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SOME FIXED POINT THEOREMS OF GENERALIZED F_t-CONTRACTION MAPPINGS IN b-METRIC SPACES

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Abstract. In this paper, two generalized contractions, the generalized F_{t_s}-contraction and the generalized (ψ,ϕ,F_{t_s})-contraction, are introduced. Two fixed point theorems were established in ordered b-metric spaces. An example is presented for illustrating the fixed point theorem of the generalized F_{t_s}-contraction.

Keywords. Fixed point; Generalized contraction mapping; Ordered b-metric space; t-property.

1. Introduction and Preliminaries

The Banach contraction principle is the most famous fixed point theorem in fixed point theory and finds various applications in nonlinear and variational analysis; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. In 2012, Wardowski [9] gave an interesting generalization of Banach contraction, which is now known as the F-contraction. In 2015, Cosentino et al. [10] further extended the F-contraction in the setting of b-metric spaces, and proved some fixed point theorems.

Definition 1.1 ([11])**.** Let X be a non-empty set and let $s \in [1,\infty)$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x,y,z \in X$ the following conditions are satisfied:

(1) $d(x,y) = 0$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$,
(3) $d(x,z) \leq s[d(x,y)+d(y,z)]$.

Then the triplet (X,d,s) is called a b-metric space.

It is obvious that a metric space is also a b-metric space while the following example shows that the converse may fail.

Example 1.1. Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}^+$ be defined by $d(x,y) = |x-y|^2$ for all $x,y \in X$. Clearly, $(X,d,2)$ is a b-metric space that is not a metric space.

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Recall that a sequence $\{x_n\}$ in a b-metric space (X,d,s) converges to a point $x \in X$ if $\lim_{n\to\infty} d(x_n,x) = 0$. A sequence $\{x_n\}$ in a b-metric space (X,d,s) is a Cauchy sequence if, for each $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for each $m,n \geq N(\varepsilon)$. A b-metric space (X,d,s) is complete if each Cauchy sequence in X converges to some point of X.

Cosentino et al. [10] modified the $\mathscr{F}$-family introduced by Wardowski [9] in the setting of b-metric spaces as follows.

Definition 1.2 ([10])**.** Let $s \geq 1$ be a real number. We denote by $\mathscr{F}_s$ the family of all functions $F : (0,+\infty) \to \mathbb{R}$ with the following properties:

(F_1) F is strictly increasing;
(F_2) for every sequence $\{\alpha_n\}$ in $(0,+\infty)$, we have $\lim_{n\to\infty} F(\alpha_n) = -\infty$ iff $\lim_{n\to\infty} \alpha_n = 0$;
(F_3) for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$ of positive numbers with $\lim_{n\to\infty} \alpha_n = 0$, there exists a number $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} (\alpha_n)^k F(\alpha_n) = -\infty$;
(F_4) for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$ of positive numbers such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$ for all $n \in \mathbb{N}$ and some $\tau \in \mathbb{R}^+$, $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$ for all $n \in \mathbb{N}$.

In 2014, Wardowski and Dung [12] introduced the notion of the F-weak contraction. In the following, we extend this definition to the b-metric spaces as follows.

Definition 1.3. Let (X,d,s) be a b-metric space. A map $T : X \to X$ is said to be an F_s-weak contraction on (X,d) if there exist $F \in \mathscr{F}_s$ and $\tau > 0$ such that, for all $x,y \in X$ satisfying $d(Tx,Ty) > 0$, the following holds:

$$\tau + F(sd(Tx,Ty)) \leq F(m(x,y)),$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

The theory of fixed points in ordered sets was started by Turinici [13] first. In 2004, Ran and Reurings [14] generalized the Banach contraction principle in the setting of ordered sets. The key feature in Ran-Reurings's theorem is that the contractive condition on the nonlinear mapping is only assumed to hold on the comparable elements instead of the whole space as in Banach contraction principle. In 2005, Nieto and Rodrguez-Lopez [15] proved a fixed point theorem by relaxing some conditions in Ran-Reurings [14]. In 2008, Suzuki [16] proved a fixed point theorem by assuming a contraction condition on those elements which satisfy the given condition. Recently, many authors studied the existence of fixed points for generalized contractions in complete b-metric spaces; see, e.g., [10, 17, 18, 19, 20, 21] and the references therein.

