q}{p}}\\
&\lesssim \sum_{k=1}^{\infty} \mu(B(a_k,r)) \left(\sup_{z\in B(a_k,r)} (1+|z|)^{np} \int_{B(z,r)} \left| f_j(\xi) e^{-\frac{|\xi|^2}{2}} \right|^p dA(\xi) \right)^{\frac{q}{p}}\\
&\lesssim \sum_{k=1}^{\infty} (1+|a_k|)^{qn} \mu(B(a_k,r)) \left(\int_{B(a_k,2r)} \left| f_j(\xi) e^{-\frac{|\xi|^2}{2}} \right|^p dA(\xi) \right)^{\frac{q}{p}}\\
&\lesssim \varepsilon.
\end{aligned}$$

So μ is a vanishing (p,q,n)-Fock Carleson measure. This completes the proof. □

Next, we characterize the (p,q,n)-Fock Carleson measure for $q < p$ by using Luecking's method via Khinchine's inequality (see [9, Theorem 1]).

Define the Rademacher functions $r_k(x)$ by

$$r_0(x) = \begin{cases} 1, & 0 \leqslant x-[x] < \frac{1}{2}, \\ -1, & \frac{1}{2} \leqslant x-[x] < 1; \end{cases}$$

$$r_k(x) = r_0(2^k x), \qquad k > 0,$$

where $[x]$ is the largest integer not greater than x. For $0 < p < \infty$, the Khinchine's inequality is

$$\int_0^1 \left| \sum_{j=1}^m c_j r_j(x) \right|^p dx \simeq \left(\sum_{j=1}^m |c_j|^2 \right)^{\frac{p}{2}}$$

for all $m \geqslant 1$ and complex numbers $c_1, c_2, \ldots, c_m$.

Theorem 3.3. *Let $0 < q < p < \infty$, n a positive integer, and μ a positive Borel measure on $\mathbb{C}$. Then the following statements are equivalent:*

(1) *μ is a (p,q,n)-Fock Carleson measure;*
(2) *μ is a vanishing (p,q,n)-Fock Carleson measure;*
(3) *$\widetilde{\mu}_{(qn,t)} \in L^{\frac{p}{p-q}}(dA,\mathbb{C})$ for some (or any) $t > 0$;*
(4) *$\mu_{(qn,r,B)} \in L^{\frac{p}{p-q}}(dA,\mathbb{C})$ for some (or any) $r > 0$;*
(5) *$\{\mu_{(qn,r,B)}(a_k)\} \in \ell^{\frac{p}{p-q}}$ for some (or any) $r > 0$, where $\{a_k\}$ is $\frac{r}{2}$-lattice in $\mathbb{C}$.*
Furthermore,

$$\|\mu\|^q \simeq \|\widetilde{\mu}_{(qn,t)}\|_{L^{\frac{p}{p-q}}(dA,\mathbb{C})} \simeq \|\mu_{(qn,r,B)}\|_{L^{\frac{p}{p-q}}(dA,\mathbb{C})} \simeq \|\{\mu_{(qn,r,B)}(a_k)\}\|_{\ell^{\frac{p}{p-q}}}. \qquad (3.9)$$

Proof. The equivalence among (3), (4) and (5) follows from Lemma 2.8.

We next prove $(1) \Leftrightarrow (4)$. To prove that (1) implies (4), we apply Luecking's method via Khinchine's inequality in [9, Theorem 1]. Let $\{c_j\} \in \ell^p$, $\{a_j\}$ be $\frac{r}{2}$-lattice with $\inf_j |a_j| \geqslant \delta_0 > 0$ and

$$f(z) = \sum_{j=1}^{\infty} c_j k_{a_j}(z).$$

By Lemma 2.9, $\|f\|_p \lesssim \|\{c_j\}\|_{\ell^p}$. Since μ is a (p,q,n)-Fock Carleson measure,

$$\int_{\mathbb{C}} \left| \sum_{j=1}^{\infty} c_j (\overline{a_j})^n k_{a_j}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) \lesssim \|\mu\|^q \|\{c_j\}\|_{\ell^p}^q. \tag{3.10}$$

Also, if $\frac{2}{q} \leqslant 1$, it is obvious that

$$\sum_{j=1}^{\infty} |c_j|^q \chi_{B(a_j,r)}(z)(1+|a_j|)^{qn} \leqslant \left(\sum_{j=1}^{\infty} |c_j|^2 (1+|a_j|)^{2n} \chi_{B(a_j,r)}(z) \right)^{\frac{q}{2}}.$$

If $\frac{2}{q} > 1$, by Hölder's inequality and Lemma 2.5, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} |c_j|^q \chi_{B(a_j,r)}(z)(1+|a_j|)^{qn} \\
&\leqslant \left(\sum_{j=1}^{\infty} |c_j|^2 (1+|a_j|)^{2n} \chi_{B(a_j,r)}(z) \right)^{\frac{q}{2}} \left(\sum_{j=1}^{\infty} \chi_{B(a_j,r)}(z) \right)^{\frac{2-q}{2}} \\
&\leqslant N^{\frac{2-q}{2}} \left(\sum_{j=1}^{\infty} |c_j|^2 (1+|a_j|)^{2n} \chi_{B(a_j,r)}(z) \right)^{\frac{q}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{j=1}^{\infty} |c_j|^q (1+|a_j|)^{qn} \mu(B(a_j,r)) \\
&= \int_{\mathbb{C}} \sum_{j=1}^{\infty} |c_j|^q \chi_{B(a_j,r)}(z)(1+|a_j|)^{qn} d\mu(z) \\
&\leqslant \max\{N^{\frac{2-q}{2}}, 1\} \int_{\mathbb{C}} \left(\sum_{j=1}^{\infty} |c_j|^2 (1+|a_j|)^{2n} \chi_{B(a_j,r)}(z) \right)^{\frac{q}{2}} d\mu(z) \\
&\lesssim \int_{\mathbb{C}} \left(\sum_{j=1}^{\infty} |c_j (\overline{a_j})^n|^2 \chi_{B(a_j,r)}(z) \right)^{\frac{q}{2}} d\mu(z) \\
&\lesssim e^{\frac{q}{2}r^2} \int_{\mathbb{C}} \left(\sum_{j=1}^{\infty} |c_j (\overline{a_j})^n|^2 e^{-|z-a_j|^2} \right)^{\frac{q}{2}} d\mu(z).
\end{aligned} \tag{3.11}$$

