J. Nonlinear Var. Anal. 5 (2021), No. 5, pp. 803-829
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.5.2021.5.12

AN INERTIAL ALGORITHM WITH A SELF-ADAPTIVE STEP SIZE FOR A SPLIT EQUILIBRIUM PROBLEM AND A FIXED POINT PROBLEM OF AN INFINITE FAMILY OF STRICT PSEUDO-CONTRACTIONS

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Abstract. In this paper, we study a split equilibrium problem and a fixed point problem with an infinite family of strict pseudo-contractive mappings. We introduce a new inertial iterative scheme with a self-adaptive step size for obtaining a common solution of the problems. Under mild conditions on the control parameters, we prove a strong convergence result of the proposed algorithm in Hilbert spaces. The implementation of our proposed algorithm does not require a prior estimate of the norm of the bounded linear operator. Finally, we present some numerical experiments to demonstrate the efficiency of the proposed algorithm in comparison with some existing results in the current literature.

Keywords. Equilibrium problem; Fixed point; Monotone operator; Variational inequality; Zero point.

1. Introduction

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, and $\mathbb{N}$ denotes the set of all positive integers. Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The Equilibrium Problem (shortly, (EP)) in the sense of Blum and Oettli [1] is to find $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0, \quad \forall\, y \in C.$$

The set of all solutions of the EP is denoted by $EP(F)$. The EP attracts considerable research efforts and serves as a unifying framework for studying many problems, such as, the nonlinear complementarity problems, variational inequality problems, fixed point problems; see, e.g., [2, 3, 4, 5, 6] and the references therein. Indeed, it also has many real applications in computer science, traffic transportation, and economics; see, e.g., [7, 8, 9, 10] and the references therein.

Let $T : C \to C$ be a nonlinear mapping. A point $x^* \in C$ is called a fixed point of T if $Tx^* = x^*$. We denote by $F(T)$ the set of all fixed points of T, i.e.,

$$F(T) = \{x^* \in C : Tx^* = x^*\}.$$

It is known that several problems in sciences and engineering can be formulated as finding fixed points of of a nonlinear mapping. Recently, the research on common solutions of fixed point

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Received September 24, 2020; Accepted January 12, 2021.

and equilibrium prolems has been investigated by some authors. More precisely, many authors studied the following problem, which consisting of find $\hat{x} \in C$ such that

$$T\hat{x} = \hat{x} \quad \text{and} \quad F(\hat{x}, y) \geq 0, \; \forall \, y \in C,$$

where $F : C \times C \to \mathbb{R}$ is a bifunction and $T : C \to C$ is a nonlinear operator. The importance and motivation for studying such a common solution problem lies in its potential application to mathematical models whose constraints can be expressed as fixed point and equilibrium problems. This arises in practical problems such as signal processing, network resource allocation, image recovery. A scenario is in network bandwidth allocation problem for two services in a heterogeneous wireless access networks in which the bandwidth of the services are mathematically related (see, e.g., [11, 12] and the references therein).

In 1994, Censor and Elfving [13] introduced the following Split Feasibility Problem (SFP) in finite-dimensional Hilbert spaces: Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The SFP is formulated as finding a point $\hat{x}$ with the property

$$\hat{x} \in C \quad \text{and} \quad A\hat{x} \in Q.$$

The SFP has been studied intensively by several authors due to its wide area of applications. The problem is applicable in intensity-modulated radiation therapy, signal processing, image restoration, and computer tomograph; see, e.g., [14, 15, 16].