Definition 1.4 ([13])**.** A sequence $\{x_n\}$ in a partially ordered set $(X,\preceq)$ is said to be increasing or ascending if, for $m < n$, $x_m \preceq x_n$. It is said to be strictly increasing if $x_m \preceq x_n$ and $x_m \neq x_n$. We denote it as $x_m \prec x_n$.

In 2019, Rashid, Khan and Aydi [22] introduced the t-property for metric spaces. We extend the t-property for b-metric space as follows.

Definition 1.5. Let $(X,d,s,\preceq)$ be any ordered b-metric space. X is said to have the t_s-property if every strictly increasing Cauchy sequence $\{x_n\}$ in X has a strict upper bound in X, i.e., there exists $u \in X$ such that $x_n \prec u$.

Example 1.2. The sets $(a,b], a,b \in \mathbb{R}$, Q, Q^c equipped with the usual ordering and $d(x,y) = |x-y|^2$ have the t_s-property with $s=2$, while they are not complete.

In 2019, Aydi et al. [23] introduced F_t-contraction mappings in partially ordered metric spaces. They also studied the existence of fixed points for self-mappings with the t-property.

Next, we extend it to the b-metric spaces.

Definition 1.6. Let $(X,d,s,\preceq)$ be an ordered metric space and let $T: X \to X$ be a self-mapping. T is said to be an F_{t_s}-contraction if there exist $F \in \mathscr{F}_s$ and $\tau > 0$ such that, for all $x,y \in X$ with $x \neq Tx$, $y \neq Ty$ and $x \prec y$,

$$\tau + F(sd(y,T(y))) \leq F(d(x,T(x))).$$

Remark that, in the case of $s=1$, the above definition reduces the F_t-contraction in [23].

Definition 1.7 ([23])**.** A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a sublinear altering distance function if it satisfies the following:

(1) ψ is monotonic increasing and continuous.
(2) $\psi(t) = 0$ iff $t = 0$.
(3) $\psi(a+b) \leq \psi(a) + \psi(b)$, for any $a,b \in [0,\infty)$.

Example 1.3. The map $\psi : [0,\infty) \to [0,\infty)$ defined by $f(x) = ax$ $(a > 0)$ is a sublinear altering function.

Note that the continuity condition in the definition of the sublinear altering function is essential because if, for example, we define

$$\psi(x) = \begin{cases} x & 0 \leq x \leq 1, \\ \frac{1}{2}x & x > 1, \end{cases}$$

then it is easy to check that ψ satisfies all the conditions of Example 1.3 except the continuity.

In the following we introduce a general form of F_{t_s}-contraction mappings.

Definition 1.8. Let $(X,d,s,\preceq)$ be an ordered metric space and let $T: X \to X$ be a self-mapping. T is said to be a (ψ,ϕ,F_{t_s})-contraction if, for $F \in \mathscr{F}_s$, there exists $\tau > 0$ such that, for all $x,y \in X$ with $x \neq Tx$, $y \neq Ty$ and $x \prec y$,

$$\tau + F[s\psi(d(y,Ty))] \leq F[\psi(d(x,Tx)) - \phi(d(x,Tx))],$$

where ψ is a sublinear altering function, and $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) = 0$ iff $t = 0$ and $\psi(t) > \phi(t)$, for all $t > 0$.

2. Main Results

In this section, two definitions and two fixed point theorems in ordered b-metric spaces are presented.

Definition 2.1. Let $(X,d,s,\preceq)$ be an ordered b-metric space and let $T: X \to X$ be a self-mapping. T is said to be a generalized F_{t_s}-contraction if there exist $F \in \mathscr{F}_s$ and $\tau > 0$ such that for all $x,y \in X$ with $x \neq Tx$, $y \neq Ty$ and $x \prec y$, we have

$$\tau + F(sd(y,T(y))) \leq F(m(x,T(x))), \tag{2.1}$$

where $m(x,y)$ is defined as in Definition 1.3.