Khinchine's inequality and Fubini's theorem yield

$$\begin{aligned}\int_{\mathbb{C}}\left(\sum_{j=1}^{\infty}|c_j(\overline{a_j})^n|^2e^{-|z-a_j|^2}\right)^{\frac{q}{2}}d\mu(z) &\lesssim \int_{\mathbb{C}}\int_0^1\left|\sum_{j=1}^{\infty}c_j(\overline{a_j})^ne^{-\frac{|z-a_j|^2}{2}}r_j(x)\right|^q dxd\mu(z)\\ &\simeq \int_0^1\int_{\mathbb{C}}\left|\sum_{j=1}^{\infty}r_j(x)c_j(\overline{a_j})^ne^{-\frac{|z-a_j|^2}{2}}\right|^q d\mu(z)dx\\ &\lesssim \|\mu\|^q\|\{c_j\}\|_{\ell^p}^q, \end{aligned} \tag{3.12}$$

where the last inequality follows from (3.10) by replacing c_j with $r_j(x)c_j$. Set $\lambda_j=|c_j|^q$. Then (3.11) and (3.12) give

$$\sum_{j=1}^{\infty}\lambda_j(1+|a_j|)^{qn}\mu(B(a_j,r))\leqslant C\|\mu\|^q\|\{\lambda_j\}\|_{\ell^{\frac{p}{q}}}.$$

By duality, we obtain

$$\{(1+|a_j|)^{qn}\mu(B(a_j,r))\}\in\ell^{\frac{p}{p-q}}. \tag{3.13}$$

So we obtain a discrete version of the condition. To see a continuous version, we observe that (3.13) still holds if $B(a_j,r)$ is replaced by $B(a_j,2r)$ since the argument is independent of r. On the other hand, it is possible to select $\frac{r}{2}$-lattice $\{a_j\}$ such that $\inf_j|a_j|>0$ (for example, by an appropriate translation). Hence, if $z\in B(a_j,r)$, then $B(z,r)\subset B(a_j,2r)$, and

$$(1+|z|)^{qn}\mu(B(z,r))\lesssim(1+|a_j|)^{qn}\mu(B(a_j,2r)).$$

It follows that

$$\begin{aligned}\int_{\mathbb{C}}\left((1+|z|)^{qn}\mu(B(z,r))\right)^{\frac{p}{p-q}}dA(z) &\lesssim \sum_{j=1}^{\infty}\int_{B(a_j,r)}\left((1+|a_j|)^{qn}\mu(B(a_j,2r))\right)^{\frac{p}{p-q}}dA(z)\\ &\lesssim \sum_{j=1}^{\infty}\left((1+|a_j|)^{qn}\mu(B(a_j,2r))\right)^{\frac{p}{p-q}}<\infty.\end{aligned}$$

This shows that (1) implies (4).

Next, we prove (4) $\Rightarrow$ (1). Lemma 2.4 and Hölder's inequality give

$$\int_{\mathbb{C}}\left|f^{(n)}(z)e^{-\frac{|z|^2}{2}}\right|^q d\mu(z)\leqslant C\|f\|_p^q\|\mu_{(qn,r,B)}\|_{L^{\frac{p}{p-q}}(dA,\mathbb{C})} \tag{3.14}$$

for any $f\in F^p(\mathbb{C})$. Hence, μ is a (p,q,n)-Fock Carleson measure.

Finally, we prove (1) $\Leftrightarrow$ (2). We only need to prove (1) $\Rightarrow$ (2). Let $\{f_j\}\subset F^p(\mathbb{C})$ be a sequence of functions such that $K_0:=\sup_j\|f_j\|_p<\infty$ and $\{f_j\}$ converges uniformly to zero on compact subsets of $\mathbb{C}$ as $j\to\infty$.

The previous proof shows that $\mu_{(qn,r,B)}\in L^{\frac{p}{p-q}}(dA,\mathbb{C})$. So, for any $\varepsilon>0$, there exists sufficiently large $R>0$ such that

$$\left(\int_{|z|>R}|\mu_{(qn,r,B)}(z)|^{\frac{p}{p-q}}dA(z)\right)^{\frac{p-q}{p}}<\frac{\varepsilon}{K_0}. \tag{3.15}$$

Let $D_R = \{z \in \mathbb{C} : |z| \leqslant R\}$, and define μ^R on $\mathbb{C}$ such that $\mu^R = \mu$ on $\mathbb{C} \setminus D_R$ and $\mu^R = 0$ on D_R. Applying (3.14) and (3.15), we have

$$\begin{aligned}\int_{\mathbb{C}\setminus D_R} \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) &= \int_{\mathbb{C}} \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu^R(z) \\ &\lesssim \|f_j\|_p^q \cdot \|\mu^R_{(qn,r,B)}\|_{L^{\frac{p}{p-q}}(dA,\mathbb{C})} \lesssim \varepsilon. \end{aligned} \tag{3.16}$$

Fix $R > 0$ such that (3.16) holds. By assumption, there exists a positive integer j_0 such that

$$\int_{D_R} \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) < \varepsilon, \tag{3.17}$$

as $j > j_0$. Combining (3.16) with (3.17), for any $\varepsilon > 0$, we have that there exists a positive integer j_0 such that

$$\int_{\mathbb{C}} \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) = \left(\int_{D_R} + \int_{\mathbb{C}\setminus D_R} \right) \left| f_j^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) \lesssim \varepsilon,$$

as $j > j_0$. Hence, μ is a vanishing (p,q,n)-Fock Carleson measure. The proof above yields the equivalence of norms in (3.9). $\square$

Finally, if $p = \infty$ in Theorem 3.3, then we have the following result.

Theorem 3.4. *Let $0 < q < \infty$, n a positive integer, and μ a positive Borel measure. Then the following statements are equivalent:*

(1) *μ is a (∞,q,n)-Fock Carleson measure;*
(2) *μ is a vanishing (∞,q,n)-Fock Carleson measure;*
(3) *$\widetilde{\mu}_{(qn,t)} \in L^1(dA,\mathbb{C})$ for some (or any) $t > 0$;*
(4) *$\mu_{(qn,r,B)} \in L^1(dA,\mathbb{C})$ for some (or any) $r > 0$;*
(5) *$\{\mu_{(qn,r,B)}(a_k)\} \in \ell^1$ for some (or any) $r > 0$, where $\{a_k\}$ is $\frac{r}{2}$-lattice in $\mathbb{C}$;*
(6) *$\mu_{qn}(\mathbb{C}) < \infty$.*

Furthermore,

$$\|\mu\|^q \simeq \|\widetilde{\mu}_{(qn,t)}\|_{L^1(dA,\mathbb{C})} \simeq \|\mu_{(qn,r,B)}\|_{L^1(dA,\mathbb{C})} \simeq \|\{\mu_{(qn,r,B)}(a_k)\}\|_{\ell^1} \simeq \mu_{qn}(\mathbb{C}).$$

Proof. The equivalence of (3), (4) and (5) follows from Lemma 2.8. Note that

$$\begin{aligned}\|\widetilde{\mu}_{(qn,t)}\|_{L^1(dA,\mathbb{C})} &= \int_{\mathbb{C}} \int_{\mathbb{C}} (1+|z|)^{qn} e^{-\frac{t}{2}|z-w|^2} d\mu(z) dA(w) \\ &= \int_{\mathbb{C}} (1+|z|)^{qn} \int_{\mathbb{C}} e^{-\frac{t}{2}|z-w|^2} dA(w) d\mu(z) \\ &\simeq \int_{\mathbb{C}} (1+|z|)^{qn} d\mu(z) = \mu_{qn}(\mathbb{C}).\end{aligned}$$

Then $(3) \Leftrightarrow (6)$ follows.