In 2013, Kazmi and Rizvi [17] introduced and studied the following Split Equilibrium Problem (SEP): Let $C \subseteq H_1$ and $Q \subseteq H_2$. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions. Let $A : H_1 \to H_2$ be a bounded linear operator. The SEP is to find $\hat{x} \in C$ such that

$$F_1(\hat{x}, x) \geq 0, \quad \forall \, x \in C, \tag{1.1}$$

and $\hat{y} = A\hat{x} \in Q$ solves

$$F_2(\hat{y}, y) \geq 0, \quad \forall \, y \in Q. \tag{1.2}$$

Observe that inequality (1.1) is the classical equilibrium problem. (1.1) and (1.2), which consist of a pair of equilibrium problems, is to find the image $\hat{y} = A\hat{x}$ under a given bounded linear operator A of the solution $\hat{x}$ of the problem (1.1) in H_1, which is the solution of the problem (1.2) in H_2. It is easy to see that the SEP includes the SFP as a special case. We denote the solution set of the SEP (1.1)-(1.2) by $SEP(F_1, F_2) = \{\hat{x} \in EP(F_1) : A\hat{x} \in EP(F_2)\}$.

In 2016, Suantai *et al.* [18] proposed the following iterative algorithm, Algorithm 1.1, for finding a solution of the SEP and a fixed point of a nonspreading multivalued mapping in Hilbert spaces:

$$\begin{aligned} &x_1 \in C, \\ &u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ &x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)Su_n, \quad \forall \, n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\} \subset (0,1)$, $\{r_n\} \subset (0, \infty)$ and $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of A^*A, where A^* is the adjoint of A, $C \subset H_1$, $Q \subset H_2$, $S : C \to K(C)$ is a $\frac{1}{2}$-nonspreading multivalued mapping, $K(C)$ is the collection of all nonempty compact subsets of C, and $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions. They proved that the sequence $\{x_n\}$ generated by the above scheme converges weakly to an element in $F(S) \cap SEP(F_1, F_2)$ under certain conditions.

In solving optimization problems, the strong convergence of iterative schemes are more desirable and useful than their weak convergence counterparts according to Bauschke and Combettes [19]. Hence, constructing iterative schemes that generate strong convergence sequence is necessary and helpful.

In 2017, Zhang and Gui [20] introduced the following iterative algorithm, Algorithm 1.2, for finding a solution the SEP and a common fixed point of a finite family of asymptotically nonexpansive mappings in Hilbert spaces:

$$\begin{aligned}
&x_0 \in C,\\
&u_n = T_{r_n}^{F_1}(I-\gamma A^*(I-T_{s_n}^{F_2})A)x_n,\\
&x_{n+1} = \alpha_n f(x_n) + \frac{(1-\alpha_n)}{m}\sum_{i=1}^{m} T_i^n u_n, \quad n \in \mathbb{N},
\end{aligned}$$

where $\{\alpha_n\} \subset (0,1), \{r_n\} \subset [r,\infty)$ with $r>0, \{s_n\} \subset [s,\infty)$ with $s>0$, $\gamma \in (0,\frac{1}{L^2})$, where L is the spectral radius of A^*A and A^* is the adjoint of $A, C \subset H_1, Q \subset H_2, f: C \to C$ is a ρ-contraction mapping, $\{T_i\}_{i=1}^m : C \to C$ is a finite family of asymptotically nonexpansive mappings with the same sequence $\{k_n\}$, and $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions. Under the following conditions on the control parameters:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty, \sum_{n=1}^{\infty} |s_n - s_{n-1}| < \infty$;
(iii) $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$;
(iv) $\lim_{n\to\infty} \sup_{x\in K} ||T_i^{n+1}x - T_i^n x|| = 0$;

where K is any bounded subset of C, they proved that the sequence $\{x_n\}$ generated by the above scheme converges strongly to an element of $\bigcap_{i=1}^m F(T_i) \cap SEP(F_1,F_2)$.

However, we notice that conditions (ii) is restrictive. Hence, the following question arise naturally. Can we devise an iterative scheme and obtain a strong convergence theorem under mild restrictions. In this paper, we will provide an affirmative answer to this question.