Theorem 2.1. *Let $(X,d,s,\preceq)$ be an ordered b-metric space with the t_s-property. Let $T: X \to X$ be a generalized F_{t_s}-contraction. If T is non-decreasing and there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$, then T has a fixed point in X.*

Proof. From the assumption, we have $x_0 \in X$ such that $x_0 \preceq T(x_0)$. If $x_0 = T(x_0)$, then the proof is completed. Otherwise, choose $x_1 = T(x_0)$ such that $x_0 \prec x_1$. Since T is non-decreasing, we have $T(x_0) \preceq T(x_1)$, that is, $x_1 \preceq T(x_1)$. If $x_1 = T(x_1)$, then the proof is completed. Otherwise, choose $x_2 = T(x_1)$ such that $x_1 \prec x_2$. Since T is non-decreasing, we have $T(x_1) \preceq T(x_2)$. Continuing this process, we obtain a strictly increasing sequence $\{x_n\}$ in X such that $x_{n+1} = T(x_n)$. As $x_0 \prec x_1$, we conclude from (2.1) that

$$\tau + F(sd(x_1,T(x_1))) \leq F(m(x_0,T(x_0))). \tag{2.2}$$

But

$$\begin{aligned} m(x_0,T(x_0)) &= \max\Big\{d(x_0,T(x_0)), d(x_0,T(x_0)), d(T(x_0),T(T(x_0))), \\ &\quad \tfrac{d(x_0,T(T(x_0)))+d(T(x_0),T(x_0))}{2}\Big\} \\ &\leq \max\Big\{d(x_0,T(x_0)), d(T(x_0),T(T(x_0))), \tfrac{d(x_0,T(x_0))+d(T(x_0),T(T(x_0)))}{2}\Big\} \\ &\leq \max\{d(x_0,T(x_0)), d(T(x_0),T(T(x_0)))\} \\ &= \max\{d(x_0,T(x_0)), d(x_1,T(x_1))\}. \end{aligned}$$

If $d(x_1,T(x_1)) \geq d(x_0,T(x_0))$, then

$$m(x_0,T(x_0)) \leq d(x_1,T(x_1)).$$

Since F is strictly increasing, We have

$$F(m(x_0,T(x_0))) \leq F(d(x_1,T(x_1))).$$

We obtain from (2.2) that

$$\tau + F(sd(x_1,T(x_1))) \leq F(d(x_1,T(x_1))).$$

Hence, we have $\tau \leq 0$, which is a contradiction. Thus,

$$d(x_1,T(x_1)) \leq d(x_0,T(x_0))$$

and

$$m(x_0,T(x_0)) = d(x_0,T(x_0)).$$

It follows that

$$\tau + F(sd(x_1,T(x_1))) \leq F(m(x_0,T(x_0)) = F(d(x_0,T(x_0))). \tag{2.3}$$

In view of $x_1 \prec x_2$, using the similar method as the above, and equation (2.1) and (F_4), we have

$$\tau + F(s^2d(x_2,T(x_2))) \leq F(sd(x_1,T(x_1))). \tag{2.4}$$

From equations (2.3) and (2.4), we obtain

$$F(s^2d(x_2,T(x_2))) \leq F(sd(x_1,T(x_1))) - \tau \leq F(d(x_0,T(x_0))) - 2\tau.$$

From (F_4), for $n \in \mathbb{N}$, we obtain

$$\begin{aligned} F(s^n d(x_n, T(x_n))) \leq & F(s^{n-1} d(x_{n-1}, T(x_{n-1}))) - \tau \leq \ldots \\ \leq & F(d(x_0, T(x_0))) - n\tau. \end{aligned} \tag{2.5}$$

Hence $\lim_{n\to\infty} F(s^n d(x_n, T(x_n))) = -\infty$. Using property (F_2), one has $\lim_{n\to\infty} s^n d(x_n, T(x_n)) = 0$. Thanks to (F_3), we see that there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (s^n d(x_n, T(x_n)))^k F(s^n d(x_n, T(x_n))) = 0.$$

From equation (2.5), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} (s^n d(x_n, T(x_n)))^k F(s^n d(x_n, T(x_n))) - (s^n d(x_n, T(x_n)))^k F(d(x_0, T(x_0))) \\ \leq -(s^n d(x_n, T(x_n)))^k n\tau \leq 0. \end{aligned}$$