The proof of $(1) \Rightarrow (4)$ is a modification of the proof of $(1) \Rightarrow (4)$ in Theorem 3.3. Also, we have

$$\|\mu_{(qn,r,B)}\|_{L^1(dA,\mathbb{C})} \lesssim \|\mu\|^q.$$

For (4) $\Rightarrow$ (1), let $f \in F^\infty(\mathbb{C})$. Applying Lemma 2.4, we obtain

$$\begin{aligned}\int_{\mathbb{C}} \left| f^{(n)}(z) e^{-\frac{|z|^2}{2}} \right|^q d\mu(z) &\lesssim \int_{\mathbb{C}} \left| f(w) e^{-\frac{|w|^2}{2}} \right|^q (1+|w|)^{qn} \mu(B(w,r)) dA(w) \\ &\lesssim \|f\|_\infty^q \int_{\mathbb{C}} (1+|w|)^{qn} \mu(B(w,r)) dA(w) \\ &\simeq \|f\|_\infty^q \cdot \|\mu_{(qn,r,B)}\|_{L^1(dA,\mathbb{C})}.\end{aligned}$$

Moreover, $\|\mu\|^q \lesssim \|\mu_{(qn,r,B)}\|_{L^1(dA,\mathbb{C})}$.

The proof of (1) $\Rightarrow$ (2) is similar to that of Theorem 3.3. Hence, the detail is omitted here. This completes the proof. □

4. Generalized Weighted Composition Operators on Fock Spaces

In this section, as an application of Theorems 3.3 and 3.4, we study the boundedness and compactness of generalized weighted composition operators from $F^p(\mathbb{C})$ into $F^q(\mathbb{C})$. In addition, we also study the essential norm of generalized weighted composition operators. The following lemmas will be used in this section.

Lemma 4.1. *(*[28, Proposition 2.1]*) Let f and φ be two entire functions on $\mathbb{C}$ such that $f \not\equiv 0$. Suppose that there is a positive constant C such that*

$$|f(z)|^2 e^{|\varphi(z)|^2 - |z|^2} \leqslant C$$

for all $z \in \mathbb{C}$. Then $\varphi(z) = \varphi(0) + \lambda z$ for some $|\lambda| \leqslant 1$. If $|\lambda| = 1$, then $f(z) = f(0)e^{-\bar{\beta} z}$, where $\beta = \bar{\lambda}\varphi(0)$. Furthermore, if

$$\lim_{|z|\to\infty} |f(z)|^2 e^{|\varphi(z)|^2 - |z|^2} = 0,$$

then $\varphi(z) = \lambda z + b$ with $|\lambda| < 1$.

The following two lemmas can be found in [4, Lemma 1.2] and [5, Lemmas 2.3 and 2.4].

Lemma 4.2. *Let $0 < p, q \leqslant \infty$ and let $T : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ be a well-defined linear continuous operator. The following two assertions are equivalent:*

(1) $T : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is compact;

(2) for every bounded sequence $\{f_k\} \subset F^p(\mathbb{C})$ converging to 0 in $H(\mathbb{C})$, the sequence $\{Tf_k\}$ also converges to 0 in $F^q(\mathbb{C})$.

Lemma 4.3. *Let $1 < p < \infty$ and $1 < q \leqslant \infty$. If $T : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is compact, then, for every sequence $\{w_n\} \subset \mathbb{C}$ with $\lim_{n\to\infty} |w_n| = \infty$, $\{Tk_{w_n}\}$ converge to 0 in $F^q(\mathbb{C})$ as $n \to \infty$.*

The following two quantities play an important role in the following discussion:

$$m_{z,n}(\psi, \varphi) := |\psi(z)||\varphi(z)|^n e^{\frac{|\varphi(z)|^2 - |z|^2}{2}}, \qquad z \in \mathbb{C},$$

and

$$m_n(\psi, \varphi) := \sup_{z\in\mathbb{C}} m_{z,n}(\psi, \varphi),$$

where n is positive integer.

First of all, we have the following necessary condition.

Proposition 4.1. *Let $0 < p,q \leqslant \infty$, n a positive integer, φ an analytic self-map of $\mathbb{C}$, and ψ a nonzero entire function on $\mathbb{C}$. If $W^{(n)}_{\psi,\varphi} : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is bounded, then $\psi \in F^q(\mathbb{C})$ and $m_n(\psi,\varphi) < \infty$. In this case, $\varphi(z) = az+b$ with $|a| < 1$ and*

$$m_{z,n}(\psi,\varphi) \leqslant \left\|W^{(n)}_{\psi,\varphi}k_{\varphi(z)}\right\|_q \leqslant \left\|W^{(n)}_{\psi,\varphi}\right\| \tag{4.1}$$

for any $z \in \mathbb{C}$.

Proof. Since $f(z) = z^n \in F^p(\mathbb{C})$, the boundedness of $W^{(n)}_{\psi,\varphi}$ implies $\psi(z) \in F^q(\mathbb{C})$. Using the boundedness of $W^{(n)}_{\psi,\varphi}$ again, we obtain

$$|\psi(z)||w|^n\left|e^{\overline{w}\varphi(z)-\frac{|w|^2}{2}}\right|e^{-\frac{|z|^2}{2}} \leqslant |W^{(n)}_{\psi,\varphi}k_w(z)|e^{-\frac{|z|^2}{2}} \leqslant \|W^{(n)}_{\psi,\varphi}\|, \quad w \in \mathbb{C}. \tag{4.2}$$

Letting $w = \varphi(z)$ in (4.2), we have

$$m_{z,n}(\psi,\varphi) \leqslant \|W^{(n)}_{\psi,\varphi}\|$$

for any $z \in \mathbb{C}$. Hence, $\varphi(z) = az+b$ with $|a| \leqslant 1$. We next prove $|a| \neq 1$. In fact, if $|a| = 1$, then we conclude from Lemma 4.1 and Liouville's theorem that

$$\psi(z)(\varphi(z))^n \equiv 0.$$

In view of $\varphi(z) = az+b \not\equiv 0$, we have $\psi(z) \equiv 0$. This contradicts the assumption. $\square$

Corollary 4.1. *Let $0 < p,q \leqslant \infty$, n be a positive integer, and ψ be not identically zero in $F^q(\mathbb{C})$. If $\varphi(z) = b$, then $W^{(n)}_{\psi,\varphi} : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is compact and*

$$|b|^n e^{\frac{|b|^2}{2}}\|\psi\|_q \lesssim \|W^{(n)}_{\psi,\varphi}\| \lesssim (1+|b|)^n e^{\frac{|b|^2}{2}}\|\psi\|_q. \tag{4.3}$$

Proof. For $f \in F^p(\mathbb{C})$, it follows from Lemma 2.2 that

$$\|W^{(n)}_{\psi,\varphi}f\|_q \lesssim (1+|b|)^n e^{\frac{|b|^2}{2}}\|f\|_p\|\psi\|_q$$

and

$$\|W^{(n)}_{\psi,\varphi}k_b\|_q = |b|^n e^{\frac{|b|^2}{2}}\|\psi\|_q,$$

which implies (4.3). Also, $W^{(n)}_{\psi,\varphi}$ is a finite rank operator with rank 1, so it is compact. $\square$

Next, we consider the case $\varphi(z) = az+b$ as $0 < |a| < 1$. We define the following pull-back measure $\nu_{\psi,\varphi,q}$ on $\mathbb{C}$

$$\nu_{\psi,\varphi,q}(E) = \frac{q}{2\pi}\int_{\varphi^{-1}(E)}|\psi(z)|^q e^{-\frac{q}{2}|z|^2}dA(z)$$

for every Borel subset E of $\mathbb{C}$.