It is also important to point out that the step size γ of the above scheme plays a crucial role in the convergence analysis. The results obtained in [18] and [20], and several other related results in literature involve the step size that requires prior knowledge of operator norms. One of the limitations of such schemes is that they are usually difficult to implement. Furthermore, the step size defined by such scheme are often very small and deteriorates the rate of convergence. In practice, a larger step size can often be used to yield better numerical results. In optimization theory, the second-order dynamical system, which is called the heavy ball method, was used to accelerate the convergence rate of iterative schemes. This method, which is a two-step iterative method for minimizing a smooth convex function, was first introduced by Polyak [21]. The following algorithm introduced by Nesterov [22] is a modified heavy ball method for the improvement of the convergence rate:

$$\begin{aligned}
&y_n = x_n + \theta_n(x_n - x_{n-1}),\\
&x_{n+1} = y_n - \lambda_n \nabla f(y_n), \quad \forall n \geq 1,
\end{aligned}$$

where $\lambda_n > 0, \theta_n \in [0,1)$ is an extrapolation factor, f is a convex function and ∇f is the gradient of f. Here, the term $\theta_n(x_n - x_{n-1})$ is the inertia.

Recently, several authors have constructed some various iterative algorithms by using inertial extrapolation technique; see, e.g., [5, 9, 23, 24, 25]. Motivated by the above results and the ongoing research interest in this direction, we introduce a new inertial iterative scheme with a self-adaptive step size for finding solutions of the SEP and common fixed points of an infinite family of strict pseudo-contractive mappings. Our algorithm is designed such that its implementation does not require a prior estimate of the norm of the bounded linear operator. We prove a strong convergence theorem in Hilbert spaces. Moreover, we also consider the zero point problems of maximal monotone operators, split generalized equilibrium problems and split variational inequality problems. We present some numerical experiments to demonstrate the efficiency of the proposed scheme in comparison with some existing results in the current literature.

2. Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Let $P_C : H \to C$ be the metric projection, which assigns each $x \in H$ to its unique element $P_C x \in C$, that is,$||x - P_C x|| = \inf\{||x - z|| : z \in C\}$. It is known that P_C is nonexpansive and has the following properties:

(1) $||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y\rangle$, $\forall x, y \in C$;

(2) for any $x \in H$ and $y \in C$, $||P_C x - y||^2 + ||x - P_C x||^2 \leq ||x - y||^2$.

For any $x, y \in H$ with $y \neq 0$, let $Q = \{z \in H : \langle y, z - x\rangle \leq 0\}$. Then, for all $u \in H$, $P_Q(u)$ is given by

$$P_Q(u) = u - \max\Big\{0, \frac{\langle y, u - x\rangle}{||y||^2}\Big\}y,$$

which gives an explicit formula for computing the projection of any point onto a half-space. In what follows, we denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively, and $w_\omega(x_n)$ denotes set of weak limits of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

Now, we present some definitions and results which are necessary in our convergence analysis.

Definition 2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be:

(1) *L-Lipschitz continuous* with $L > 0$ if

$$||Tx - Ty|| \leq L||x - y||, \quad \forall\, x, y \in C;$$

if $L \in [0, 1)$, then T is called a *contraction mapping;*

(2) *nonexpansive* if T is $1-$Lipschitz continuous;

(3) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n||x - y||, \quad \forall\, x, y \in C;$$

(4) *k-strictly pseudo-contractive* if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \quad \forall\, x, y \in C;$$

(5) *monotone* if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall\, x, y \in C.$$

Observe that the class of k-strictly pseudo-contractive mappings properly contains the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. It is known that if T is a k-strict pseudo-contraction and $F(T) \neq \emptyset$, then $F(T)$ is a closed convex subset of H (see [26]). Strict pseudo-contractions have many applications due to their ties with inverse-strongly monotone operators. So, one can recast the problem of zeros for variational inequalities with inverse-strongly monotone operators as a fixed point problem of strict pseudo-contractions.

The following equalities are trivial in Hilbert spaces

(i) $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle$;
(ii) $||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$;
(iii) $||\delta x + (1-\delta)y||^2 = \delta||x||^2 + (1-\delta)||y||^2 - \delta(1-\delta)||x-y||^2$.

Each Hilbert space H satisfies the Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$ holds for every $y \in H$ with $y \neq x$.