Letting $n \to \infty$, we arrive at

$$\lim_{n\to\infty} n(s^n d(x_n, T(x_n)))^k = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that $n(s^n d(x_n, T(x_n)))^k \leq 1$ for all $n \geq n_1$, that is,

$$s^n d(x_n, T(x_n)) \leq \frac{1}{n^{1/k}}. \tag{2.6}$$

We show that $\{x_n\}$ is a Cauchy sequence. Let $n, m \in \mathbb{N}$ with $m > n \geq n_1$. Using equation (2.6), one has

$$\begin{aligned} d(x_n, x_m) &\leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^{m-n} d(x_{m-1}, x_m) \\ &= s d(x_n, T(x_n)) + s^2 d(x_{n+1}, T(x_{n+1})) + \ldots + s^{m-n} d(x_{m-1}, T(x_{m-1})) \\ &= \sum_{i=n}^{m-1} s^{i-n+1} d(x_i, T(x_i)) \leq \sum_{i=n}^{m-1} s^i d(x_i, T(x_i)) \\ &\leq \sum_{i=n}^{\infty} s^i d(x_i, T(x_i)) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Taking $n \to \infty$, we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Consequently, $\{x_n\}$ is a strictly increasing Cauchy sequence in X. Hence the t-property implies that there exists $u \in X$ such that $x_n \prec u$. If $T(u) = u$, the proof is completed. Otherwise, from equation (2.1), we have

$$\tau + F(sd(u, T(u))) \leq F(m(x_n, T(x_n))), \tag{2.7}$$

where

$$\begin{aligned} m(x_n,T(x_n)) &= \max\Big\{d(x_n,T(x_n)),d(x_n,T(x_n)),d(Tx_n,T(T(x_n))), \\ &\quad \tfrac{d(x_n,T(T(x_n)))+d(Tx_n,T(x_n))}{2}\Big\} \\ &\le \max\Big\{d(x_n,T(x_n)),d(T(x_n),T(T(x_n))), \\ &\quad \tfrac{d(x_n,T(x_n))+d(T(x_n),T(T(x_n)))}{2}\Big\} \\ &\le \max\{d(x_n,T(x_n)),d(T(x_n),T(Tx_n))\} \\ &= \max\{d(x_n,T(x_n)),d(x_{n+1},T(x_{n+1}))\}. \end{aligned}$$

If $d(x_{n+1},T(x_{n+1})) \ge d(x_n,T(x_n))$, then we obtain from (2.7) and (F_4) that

$$\tau + F(s^{n+2}d(u,T(u))) \le F(s^{n+1}d(x_{n+1},T(x_{n+1}))).$$

By using equation (2.5), we obtain

$$F(s^{n+2}d(u,T(u))) \le F(d(x_0,T(x_0))) - (n+2)\tau.$$

If $d(x_{n+1},T(x_{n+1})) \le d(x_n,T(x_n))$, then it follows from (2.7) and (F_4) that

$$\tau + F(s^{n+1}d(u,T(u))) \le F(s^n d(x_n,T(x_n))).$$

By using equation (2.5), we have

$$F(s^{n+1}d(u,T(u))) \le F(d(x_0,T(x_0))) - (n+1)\tau.$$

Therefore it follows from both cases that

$$F(s^i d(u,T(u))) \le F(d(x_0,T(x_0))) - i\tau, \quad \forall i \in \mathbb{N}.$$

By letting i to infinity, we have $\lim_{i\to\infty} F(s^i d(u,T(u))) = -\infty$. From (F_2), we have $\lim_{i\to\infty} s^i d(u,T(u)) = 0$. From $s \ne 0$, one has $T(u) = u$. Thus, u is a fixed point of T in X and the proof is completed. □

The following example illustrates Theorem 2.1.