Theorem 4.1. *Let $0 < q < p < \infty$ and n be a positive integer. If $\psi \in F^q(\mathbb{C})$ is not identical zero, and $\varphi(z) = az+b$ with $0 < |a| < 1$, then the following statements are equivalent:*

(1) *$W^{(n)}_{\psi,\varphi} : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is bounded;*

(2) *$W^{(n)}_{\psi,\varphi} : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is compact;*

(3) *$m_{z,n}(\psi,\varphi) \in L^{\frac{pq}{p-q}}(dA,\mathbb{C})$.*

Furthermore,

$$|a|^{\frac{2(p-q)}{pq}} \left\|m_{z,n}(\psi,\varphi)\right\|_{L^{\frac{pq}{p-q}}(dA,\mathbb{C})} \lesssim \|W_{\psi,\varphi}^{(n)}\| \lesssim |a|^{-\frac{2}{p}} \left(\|\psi\|_q^{\frac{pq}{p-q}} + \left\|m_{z,n}(\psi,\varphi)\right\|_{L^{\frac{pq}{p-q}}(dA,\mathbb{C})}^{\frac{pq}{p-q}} \right)^{\frac{p-q}{pq}}.$$

Proof. We only need to prove $(1) \Rightarrow (3) \Rightarrow (2)$.

$(1) \Rightarrow (3)$. For $f \in F^p(\mathbb{C})$, we deduce that

$$\begin{aligned}
\|W_{\psi,\varphi}^{(n)}\| \|f\|_p \geqslant \|W_{\psi,\varphi}^{(n)}(f)\|_q & \\
&= \left(\int_{\mathbb{C}} \left|f^{(n)}(z)\right|^q d\nu_{\psi,\varphi,q}(z) \right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{C}} \left|f^{(n)}(z)\right|^q e^{-\frac{q}{2}|z|^2} d\mu_{\psi,\varphi,q}(z) \right)^{\frac{1}{q}},
\end{aligned}$$

where $d\mu_{\psi,\varphi,q}(z) = e^{\frac{q}{2}|z|^2} d\nu_{\psi,\varphi,q}(z)$. This shows that $\mu_{\psi,\varphi,q}$ is a (p,q,n)-Fock Carleson measure. Applying Theorem 3.3, we obtain

$$\begin{aligned}
(\widetilde{\mu_{\psi,\varphi,q}})_{(qn,q)}(w) &= \int_{\mathbb{C}} (1+|z|)^{qn} |k_w(z)|^q d\nu_{\psi,\varphi,q}(z) \\
&= \frac{q}{2\pi} \int_{\mathbb{C}} |\psi(z)|^q (1+|\varphi(z)|)^{qn} |k_w(\varphi(z))|^q e^{-\frac{q}{2}|z|^2} dA(z) \\
&\geqslant \frac{q}{2\pi} \int_{\mathbb{C}} |\psi(z)|^q |\varphi(z)|^{qn} |k_w(\varphi(z))|^q e^{-\frac{q}{2}|z|^2} dA(z).
\end{aligned}$$

In particular, if $w = \varphi(z)$, then

$$\begin{aligned}
(\widetilde{\mu_{\psi,\varphi,q}})_{(qn,q)}(\varphi(z)) &\geqslant \left|W_{\psi,\varphi}^{(n)} k_{\varphi(z)}(z)\right|^q e^{-\frac{q}{2}|z|^2} \\
&= |\psi(z)|^q |\varphi(z)|^{qn} |k_{\varphi(z)}(\varphi(z))|^q e^{-\frac{q}{2}|z|^2} = \left(m_{z,n}(\psi,\varphi)\right)^q
\end{aligned}$$

for any $z \in \mathbb{C}$. Hence,

$$\begin{aligned}
\int_{\mathbb{C}} \left(m_{z,n}(\psi,\varphi)\right)^{\frac{pq}{p-q}} dA(z) &\leqslant \int_{\mathbb{C}} \left((\widetilde{\mu_{\psi,\varphi,q}})_{(qn,q)}(\varphi(z))\right)^{\frac{p}{p-q}} dA(z) \\
&= |a|^{-2} \int_{\mathbb{C}} \left((\widetilde{\mu_{\psi,\varphi,q}})_{(qn,q)}(z)\right)^{\frac{p}{p-q}} dA(z) < \infty. \qquad (4.4)
\end{aligned}$$

Therefore, $m_{z,n}(\psi,\varphi) \in L^{\frac{pq}{p-q}}(dA,\mathbb{C})$. Furthermore, applying (3.9) of Theorem 3.3, we have

$$\left\|W_{\psi,\varphi}^{(n)}\right\|^q = \|\mu_{\psi,\varphi,q}\|^q \simeq \|(\widetilde{\mu_{\psi,\varphi,q}})_{(qn,q)}\|_{L^{\frac{p}{p-q}}(dA,\mathbb{C})}. \qquad (4.5)$$

Combining (4.4) with (4.5), we obtain

$$\begin{aligned}
\left\|m_{z,n}(\psi,\varphi)\right\|_{L^{\frac{pq}{p-q}}(dA,\mathbb{C})} &\leqslant \left(|a|^{-2} \int_{\mathbb{C}} \left((\widetilde{\mu_{\psi,\varphi,q}})_{(qn,q)}(z)\right)^{\frac{p}{p-q}} dA(z) \right)^{\frac{p-q}{pq}} \\
&\lesssim |a|^{-\frac{2(p-q)}{pq}} \left\|W_{\psi,\varphi}^{(n)}\right\|. \qquad (4.6)
\end{aligned}$$

$(3) \Rightarrow (2)$. For $f \in F^p(\mathbb{C})$, Hölder's inequality gives

$$\|W_{\psi,\varphi}^{(n)} f\|_q^q \leqslant \frac{q}{2\pi}\left(\int_{\mathbb{C}}\left(|\psi(z)|(1+|\varphi(z)|)^n e^{\frac{|\varphi(z)|^2-|z|^2}{2}}\right)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}} \times \left(\int_{\mathbb{C}}\left(\frac{|(f^{(n)}\circ\varphi)(z)|e^{-\frac{|\varphi(z)|^2}{2}}}{(1+|\varphi(z)|)^n}\right)^p dA(z)\right)^{\frac{q}{p}}. \tag{4.7}$$