Recall that a space is said to satisfy the Opial's condition if for any sequence $\{x_n\}$ in the space with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$ holds for every $y \in H$ with $y \neq x$. It is known that Hilbert spaces enjoy the Opial's condition. Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a bounded linear operator $D : C \to H$ is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$ such that $\langle Dx, x\rangle \geq \bar{\gamma}||x||^2$, for all $x \in C$.

Definition 2.2. [27] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of k_n-strict pseudo-contractions. Define $S'_n = t_n I + (1-t_n)S_n, t_n \in [k_n, 1)$. Consider the mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n S'_n U_{n,n+1} + (1-\zeta_n)I, \\ U_{n,n-1} = \zeta_{n-1} S'_{n-1} U_{n,n} + (1-\zeta_{n-1})I, \\ \cdots, \\ U_{n,k} = \zeta_k S'_k U_{n,k+1} + (1-\zeta_k)I, \\ U_{n,k-1} = \zeta_{k-1} S'_{k-1} U_{n,k} + (1-\zeta_{k-1})I, \\ \cdots, \\ U_{n,2} = \zeta_2 S'_2 U_{n,3} + (1-\zeta_2)I, \\ W_n = U_{n,1} = \zeta_1 S'_1 U_{n,2} + (1-\zeta_1)I, \end{cases} \tag{2.1}$$

where $\{\zeta_i\}$ is a sequence of real numbers such that $0 \leq \zeta_i \leq 1$ for all $i \geq 1$. For each $n \geq 1$, such a mapping W_n is nonexpansive.

We have the following lemmas related to the mapping W_n.

Lemma 2.1. [28] *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{S'_i\}$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(S'_i) \neq \emptyset$ and let $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq b < 1$ for all $i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(S'_i)$ for each $n \geq 1$;*

(2) *for each $x \in C$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}x$ exists;*
(3) *the mapping W of C into itself defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i')$, which is called the modified W-mapping generated by $S_1, S_2, \cdots, \zeta_1, \zeta_2, \cdots$ and $t_1, t_2, \cdots$.

From [26], we have from Lemma 2.1 that $F(W) = \bigcap_{i=1}^{\infty} F(S_i') = \bigcap_{i=1}^{\infty} F(S_i)$.

Lemma 2.2. [29] *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{S_i'\}$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(S_i') \neq \emptyset$ and let $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq b < 1$ for all $i \geq 1$, where b is a positive real number. If K is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in K} ||Wx - W_n x|| = 0$.*

Lemma 2.3. [26] *Let C be a nonempty closed convex subset of a Hilbert space H. If S is a k-strict pseudo-contraction defined on C, then $I - S$ is demiclosed at any point $y \in H$.*

Lemma 2.4. [30] *Let $\{a_n\}$ be a sequence of non-negative real numbers. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and let $\{b_n\}$ be a sequence of real numbers. Let $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n$, for all $n \geq 1$. If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.5. [31] *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}_+, \{\sigma_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that $a_{n+1} \leq (1-\sigma_n)a_n + b_n + c_n$ for all $n \geq 0$. Assume $\sum_{n=0}^{\infty} |c_n| < \infty$. Then the following results hold.*
(1) If $b_n \leq \beta\sigma_n$ for some $\beta \geq 0$, then $\{a_n\}$ is a bounded sequence.
(2) If $\sum_{n=0}^{\infty} \sigma_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\sigma_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Assumption 2.1. For solving the equilibrium problem, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) F is upper hemicontinuous, that is, for all $x, y, z \in C$, $\lim_{t\downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x,y)$;
(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.6. [1] *Let C be a nonempty closed convex subset of a Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\, y \in C\}.$$

Then T_r^F is well defined and the following hold:

(1) for each $x \in H, T_r^F(x) \neq \emptyset$;
(2) T_r^F is single-valued;
(3) $\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle$, for any $x, y \in H$, that is, T_r^F is a firmly nonexpansive mapping.
(4) $F(T_r^F) = EP(F)$;
(5) $EP(F)$ is closed and convex.

3. Main Results

In this section, we present our proposed iterative scheme and discuss its convergence.