Example 2.1. Let $A = \{a_n : a_{n+1} = 4a_n + 1 \; for\; n \ge 0 \; and \; a_0 = -1\}$ and $B = (-1,0] \cap \mathbb{Q}$. Take $X = A \cup B$, i.e., $X = \{\dots,-43,-11,-3,-1\} \cup B$. Endow X with b-metric $d(x,y) = |x-y|^2$ for all $x,y \in X$, where $s = 2$ and the natural ordering $\le$. Clearly $(X,d,\preceq)$ is not complete but has the t-property. Define $T : X \to X$ by

$$T(x) = \begin{cases} 4x+1 & if\; x \in A, \\ x & if\; x \in B. \end{cases}$$

Obviously, T is non-decreasing. Now, it remains to prove that T satisfies equation (2.1). Letting $x,y \in X$ with $x < y$, $x \ne Tx$ and $y \ne T(y)$, we have

$$m(x,T(x)) = \max\left\{(3x+1)^2,(12x+4)^2,\frac{(15x+5)^2}{2}\right\} = (12x+4)^2$$

and

$$d(y,T(y)) = (3y+1)^2, \;\; 8x^2 - y \ge 1.$$

If we take $F(\alpha) = \ln(\alpha) + \alpha \in \mathscr{F}$ and $\tau = 2 > 0$, then

$$\begin{aligned} F(m(x,Tx)) - F(sd(y,Ty)) = &\ln(12x+4)^2 + (12x+4)^2 - \ln(2(3xy+1)^2) \\ &-2(3y+1)^2 = \ln\frac{(12x+4)^2}{2(3y+1)^2} + 18(8x^2-y) \\ &+12(8x^2-y) + 14 \geq 14 > 2 = \tau. \end{aligned}$$

Hence,

$$\tau + F(d(y,T(y)) \leq F(d(x,T(x)),$$

i.e., T is a generalized F_t-contraction. Consequently, all the conditions of Theorem 2.1 are satisfied. So, T has fixed point. It is clear that the fixed points set T equals B.

Remark 2.1. It is obvious that Theorem 2.1 is a generalization of the Theorem 1 in [23]. Indeed, if $m(x,y) = d(x,y)$ and $s = 1$, then we obtain the Theorem 1 in [23] immediately.

The following definition is a generalization of Definition1.8.

Definition 2.2. Let $(X,d,s,\preceq)$ be an ordered b-metric space and let $T : X \to X$ be a self-mapping. We say that T is a generalized (ψ,ϕ,F_{t_s})-contraction if there exist $F \in \mathscr{F}_s$ and $\tau > 0$ such that, for all $x,y \in X$ with $x \neq Tx$, $y \neq Ty$ and $x \prec y$,

$$\tau + F[\psi(sd(y,Ty))] \leq F[\psi(m(x,Tx)) - \phi(m(x,Tx))], \tag{2.8}$$

where ψ is a sublinear altering function, $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) = 0$ iff $t = 0$ and $\psi(t) > \phi(t)$, and $m(x,y)$ is defined as in Definition 1.3.

Theorem 2.2. *Let $(X,d,s,\preceq)$ be an ordered b-metric space with the t_s-property. Let $T : X \to X$ be a generalized (ψ,ϕ,F_{t_s})-contraction mapping, where ψ and ϕ are defined as in Definition 2.2, and $m(x,y)$ is defined as in Definition 1.3. Suppose that T is non-decreasing and there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$. Then, T has a fixed point in X.*

Proof. Let $x_0 \in X$ be such that $x_0 \preceq T(x_0)$. If $x_0 = T(x_0)$, then the proof is completed. Otherwise, choose $x_1 = T(x_0)$ such that $x_0 \prec x_1$. Proceeding similarly as Theorem 2.1, we obtain a strictly increasing sequence $\{x_n\}$ in X such that $x_{n+1} = T(x_n)$. As $x_0 \prec x_1$, we conclude from (2.8) that

$$\tau + F[\psi(sd(x_1,T(x_1)))] \leq F[\psi(m(x_0,T(x_0))) - \phi(m(x_0,T(x_0)))]. \tag{2.9}$$