For the first integral, we split it into two pieces

$$\left(\int_{|\varphi(z)|<1}+\int_{|\varphi(z)|\geqslant 1}\right)\left(|\psi(z)|(1+|\varphi(z)|)^n e^{\frac{|\varphi(z)|^2-|z|^2}{2}}\right)^{\frac{pq}{p-q}} dA(z) := I_1+I_2.$$

By Proposition 4.1 and Lemma 2.1, for $\psi \in F^q(\mathbb{C})$, we have

$$I_1 \lesssim \int_{|\varphi(z)|<1}\left(|\psi(z)|e^{-\frac{|z|^2}{2}}\right)^{\frac{pq}{p-q}} dA(z) \lesssim \|\psi\|_q^{\frac{pq}{p-q}}. \tag{4.8}$$

It is easy to check that

$$I_2 \lesssim \int_{\mathbb{C}} \left(m_{z,n}(\psi,\varphi)\right)^{\frac{pq}{p-q}} dA(z). \tag{4.9}$$

For the second integral in (4.7), by Lemma 2.10, we have

$$\begin{aligned}\int_{\mathbb{C}}\left(\frac{|(f^{(n)}\circ\varphi)(z)|e^{-\frac{|\varphi(z)|^2}{2}}}{(1+|\varphi(z)|)^n}\right)^p dA(z) &= |a|^{-2}\int_{\mathbb{C}}\left(\frac{|f^{(n)}(z)|e^{-\frac{|z|^2}{2}}}{(1+|z|)^n}\right)^p dA(z)\\ &\lesssim |a|^{-2}\|f\|_p^p.\end{aligned} \tag{4.10}$$

Combining (4.7) with (4.8), (4.9), and (4.10), we obtain

$$\|W_{\psi,\varphi}^{(n)} f\|_q \lesssim |a|^{-\frac{2}{p}}\left(\|\psi\|_q^{\frac{pq}{p-q}}+\|m_{z,n}(\psi,\varphi)\|_{L^{\frac{pq}{p-q}}(dA,\mathbb{C})}^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}\|f\|_p.$$

Hence

$$\|W_{\psi,\varphi}^{(n)}\| \lesssim |a|^{-\frac{2}{p}}\left(\|\psi\|_q^{\frac{pq}{p-q}}+\|m_{z,n}(\psi,\varphi)\|_{L^{\frac{pq}{p-q}}(dA,\mathbb{C})}^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}. \tag{4.11}$$

Let $\{f_j\} \subset F^p(\mathbb{C})$ be any bounded sequence converging uniformly to zero on compact subsets of $\mathbb{C}$ as $j \to \infty$. Let $K_1 := \sup_j \|f_j\|_p^q < \infty$. Since $m_{z,n}(\psi,\varphi) \in L^{\frac{pq}{p-q}}(dA,\mathbb{C})$, then, for any $\varepsilon > 0$, there exists $R > 0$ such that, as $|z| > R$, $|\varphi(z)| > 1$ and

$$\left(\int_{|z|>R}\left(m_{z,n}(\psi,\varphi)\right)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}} < \frac{\varepsilon}{K_1|a|^{-\frac{2q}{p}}}.$$

Thus

$$\begin{aligned}
&\int_{|z|>R} |\psi(z)|^q |(f_j^{(n)} \circ \varphi)(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \\
\lesssim &\left(\int_{|z|>R} \left(|\psi(z)|(1+|\varphi(z)|)^n e^{\frac{|\varphi(z)|^2-|z|^2}{2}} \right)^{\frac{pq}{p-q}} dA(z) \right)^{\frac{p-q}{p}} \times \\
&\left(\int_{|z|>R} \left(\frac{|(f_j^{(n)} \circ \varphi)(z)| e^{-\frac{|\varphi(z)|^2}{2}}}{(1+|\varphi(z)|)^n} \right)^p dA(z) \right)^{\frac{q}{p}} \\
\lesssim &|a|^{-\frac{2q}{p}} \|f_j\|_p^q \left(\int_{|z|>R} \left(m_{z,n}(\psi,\varphi) \right)^{\frac{pq}{p-q}} dA(z) \right)^{\frac{p-q}{p}} \lesssim \varepsilon.
\end{aligned}$$

By the assumption of $\{f_j\}$, there exists a positive integer j_0 such that when $j > j_0$

$$\int_{|z|\leqslant R} |\psi(z)|^q |(f_j^{(n)} \circ \varphi)(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \leqslant \|\psi\|_q^q \cdot \max_{|z|\leqslant R} |(f_j^{(n)} \circ \varphi)(z)|^q < \varepsilon.$$

Hence, for any $\varepsilon > 0$, there exists a positive integer j_0 such that when $j > j_0$

$$\|W_{\psi,\varphi}^{(n)} f_j\|_q^q = \frac{q}{2\pi} \left(\int_{|z|\leqslant R} + \int_{|z|>R} \right) |\psi(z)|^q |(f_j^{(n)} \circ \varphi)(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \lesssim \varepsilon.$$

Applying Lemma 4.2, $W_{\psi,\varphi}^{(n)}$ is compact. Furthermore, combining (4.6) with (4.11), we have the norm estimates about $W_{\psi,\varphi}^{(n)}$. This completes the proof. □

Theorem 4.2. *Let $0 < q < \infty$ and n be a positive integer. If $\psi \in F^q(\mathbb{C})$ is a nonzero function and $\varphi(z) = az+b$ with $0 < |a| < 1$. Then the following statements are equivalent:*
(1) $W_{\psi,\varphi}^{(n)} : F^\infty(\mathbb{C}) \to F^q(\mathbb{C})$ is bounded;
(2) $W_{\psi,\varphi}^{(n)} : F^\infty(\mathbb{C}) \to F^q(\mathbb{C})$ is compact;
(3) $m_{z,n}(\psi,\varphi) \in L^q(dA,\mathbb{C})$.
Moreover,

$$|a|^{\frac{2}{q}} \|m_{z,n}(\psi,\varphi)\|_{L^q(dA,\mathbb{C})} \lesssim \|W_{\psi,\varphi}^{(n)}\| \lesssim \left(\|\psi\|_q^q + \|m_{z,n}(\psi,\varphi)\|_{L^q(dA,\mathbb{C})}^q \right)^{\frac{1}{q}}.$$

Proof. The proof is similar to Theorem 4.1 with a simple modification. So we omit the details here. □

Theorem 4.3. *Let $0 < p \leqslant q < \infty$ and n be a positive integer. If $\psi \in F^q(\mathbb{C})$ is a nonzero function and $\varphi(z) = az+b$ with $0 < |a| < 1$.*
(1) $W_{\psi,\varphi}^{(n)} : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is bounded if and only if $m_n(\psi,\varphi) < \infty$. Moreover,

$$m_n(\psi,\varphi) \leqslant \|W_{\psi,\varphi}^{(n)}\| \lesssim |a|^{-\frac{2}{q}} \left(\|\psi\|_q^q + \left(m_n(\psi,\varphi)\right)^q \right)^{\frac{1}{q}}. \tag{4.12}$$

(2) $W_{\psi,\varphi}^{(n)} : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is compact if and only if $\lim_{|z|\to\infty} m_{z,n}(\psi,\varphi) = 0$.