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint $A^* : H_2 \to H_1$. Suppose that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Assumption 2.1 with F_2 being upper semicontinuous in the first argument, and let $\{W_n\}$ be the sequence defined by (2.1). Let $D : H \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma}$, and let $f : H \to H$ be a contraction with coefficient $\rho \in (0,1)$ such that $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Suppose that the solution set denoted by $\Omega = SEP(F_1,F_2) \cap \bigcap_{i=1}^{\infty} F(S_i)$ is nonempty, where $S_i : C \to C$ is an infinite family of k_i-strict pseudo-contractions. It is known that $SEP(F_1,F_2)$ and $\bigcap_{i=1}^{\infty} F(S_i)$ are closed and convex. Hence, it follows that the solution set Ω is closed and convex and the projection P_Ω is well defined. We establish the convergence of the scheme under the following conditions on the control parameters:

(C1) Let $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset (0,1)$;

(C2) Let $\theta > 0$, $\{\delta_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\delta_n}{\alpha_n} = 0$, and $0 < a \leq \tau_n \leq b < 1$;

(C3) $\{r_n\} \subset (0,\infty)$ and $\{s_n\} \subset (0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$ and $\liminf_{n\to\infty} s_n > 0$.

Now, our main algorithm is presented as follows.

Algorithm 3.1.

Step 0. Let $x_0, x_1 \in C$ be two arbitrary initial points and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\delta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}. \end{cases} \tag{3.1}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = T_{r_n}^{F_1}(w_n + \gamma_n A^*(T_{s_n}^{F_2} - I)Aw_n),$$

where

$$\gamma_n := \begin{cases} \tau_n \frac{||(T_{s_n}^{F_2} - I)Aw_n||^2}{||A^*(T_{s_n}^{F_2} - I)Aw_n||^2}, & \text{if } Aw_n \neq T_{s_n}^{F_2}Aw_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases} \tag{3.2}$$

Step 4. Compute

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n D)[(1-\beta_n)z_n + \beta_n W_n z_n].$$

Set $n := n+1$ and return to **Step 1**.

Remark 3.1. From conditions (C1) and (C2), one can easily verify from (3.1) that

$$\lim_{n\to\infty} \theta_n ||x_n - x_{n-1}|| = 0 \text{ and } \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0.$$

Also, observe that, in (3.2), the choice of γ_n is independent on the norm of operator $||A||$. The value of λ does not influence the considered algorithm but was introduced for clarity.

We first establish the following lemmas which are essential in proving our strong convergence result.

Lemma 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\{x_n\}$ is bounded.*

Proof. First, we show that $P_\Omega(I-D+\gamma f)$ is a contraction on H_1. For all $x,y\in H_1$, we have that

$$\begin{aligned}
&||P_\Omega(I-D+\gamma f)(x)-P_\Omega(I-D+\gamma f)(y)||\\
&\leq ||(I-D)x-(I-D)y||+\gamma||fx-fy||\\
&\leq (1-(\bar{\gamma}-\gamma\rho))||x-y||.
\end{aligned}$$

This shows that $P_\Omega(I-D+\gamma f)$ is a contraction. Hence, there exists an element $p\in\Omega$ such that $p=P_\Omega(I-D+\gamma f)(p)$. Since $p\in\Omega$, then $p=T_{r_n}^{F_1}p$ and $Ap=T_{s_n}^{F_2}(Ap)$. Also, since $T_{r_n}^{F_1}$ is nonexpansive, we have

$$\begin{aligned}
||z_n-p||^2 &= ||T_{r_n}^{F_1}(w_n+\gamma_n A^*(T_{s_n}^{F_2}-I)Aw_n)-p||^2\\
&\leq ||w_n+\gamma_n A^*(T_{s_n}^{F_2}-I)Aw_n-p||^2 \qquad (3.3)\\
&= ||w_n-p||^2+\gamma_n^2||A^*(T_{s_n}^{F_2}-I)Aw_n||^2+2\gamma_n\langle w_n-p,A^*(T_{s_n}^{F_2}-I)Aw_n\rangle. \qquad (3.4)
\end{aligned}$$