Since $\phi(m(d(x_0,T(x_0)))) \geq 0$, and for each $t > 0$, $\psi(t) > \phi(t)$, one has

$$\psi(m(x_0,T(x_0))) - \phi(m(x_0,T(x_0))) < \psi(m(x_0,T(x_0))).$$

Also, as F is strictly increasing, we obtain

$$F[\psi(m(x_0,T(x_0))) - \phi(m(x_0,T(x_0)))] < F[\psi(m(x_0,T(x_0)))].$$

It follows from (2.9) that

$$\tau + F[\psi(sd(x_1,T(x_1)))] \leq F[\psi(m(x_0,T(x_0)))]. \tag{2.10}$$

But

$$\begin{aligned}
m(x_0,T(x_0)) &= \max\Big\{d(x_0,T(x_0)),d(x_0,T(x_0)),d(T(x_0),T(T(x_0))) \\
&\quad ,\tfrac{d(x_0,T(T(x_0)))+d(T(x_0),T(x_0))}{2}\Big\} \\
&\leq \max\Big\{d(x_0,T(x_0)),d(T(x_0),T(T(x_0))) \\
&\quad ,\tfrac{d(x_0,T(x_0))+d(T(x_0),T(T(x_0)))}{2}\Big\} \\
&\leq \max\{d(x_0,T(x_0)),d(T(x_0),T(T(x_0)))\} \\
&= \max\{d(x_0,T(x_0)),d(x_1,T(x_1))\}.
\end{aligned}$$

If $d(x_1,T(x_1)) \geq d(x_0,T(x_0))$, then

$$m(x_0,T(x_0)) \leq d(x_1,T(x_1)).$$

Since ψ and F are strictly increasing, one has

$$F[\psi(m(x_0,T(x_0)))] \leq F[\psi(d(x_1,T(x_1)))].$$

It follow from (2.10) that

$$\tau + F[\psi(sd(x_1,T(x_1)))] \leq F[\psi(d(x_1,T(x_1)))].$$

So $\tau \leq 0$, which is a contradiction. Hence $d(x_1,T(x_1)) < d(x_0,T(x_0))$ and $m(x_0,T(x_0)) = d(x_0,T(x_0))$. Thus

$$F[\psi(sd(x_1,T(x_1)))] \leq F[\psi(d(x_0,T(x_0)))] - \tau. \tag{2.11}$$

Since $x_1 \prec x_2$, we conclude from (2.8) that

$$\tau + F[\psi(sd(x_2,T(x_2)))] \leq F[\psi(m(x_1,T(x_1))) - \phi(m(x_1,T(x_1)))].$$

In view of (F_4), one has

$$F[\psi(s^2 d(x_2,T(x_2)))] \leq F[\psi(d(x_0,T(x_0)))] - 2\tau.$$

Continuing in this process, for $n \in \mathbb{N}$, one has

$$F[\psi(s^n d(x_n,T(x_n)))] \leq F[\psi(d(x_0,T(x_0)))] - n\tau. \tag{2.12}$$

Hence,

$$\lim_{n\to\infty} F[\psi(s^n d(x_n,T(x_n)))] = -\infty.$$

Using property (F_2), one has

$$\lim_{n\to\infty} \psi(s^n d(x_n,T(x_n))) = 0. \tag{2.13}$$

Thanks to (F_3), we find that there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (\psi(s^n d(x_n,T(x_n))))^k F[\psi(s^n d(x_n,T(x_n)))] = 0. \tag{2.14}$$

From equation (2.12), we obtain

$$\begin{aligned}
(\psi(s^n d(x_n,T(x_n))))^k F[\psi(s^n d(x_n,T(x_n)))] - (\psi(s^n d(x_n,T(x_n))))^k F[d(x_0,T(x_0))] \\
\leq -(\psi(s^n d(x_n,T(x_n))))^k n\tau \leq 0.
\end{aligned} \tag{2.15}$$