Proof. (1) The proof of necessity follows from Proposition 4.1. Conversely, for every $f \in F^p(\mathbb{C})$, applying Lemmas 2.2, 2.10 and 2.1, we deduce that

$$\begin{aligned}\|W^{(n)}_{\psi,\varphi}f\|_q^q &\lesssim \left(\max_{|\varphi(z)|\leqslant 1}|\psi(z)|^q e^{-\frac{q}{2}|z|^2}+\big(m_n(\psi,\varphi)\big)^q\right)\int_{\mathbb{C}}\frac{|(f^{(n)}\circ\varphi)(z)|^q e^{-\frac{q}{2}|\varphi(z)|^2}}{(1+|\varphi(z)|)^{qn}}dA(z)\\ &\lesssim |a|^{-2}\big(\|\psi\|_q^q+\big(m_n(\psi,\varphi)\big)^q\big)\|f\|_p^q.\end{aligned}$$

Combining this upper bound with inequality (4.1), we have (4.12).

(2) Assume that $W^{(n)}_{\psi,\varphi}$ is compact. For every sequence $\{z_j\}\subset\mathbb{C}$ such that $|z_j|\to\infty$ and $k_{\varphi(z_j)}\to 0$ as $j\to\infty$, we obtain from (4.1) and Lemma 4.2 that

$$m_{z_j,n}(\psi,\varphi)\leqslant \big\|W^{(n)}_{\psi,\varphi}k_{\varphi(z_j)}\big\|_q\to 0$$

as $j\to 0$. This implies that $m_{z,n}(\psi,\varphi)\to 0$ as $|z|\to\infty$.

Conversely, assume that $\lim\limits_{|z|\to\infty} m_{z,n}(\psi,\varphi)=0$. We see that $W^{(n)}_{\psi,\varphi}$ is bounded by part (1). Let $\{f_j\}\subset F^p(\mathbb{C})$ such that f_j converges uniformly to zero on compact subsets of $\mathbb{C}$ as $j\to\infty$. Let $K_2:=\sup_j\|f_j\|_p^q<\infty$. Since $\lim_{|z|\to\infty}m_{z,n}(\psi,\varphi)=0$, for any $\varepsilon>0$, there exists $R>0$ such that as $|z|>R$ $|\varphi(z)|>1$ and

$$m_{z,n}(\psi,\varphi)<\left(\frac{\varepsilon}{K_2|a|^{-2}}\right)^{\frac{1}{q}}.$$

In addition, there exists a positive integer j_0 such that when $j>j_0$

$$\|\psi\|_q^q\max_{|z|\leqslant R}\big|\big(f_j^{(n)}\circ\varphi\big)(z)\big|^q<\varepsilon.$$

When $j>j_0$, we have

$$\begin{aligned}&\big\|W^{(n)}_{\psi,\varphi}f_j\big\|_q^q\\ =&\frac{q}{2\pi}\left(\int_{|z|\leqslant R}+\int_{|z|>R}\right)|\psi(z)|^q\big|\big(f_j^{(n)}\circ\varphi\big)(z)\big|^q e^{-\frac{q}{2}|z|^2}dA(z)\\ \leqslant&\frac{q}{2\pi}\max_{|z|\leqslant R}\big|\big(f_j^{(n)}\circ\varphi\big)(z)\big|^q\int_{|z|\leqslant R}|\psi(z)|^q e^{-\frac{q}{2}|z|^2}dA(z)\\ &+\frac{q}{2\pi}\max_{|z|>R}\left(|\psi(z)|(1+|\varphi(z)|)^n e^{\frac{|\varphi(z)|^2-|z|^2}{2}}\right)^q\int_{|z|>R}\frac{\big|\big(f_j^{(n)}\circ\varphi\big)(z)\big|^q e^{-\frac{q}{2}|\varphi(z)|^2}}{(1+|\varphi(z)|)^{qn}}dA(z)\\ \lesssim&\|\psi\|_q^q\max_{|z|\leqslant R}\big|\big(f_j^{(n)}\circ\varphi\big)(z)\big|^q+|a|^{-2}\sup_j\|f_j\|_p^q\sup_{|z|>R}\big(m_{z,n}(\psi,\varphi)\big)^q\lesssim\varepsilon.\end{aligned}$$

So $W^{(n)}_{\psi,\varphi}$ is compact from $F^p(\mathbb{C})$ into $F^q(\mathbb{C})$. □

Theorem 4.4. *Let $0<p\leqslant\infty$ and n be a positive integer. Let ψ be a nonzero function in $F^\infty(\mathbb{C})$ and $\varphi(z)=az+b$ with $0<|a|<1$.*

(1) $W^{(n)}_{\psi,\varphi}:F^p(\mathbb{C})\to F^\infty(\mathbb{C})$ is bounded if and only if $m_n(\psi,\varphi)<\infty$. Moreover,

$$m_n(\psi,\varphi)\leqslant\big\|W^{(n)}_{\psi,\varphi}\big\|\lesssim\|\psi\|_\infty+m_n(\psi,\varphi).$$

(2) $W^{(n)}_{\psi,\varphi}:F^p(\mathbb{C})\to F^\infty(\mathbb{C})$ is compact if and only if $\lim\limits_{|z|\to\infty}m_{z,n}(\psi,\varphi)=0$.

Proof. Since the proof is similar to that of Theorem 4.3, we sketch the theorem briefly. First, we consider the case $0 < p < \infty$.

(1) The necessity follows from (4.1). If $m_n(\psi,\varphi) < \infty$, then, for every $f \in F^p(\mathbb{C})$, we conclude from Lemmas 2.10 and 2.2 that

$$\begin{aligned}\|W^{(n)}_{\psi,\varphi}f\|_\infty &\lesssim \Big(\sup_{|\varphi(z)|<1} + \sup_{|\varphi(z)|\geqslant 1}\Big)|\psi(z)|(1+|\varphi(z)|)^n e^{\frac{|\varphi(z)|^2-|z|^2}{2}} \cdot \|f\|_\infty \\ &\lesssim \big(\|\psi\|_\infty + m_n(\psi,\varphi)\big)\cdot\|f\|_p.\end{aligned}$$

Hence, $W^{(n)}_{\psi,\varphi}$ is bounded.