From the nonexpansivity of $T_{s_n}^{F_2}$, we obtain

$$\begin{aligned}
&\langle w_n-p,A^*(T_{s_n}^{F_2}-I)Aw_n\rangle\\
&=\langle T_{s_n}^{F_2}Aw_n-Ap-(T_{s_n}^{F_2}-I)Aw_n,(T_{s_n}^{F_2}-I)Aw_n\rangle\\
&=\langle T_{s_n}^{F_2}Aw_n-Ap,(T_{s_n}^{F_2}-I)Aw_n\rangle-\langle(T_{s_n}^{F_2}-I)Aw_n,(T_{s_n}^{F_2}-I)Aw_n\rangle\\
&=\frac{1}{2}\left[||T_{s_n}^{F_2}Aw_n-Ap||^2+||(T_{s_n}^{F_2}-I)Aw_n||^2-||T_{s_n}^{F_2}Aw_n-Ap-(T_{s_n}^{F_2}-I)Aw_n||^2\right]\\
&-||(T_{s_n}^{F_2}-I)Aw_n||^2\\
&=\frac{1}{2}\left[||T_{s_n}^{F_2}Aw_n-Ap||^2-||Aw_n-Ap||^2-||(T_{s_n}^{F_2}-I)Aw_n||^2\right]\\
&\leq -\frac{1}{2}||(T_{s_n}^{F_2}-I)Aw_n||^2. \qquad (3.5)
\end{aligned}$$

Substituting (3.5) into (3.4), and using the definition of γ_n and the condition on τ_n, we obtain

$$\begin{aligned}
||z_n-p||^2 &\leq ||w_n-p||^2+\gamma_n^2||A^*(T_{s_n}^{F_2}-I)Aw_n||^2-\gamma_n||(T_{s_n}^{F_2}-I)Aw_n||^2\\
&= ||w_n-p||^2-\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2 \qquad (3.6)\\
&\leq ||w_n-p||^2. \qquad (3.7)
\end{aligned}$$

Note that

$$||w_n-p||\leq||x_n-p||+\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||. \qquad (3.8)$$

From Remark 3.1, and $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||=0$, we have that there exists a constant $M_1>0$ such that $\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\leq M_1$ for all $n\geq 1$. Hence, it follows from (3.8) that

$$||w_n-p||\leq||x_n-p||+\alpha_n M_1. \qquad (3.9)$$

Define $U_n = (1-\beta_n)I + \beta_n W_n$ for each $n \geq 1$. It follows that

$$||U_n z_n - p|| = ||(1-\beta_n)(z_n - p) + \beta_n(W_n z_n - p)|| \leq ||z_n - p||. \tag{3.10}$$

Hence, by using (3.7), (3.9) and (3.10), we have

$$\begin{aligned}
||x_{n+1} - p|| &= ||\alpha_n(\gamma f(x_n) - Dp) + (I - \alpha_n D)(U_n z_n - p)|| \\
&\leq \alpha_n ||\gamma(f(x_n) - f(p)) + (\gamma f(p) - Dp)|| + (1 - \alpha_n\bar{\gamma})||z_n - p|| \\
&\leq \alpha_n \gamma\rho ||x_n - p|| + \alpha_n ||\gamma f(p) - Dp|| + (1 - \alpha_n\bar{\gamma})(||x_n - p|| + \alpha_n M_1) \\
&= (1 - \alpha_n(\bar{\gamma} - \gamma\rho))||x_n - p|| + \alpha_n(\bar{\gamma} - \gamma\rho)\Big\{\frac{||\gamma f(p) - Dp||}{\bar{\gamma} - \gamma\rho} + \frac{(1 - \alpha_n\bar{\gamma})}{\bar{\gamma} - \gamma\rho}M_1\Big\} \\
&\leq (1 - \alpha_n(\bar{\gamma} - \gamma\rho))||x_n - p|| + \alpha_n(\bar{\gamma} - \gamma\rho)M^*,
\end{aligned}$$

where

$$M^* := \sup_{n\in\mathbb{N}}\Big\{\frac{||\gamma f(p) - Dp||}{\bar{\gamma} - \gamma\rho} + \frac{(1 - \alpha_n\bar{\gamma})}{\bar{\gamma} - \gamma\rho}M_1\Big\}.$$