Letting $n \to \infty$ in equation (2.15) and using equation (2.13) and (2.14), we obtain

$$\lim_{n\to\infty} n(\psi(s^n d(x_n, T(x_n))))^k = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that $n(\psi(s^n d(x_n, T(x_n))))^k \leq 1$ for all $n \geq n_1$, that is,

$$\psi(s^n d(x_n, T(x_n))) \leq \frac{1}{n^{1/k}}. \tag{2.16}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Let $n, m \in \mathbb{N}$ with $m > n \geq n_1$. Using the triangular inequality, properties of ψ, and equation (2.16), we have

$$\begin{aligned}
\psi(d(x_n, x_m)) &\leq \psi[sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^{m-n} d(x_{m-1}, x_m)] \\
&\leq \psi[sd(x_n, x_{n+1})] + \psi[s^2 d(x_{n+1}, x_{n+2})] + \ldots + \psi[s^{m-n} d(x_{m-1}, x_m)] \\
&= \psi(sd(x_n, T(x_n))) + \psi(s^2 d(x_{n+1}, T(x_{n+1}))) + \ldots \\
&+ \psi(s^{m-n} d(x_{m-1}, T(x_{m-1}))) \\
&= \textstyle\sum_{i=n}^{m-1} \psi(s^{i-n+1} d(x_i, T(x_i))) \leq \sum_{i=n}^{m-1} \psi(s^i d(x_i, T(x_i))) \\
&\leq \textstyle\sum_{i=n}^{\infty} \psi(s^i d(x_i, T(x_i))) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned}$$

Taking $n \to \infty$, we get $\lim_{n,m\to\infty} \psi[d(x_n, x_m)] = 0$. By properties of ψ, we obtain $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Thus, $\{x_n\}$ is a strictly increasing Cauchy sequence in X, which has the t-property. Therefore, there exists $u \in X$ such that $x_n \prec u$. If $T(u) = u$, the proof is completed. Otherwise, by Equation (2.8), we have

$$\tau + F(\psi[sd(u, T(u)))] \leq F[\psi(m(x_n, T(x_n)))) - \phi(m(x_n, T(x_n))))].$$

Since $\phi(m(x_n, T(x_n))) \geq 0$ and for each $t > 0$, $\psi(t) > \phi(t)$, we have

$$\tau + F[\psi(sd(u, T(u)))] \leq F[\psi(m(x_n, T(x_n))))]. \tag{2.17}$$

where

$$\begin{aligned}
m(x_n, T(x_n)) &= \max\Big\{ d(x_n, T(x_n)), d(x_n, T(x_n)), d(Tx_n, T(T(x_n))), \\
&\qquad \tfrac{d(x_n, T(T(x_n))) + d(Tx_n, T(x_n))}{2} \Big\} \\
&\leq \max\Big\{ d(x_n, T(x_n)), d(T(x_n), T(T(x_n))), \\
&\qquad \tfrac{d(x_n, T(x_n)) + d(T(x_n), T(T(x_n)))}{2} \Big\} \\
&\leq \max\{d(x_n, T(x_n)), d(T(x_n), T(Tx_n))\} \\
&= \max\{d(x_n, T(x_n)), d(x_{n+1}, T(x_{n+1}))\}.
\end{aligned}$$

If $d(x_{n+1}, T(x_{n+1})) \geq d(x_n, T(x_n))$, then

$$m(x_n, T(x_n)) \leq d(x_{n+1}, T(x_{n+1})).$$

Since ψ and F are strictly increasing, one has

$$F[\psi(m(x_n, T(x_n)))] \leq F[\psi(d(x_{n+1}, T(x_{n+1})))].$$

So, it follow from (2.17) and (F_4) that

$$\tau+F[\psi(s^{n+2}d(u,T(u)))]\leq F[\psi(s^{n+1}d(x_{n+1},T(x_{n+1}))].$$

By using equation (2.12), we obtain

$$F[\psi(s^{n+2}d(u,T(u)))]\leq F[\psi(d(x_0,T(x_0))]-(n+2)\tau.$$

If $d(x_{n+1},T(x_{n+1}))\leq d(x_n,T(x_n))$, we conclude from equation (2.17) that

$$\tau+F[\psi(s^{n+1}d(u,T(u)))]\leq F[\psi(s^{n}d(x_{n},T(x_{n})))].$$

By using equation (2.12), we obtain

$$F[\psi(s^{n+1}d(u,T(u)))]\leq F[\psi(d(x_0,T(x_0))]-(n+1)\tau.$$

Therefore, in both cases, for each $i\in\mathbb{N}$, we have $\lim_{i\to\infty}F[\psi(s^id(u,T(u)))]=-\infty$. From (F_2), we have $\lim_{i\to\infty}\psi(s^id(u,T(u)))=0$. This implies that $d(u,T(u))=0$, i.e., $T(u)=u$. Thus, u is a fixed point of T in X. □

Remark 2.2. Theorem 2.2 is a generalization of the Theorem 2 in [23]. Indeed, if $m(x,y)=d(x,y)$ and $s=1$, then we have from Theorem 2.2 the Theorem 2 in [23] immediately.