(2) Assume that $W^{(n)}_{\psi,\varphi}$ is compact, let $\{z_j\} \subset \mathbb{C}$ such that $|z_j| \to \infty$ and $k_{\varphi(z_j)} \to 0$ as $j \to \infty$. It follows from Proposition 4.1 and Lemma 4.2 that

$$m_{z_j,n}(\psi,\varphi) \leqslant \|W^{(n)}_{\psi,\varphi}k_{\varphi(z_j)}\|_\infty \to 0$$

as $j \to \infty$. So $\lim_{|z|\to\infty} m_{z,n}(\psi,\varphi) = 0$.

Conversely, $\lim_{|z|\to\infty} m_{z,n}(\psi,\varphi) = 0$. We know that $W^{(n)}_{\psi,\varphi}$ is bounded by part (1). Let $\{f_j\} \subset F^p(\mathbb{C})$ be a bounded sequence and $\{f_j\}$ converge uniformly to zero on compact subsets of $\mathbb{C}$ as $j \to \infty$. Then, for any $\varepsilon > 0$, there exist $R > 0$ and positive integer j_0 such that, as $j > j_0$, $|\varphi(z)| > 1$ and

$$\|W^{(n)}_{\psi,\varphi}f_j\|_\infty \lesssim \|\psi\|_\infty \sup_{|z|\leqslant R}\big|\big(f_j^{(n)}\circ\varphi\big)(z)\big| + \sup_j \|f_j\|_p \sup_{|z|>R} m_{z,n}(\psi,\varphi) \lesssim \varepsilon.$$

Hence, $W^{(n)}_{\psi,\varphi}$ is compact. The proof of $p = \infty$ is similar by a simple modification. This completes the proof. □

As an application of Theorems 4.1, 4.2, 4.3 and 4.4, we can easily obtain the following corollary about the boundedness and compactness of generalized composition operator $C^{(n)}_\varphi$, where $C^{(n)}_\varphi$ is the special case of $W^{(n)}_{\psi,\varphi}$ when $\psi \equiv 1$. We see that when $\varphi(z)$ has the form $\varphi(z) = az+b$ with $|a| < 1$ the boundedness and compactness of $C^{(n)}_\varphi$ between different Fock spaces are equivalent.

Corollary 4.2. *For $0 < p,q \leqslant \infty$, let n be a positive integer. The following statements are equivalent:*

(1) $C^{(n)}_\varphi : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ *is bounded;*
(2) $C^{(n)}_\varphi : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ *is compact;*
(3) $\varphi(z) = az+b$ *with* $|a| < 1$.

Finally, we study the essential norm of $W^{(n)}_{\psi,\varphi} : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ when $1 < p \leqslant q < \infty$ and $1 < p < \infty$, $q = \infty$ with $\varphi(z) = az+b$, and $0 < |a| < 1$. We recall the concept of essential norms for bounded linear operators. Let X and Y be Banach spaces, and let $\mathscr{K}(X,Y)$ be the set of all compact operators from X into Y. The essential norm of a bounded linear operator $L : X \to Y$, denoted by $\|L\|_e$, is defined as

$$\|L\|_e = \inf\{\|L-T\| : T \in \mathscr{K}(X,Y)\}.$$

So L is compact if and only if $\|L\|_e = 0$. The idea of proofs of the following two theorems follow from [5, Theorem 3.8].

Theorem 4.5. *Let $1 < p \leqslant q < \infty$. Let $\psi \in F^q(\mathbb{C})$ be a nonzero function and $\varphi(z) = az+b$ with $0 < |a| < 1$. If $W^{(n)}_{\psi,\varphi} : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ is bounded, then*

$$\limsup_{|z|\to\infty} m_{z,n}(\psi,\varphi) \leqslant \big\|W^{(n)}_{\psi,\varphi}\big\|_e \lesssim |a|^{-\frac{2}{q}} \limsup_{|z|\to\infty} m_{z,n}(\psi,\varphi).$$

Proof. From (4.1), $\limsup_{|z|\to\infty} m_{z,n}(\psi,\varphi) < \infty$. We assume that

$$\big\|W^{(n)}_{\psi,\varphi}\big\|_e < \limsup_{|z|\to\infty} m_{z,n}(\psi,\varphi).$$

Then there exist positive constants $c_1 < c_2$ and a compact operator $T : F^p(\mathbb{C}) \to F^q(\mathbb{C})$ such that

$$\big\|W^{(n)}_{\psi,\varphi} - T\big\| < c_1 < c_2 < \limsup_{|z|\to\infty} m_{z,n}(\psi,\varphi).$$

In particular, there is $\{z_j\}$, $|z_j| \to \infty$ as $j \to \infty$ such that

$$\lim_{j\to\infty} m_{z_j,n}(\psi,\varphi) = \limsup_{|z|\to\infty} m_{z,n}(\psi,\varphi) > c_2. \tag{4.13}$$

On the other hand, applying (4.1), we obtain

$$\begin{aligned} \big\|W^{(n)}_{\psi,\varphi} - T\big\| &\geqslant \big\|W^{(n)}_{\psi,\varphi} k_{\varphi(z_j)} - T k_{\varphi(z_j)}\big\|_q \\ &\geqslant \big\|W^{(n)}_{\psi,\varphi} k_{\varphi(z_j)}\big\|_q - \big\|T k_{\varphi(z_j)}\big\|_q \\ &\geqslant m_{z_j,n}(\psi,\varphi) - \big\|T k_{\varphi(z_j)}\big\|_q. \end{aligned} \tag{4.14}$$

Since $|\varphi(z_j)| \to \infty$ as $j \to \infty$, then $\big\|T k_{\varphi(z_j)}\big\|_q \to 0$ as $j \to \infty$ by Lemma 4.3. Combining (4.13) with (4.14), we see that, when j is sufficiently large,

$$c_1 > \big\|W^{(n)}_{\psi,\varphi} - T\big\| \geqslant m_{z_j,n}(\psi,\varphi) > c_2.$$

This is a contradiction. So

$$\limsup_{|z|\to\infty} m_{z,n}(\psi,\varphi) \leqslant \big\|W^{(n)}_{\psi,\varphi}\big\|_e.$$

For the upper bound, let k be a positive integer, $U_{\varepsilon_k} f(z) = (\varepsilon_k)^{-n} V_{\varepsilon_k} f(z)$, and $V_{\varepsilon_k} f(z) = f(\varepsilon_k z)$ with $\varepsilon_k = \frac{k}{k+1}$. Then according to the proof of Theorem 3.8 in [5], we have U_{ε_k} is compact on $F^p(\mathbb{C})$ and