Setting $a_n := ||x_n - p||$, $b_n := \alpha_n(\bar{\gamma} - \gamma\rho)M^*$, $c_n := 0$, and $\sigma_n := \alpha_n(\bar{\gamma} - \gamma\rho)$. From Lemma 2.5 and the assumptions on the control sequences, we have that $\{||x_n - p||\}$ is bounded and this implies that $\{x_n\}$ is bounded. Consequently, $\{w_n\}$ and $\{z_n\}$ are also bounded. $\square$

Lemma 3.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1 and $p \in \Omega$. Then, under conditions* (C1)-(C3)*,*

$$\begin{aligned}
||x_{n+1} - p||^2 \leq \Big(1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_n\gamma\rho)}\Big)||x_n - p||^2 + \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\bar{\gamma}^2 M}{2(\bar{\gamma} - \gamma\rho)} \\
+ \frac{3M_2(1 - \alpha_n\bar{\gamma})^2}{2(\bar{\gamma} - \gamma\rho)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| + \frac{1}{(\bar{\gamma} - \gamma\rho)}\langle\gamma f(p) - Dp, x_{n+1} - p\rangle\Big\} \\
- \frac{(1 - \alpha_n\bar{\gamma})^2}{(1 - \alpha_n\gamma\rho)}\big[\gamma_n(1 - \tau_n)||(T^{F_2}_{s_n} - I)Aw_n||^2 + \beta_n(1 - \beta_n)||W_n z_n - z_n||^2\big].
\end{aligned}$$

Proof. Let $p \in \Omega$. From the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
||w_n - p||^2 &= ||x_n - p||^2 + \theta_n^2||x_n - x_{n-1}||^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq ||x_n - p||^2 + \theta_n||x_n - x_{n-1}||(\theta_n||x_n - x_{n-1}|| + 2||x_n - p||) \\
&\leq ||x_n - p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||,
\end{aligned} \tag{3.11}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{||x_n - p||, \theta_n||x_n - x_{n-1}||\} > 0$.

Next, by using (3.6) and (3.11), we obtain

$$\begin{aligned}
||U_n z_n - p||^2 &\leq (1 - \beta_n)||z_n - p||^2 + \beta_n||W_n z_n - p||^2 - \beta_n(1 - \beta_n)||W_n z_n - z_n||^2 \\
&\leq ||z_n - p||^2 - \beta_n(1 - \beta_n)||W_n z_n - z_n||^2 \\
&\leq ||x_n - p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| - \gamma_n(1 - \tau_n)||(T^{F_2}_{s_n} - I)Aw_n||^2 \\
&\quad - \beta_n(1 - \beta_n)||W_n z_n - z_n||^2.
\end{aligned} \tag{3.12}$$