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References

[1] T.H. Cuong, J.C. Yao, N.D. Yen, Qualitative properties of the minimum sum-of-squares clustering problem, Optimization, 69 (2020), 2131-2154.
[2] X. Qin, S.Y. Cho, L. Wang, Iterative algorithms with errors for zero points of m-accretive operators, Fixed Point Theory Appl. 2013 (2013), 148.
[3] P. Cubiotti, J.C. Yao, On the Cauchy problem for a class of differential inclusions with applications, Appl. Anal. 99 (2020), 2543-2554.
[4] S.Y. Cho, A monotone Bregan projection algorithm for fixed point and equilibrium problems in a reflexive Banach space, Filomat, 34 (2020), 1487-1497.
[5] L.V. Nguyen, Q.H. Ansari, X. Qin, Weak sharpness and finite convergence for solutions of nonsmooth variational inequalities in Hilbert spaces, Appl. Math. Optim. 84 (2021), 807-828.
[6] L.V. Nguyen, X. Qin, The minimal time function associated with a collection of sets, ESAIM Control Optim. Calc. Var. 26 (2020), 93.
[7] S.Y. Cho, A convergence theorem for generalized mixed equilibrium problems and multivalued asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 21 (2020), 1017-1026.
[8] Q.H. Ansari, M. Islam, J.C. Yao, Nonsmooth variational inequalities on Hadamard manifolds, Appl. Anal. 99 (2020), 340-358.
[9] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), 94.
[10] M. Cosentino, M. Jleli, B. Samet, C. Vetro, Solvability of integro-differential problem via fixed point theory in *b*-metric spaces, Fixed Point Theory Appl. 2015 (2015), 70.
[11] S. Czerwik, Contraction mappings in *b*-metric spaces, Acta Math. Info. Univ. Ostraviensis. 1 (1993), 5-11.
[12] D. Wardowski, N.V. Dung, Fixed points of F-weak contractions on complete metric spaces, Demonstr. Math. 47 (2014), 146-155.
[13] M. Turinici, Abstract comparison principles and multivariable Gronwall-Bellman inequalities, J. Math. Anal. Appl. 117 (1986), 100-127.

[14] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.

[15] J.J. Nieto, R. Rodrguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Orderd 22 (2005), 223-239.

[16] T. Suzuki, A generalized Banach contraction principle that characterizesmetric completeness, Proc. Amer. Math. Soc. 136 (2008), 1861-1869.

[17] A. Aghajani, M. Abbas, J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca. 64 (2014), 941-960.

[18] H.S. Ding, M. Imdad, S. Radenovic, J. Vujakovic, On some fixed point results in b-metric, rectangular and b-rectangular metric spaces, Arab. J. Math. Sci. 22 (2016), 151-164.

[19] S. Radenovic, T. Došenović, T.A. Lampertc, Z. Golubovíc, A note on some recent fixed point results for cyclic contractions in b-metrics paces and an application to integral equations, Appl. Math. Comput. 273 (2016), 155-164.

[20] N. Goswami, N. Haokip, V.N. Mishra, F-contractive type mappings in b-metric spaces and some related fixed point results, Fixed Point Theory Appl. 2019 (2019), 13.

[21] M.U. Ali, T. Kamran, M. Postolache, Solution of volterra integral inclusion in b-metric spaces via new fixed point theorem, Nonlinear Anal. 22 (2017), 17-30.

[22] T. Rashid, Q. H. Khan, H. Aydi, t-property of metric spaces and fixed point theorems, Ital. J. Pure Appl. Math. 41 (2019), 1-12.

[23] H. Aydi, T. Rashid, Q.H. Khan, Z. Mustafa, M.M. Jaradat, Fixed point results using F_t-contractions in ordered metric spaces having t-property, Symmetry, 11 (2019), 313.