$$\|U_{\varepsilon_k}\| \leqslant \varepsilon_k^{-\frac{2}{p}-n}$$

for k. Fixing $R>0$ such that $|\varphi(z)|>1$, $|z|>R$, we have

$$
\begin{aligned}
\|W^{(n)}_{\psi,\varphi}\|_e \leqslant & \|W^{(n)}_{\psi,\varphi}-W^{(n)}_{\psi,\varphi}\circ U_{\varepsilon_k}\| \\
= & \sup_{\|f\|_p\leqslant 1}\|W^{(n)}_{\psi,\varphi}\circ(I-U_{\varepsilon_k})f\|_q \\
\leqslant & \sup_{\|f\|_p\leqslant 1}\left(\frac{q}{2\pi}\int_{|z|\leqslant R}\left|W^{(n)}_{\psi,\varphi}\circ(I-U_{\varepsilon_k})f(z)\right|^q e^{-\frac{q}{2}|z|^2}dA(z)\right)^{\frac{1}{q}} \\
& + \sup_{\|f\|_p\leqslant 1}\left(\frac{q}{2\pi}\int_{|z|>R}\left|W^{(n)}_{\psi,\varphi}\circ(I-U_{\varepsilon_k})f(z)\right|^q e^{-\frac{q}{2}|z|^2}dA(z)\right)^{\frac{1}{q}} \\
:= & J_{1,k}+J_{2,k},
\end{aligned}
$$

where I is the identity operator on $F^p(\mathbb{C})$. Applying Lemmas 2.1 and 2.10, we have

$$
\begin{aligned}
J_{2,k}\lesssim & \left(\sup_{|z|>R} m_{z,n}(\psi,\varphi)\right)\times \\
& \sup_{\|f\|_p\leqslant 1}\left(\frac{q}{2\pi}\int_{|z|>R}\frac{\left|\left(((I-U_{\varepsilon_k})f)^{(n)}\circ\varphi\right)(z)\right|^q e^{-\frac{q}{2}|\varphi(z)|^2}}{(1+|\varphi(z)|)^{qn}}dA(z)\right)^{\frac{1}{q}} \\
\lesssim & |a|^{-\frac{2}{q}}\left(\sup_{|z|>R} m_{z,n}(\psi,\varphi)\right)\cdot\sup_{\|f\|_p\leqslant 1}\|(I-U_{\varepsilon_k})f\|_q \\
\lesssim & |a|^{-\frac{2}{q}}\left(\sup_{|z|>R} m_{z,n}(\psi,\varphi)\right)\cdot\sup_{\|f\|_p\leqslant 1}\|(I-U_{\varepsilon_k})f\|_p \\
\lesssim & |a|^{-\frac{2}{q}}\|I-U_{\varepsilon_k}\|\cdot\left(\sup_{|z|>R} m_{z,n}(\psi,\varphi)\right) \\
\lesssim & |a|^{-\frac{2}{q}}\left(1+\varepsilon_k^{-\frac{2}{q}-n}\right)\cdot\left(\sup_{|z|>R} m_{z,n}(\psi,\varphi)\right).
\end{aligned}
$$

For $J_{1,k}$, we have

$$
\begin{aligned}
J_{1,k}\leqslant & \left(\sup_{|z|\leqslant R}|\psi(z)|\right)\cdot\sup_{\|f\|_p\leqslant 1}\left(\frac{q}{2\pi}\int_{|z|\leqslant R}\left|(I-V_{\varepsilon_k})f^{(n)}(\varphi(z))\right|^q e^{-\frac{q}{2}|z|^2}dA(z)\right)^{\frac{1}{q}} \\
\leqslant & \left(\frac{q}{2\pi}\int_{|z|\leqslant R}e^{-\frac{q}{2}|z|^2}dA(z)\right)^{\frac{1}{q}}\cdot\left(\sup_{|z|\leqslant R}|\psi(z)|\right)\cdot\sup_{\|f\|_p\leqslant 1}\sup_{|z|\leqslant R}\left|(I-V_{\varepsilon_k})f^{(n)}(\varphi(z))\right| \\
\leqslant & \left(\sup_{|z|\leqslant R}|\psi(z)|\right)\cdot\sup_{\|f\|_\infty\leqslant 1}\sup_{|z|\leqslant R}\left|(I-V_{\varepsilon_k})f^{(n)}(\varphi(z))\right|.
\end{aligned}
$$

The last inequality follows from the fact $\|f\|_\infty\leqslant\|f\|_p$. For each $f(z)=\sum_{j=0}^\infty a_j z^j$ with $\|f\|_\infty\leqslant 1$, by the estimate in the proof of Theorem 3.8 about a_j in [5], we have

$$|a_j|\leqslant\left(\frac{e}{j}\right)^{\frac{j}{2}}$$

for all $j \geqslant 1$. Setting $R_\varphi = \max_{|z|\leqslant R} |\varphi(z)|$, we obtain

$$\begin{aligned}
J_{1,k} &\leqslant \left(\sup_{|z|\leqslant R}|\psi(z)|\right)\cdot \sup_{\|f\|_\infty\leqslant 1}\sup_{|z|\leqslant R_\varphi}\left|(I-V_{\varepsilon_k})f^{(n)}(z)\right| \\
&\leqslant \left(\sup_{|z|\leqslant R}|\psi(z)|\right) \\
&\quad \cdot \sup_{\|f\|_\infty\leqslant 1}\sup_{|z|\leqslant R_\varphi}\sum_{j=n+1}^{\infty}|a_j|j(j-1)\cdots(j-n+1)\left(1-\left(\frac{k}{k+1}\right)^{j-n}\right)|z|^{j-n} \\
&= \left(\sup_{|z|\leqslant R}|\psi(z)|\right)\cdot \sup_{\|f\|_\infty\leqslant 1}\sup_{|z|\leqslant R_\varphi}\sum_{i=1}^{\infty}|a_{i+n}|\frac{(i+n)!}{i!}\left(1-\left(\frac{k}{k+1}\right)^{i}\right)|z|^{i} \\
&\leqslant \frac{1}{k+1}\left(\sup_{|z|\leqslant R}|\psi(z)|\right)\cdot\sum_{i=1}^{\infty}\frac{(i+n)!}{i!}R_\varphi^i\left(\frac{e}{i+n}\right)^{\frac{i+n}{2}},
\end{aligned}$$

where the series in the last equation is convergent. Then,

$$\begin{aligned}
\left\|W^{(n)}_{\psi,\varphi}\right\|_e &\leqslant \limsup_{k\to\infty}\left\|W^{(n)}_{\psi,\varphi}-W^{(n)}_{\psi,\varphi}\circ U_{\varepsilon_k}\right\| \\
&\leqslant \limsup_{k\to\infty} J_{1,k} + \limsup_{k\to\infty} J_{2,k} \\
&\lesssim |a|^{-\frac{2}{q}}\sup_{|z|>R} m_{z,n}(\psi,\varphi).
\end{aligned}$$

Hence, the desired estimate follows by letting $R \to \infty$. □

Theorem 4.6. *Let $1 < p < \infty$, $\psi \in F^\infty(\mathbb{C})$ be a nonzero function, and $\varphi(z) = az + b$ with $0 < |a| < 1$. If $W^{(n)}_{\psi,\varphi} : F^p(\mathbb{C}) \to F^\infty(\mathbb{C})$ is bounded, then*

$$\left\|W^{(n)}_{\psi,\varphi}\right\|_e \simeq \limsup_{|z|\to\infty} m_{z,n}(\psi,\varphi).$$

Proof. Since the proof is similar to Theorem 4.5 with a simple modification, we omit the proof here. □

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