In view of (3.12), we conclude

$$\begin{aligned}
&||x_{n+1}-p||^2\\
&=||\alpha_n(\gamma f(x_n)-Dp)+(I-\alpha_n D)(U_n z_n-p)||^2\\
&\le(1-\alpha_n\bar{\gamma})^2\Big\{||x_n-p||^2+3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||-\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2\\
&\quad-\beta_n(1-\beta_n)||W_n z_n-z_n||^2\Big\}+2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle\\
&+2\alpha_n\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&\le(1-\alpha_n\bar{\gamma})^2||x_n-p||^2+3M_2(1-\alpha_n\bar{\gamma})^2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\\
&\quad-(1-\alpha_n\bar{\gamma})^2\Big\{\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2+\beta_n(1-\beta_n)||W_n z_n-z_n||^2\Big\}\\
&+2\alpha_n\gamma\rho||x_n-p||||x_{n+1}-p||+2\alpha_n\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&\le(1-\alpha_n\bar{\gamma})^2||x_n-p||^2+3M_2(1-\alpha_n\bar{\gamma})^2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\\
&\quad-(1-\alpha_n\bar{\gamma})^2\Big\{\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2+\beta_n(1-\beta_n)||W_n z_n-z_n||^2\Big\}\\
&\quad+\alpha_n\gamma\rho(||x_n-p||^2+||x_{n+1}-p||^2)+2\alpha_n\langle\gamma f(p)-Dp,x_{n+1}-p\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
&||x_{n+1}-p||^2\\
&\le\frac{(1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}||x_n-p||^2+3M_2\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\\
&+\frac{2\alpha_n}{(1-\alpha_n\gamma\rho)}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle-\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2\\
&+\beta_n(1-\beta_n)||W_n z_n-z_n||^2\Big\}\\
&=\frac{(1-2\alpha_n\bar{\gamma}+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}||x_n-p||^2+\frac{(\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}||x_n-p||^2\\
&+3M_2\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{2\alpha_n}{(1-\alpha_n\gamma\rho)}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&\quad-\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2+\beta_n(1-\beta_n)||W_n z_n-z_n||^2\Big\}\\
&\le\Big(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big)||x_n-p||^2+\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\rho)}M\\
&\quad+\frac{3M_2(1-\alpha_n\bar{\gamma})^2}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{1}{(\bar{\gamma}-\gamma\rho)}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\Big\}\\
&\quad-\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\gamma_n(1-\tau_n)||(T_{s_n}^{F_2}-I)Aw_n||^2+\beta_n(1-\beta_n)||W_n z_n-z_n||^2\Big\},
\end{aligned}$$

where $M:=\sup\{||x_n-p||^2:n\in\mathbb{N}\}$. This completes the proof. $\square$

Lemma 3.3. *The following inequality holds*

$$||x_{n+1}-p||^2 \le (1-\alpha_n\bar{\gamma})^2\big\{||x_n-p||^2+3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||-||z_n-w_n||^2 \\ +2\gamma_n||z_n-w_n||||A^*(T_{s_n}^{F_2}-I)Aw_n||\big\}+2\alpha_n\langle\gamma f(x_n)-Dp,x_{n+1}-p\rangle, \quad \forall p\in\Omega.$$

Proof. Let $p\in\Omega$. From (3.3) and (3.7), we observe that

$$||w_n+\gamma_n A^*(T_{s_n}^{F_2}-I)Aw_n-p||^2 \le ||w_n-p||^2.$$

From the firm nonexpansivity of $T_{r_n}^{F_1}$, we have

$$\begin{aligned}
||z_n-p||^2 &\le \langle z_n-p, w_n+\gamma_n A^*(T_{s_n}^{F_2}-I)Aw_n-p\rangle \\
&= \frac{1}{2}\big\{||z_n-p||^2+||w_n+\gamma_n A^*(T_{s_n}^{F_2}-I)Aw_n-p||^2-||(z_n-p) \\
&\quad -(w_n+\gamma_n A^*(T_{s_n}^{F_2}-I)Aw_n-p)||^2\big\} \\
&\le \frac{1}{2}\big\{||z_n-p||^2+||w_n-p||^2-||z_n-w_n-\gamma_n A^*(T_{s_n}^{F_2}-I)Aw_n||^2\big\} \\
&= \frac{1}{2}\big\{||z_n-p||^2+||w_n-p||^2-\big(||z_n-w_n||^2+\gamma_n^2||A^*(T_{s_n}^{F_2}-I)Aw_n||^2 \\
&\quad -2\gamma_n\langle z_n-w_n, A^*(T_{s_n}^{F_2}-I)Aw_n\rangle\big)\big\} \\
&\le \frac{1}{2}\big\{||z_n-p||^2+||w_n-p||^2-||z_n-w_n||^2-\gamma_n^2||A^*(T_{s_n}^{F_2}-I)Aw_n||^2 \\
&\quad +2\gamma_n||z_n-w_n||||A^*(T_{s_n}^{F_2}-I)Aw_n||\big\} \\
&\le \frac{1}{2}\big\{||z_n-p||^2+||w_n-p||^2-||z_n-w_n||^2+2\gamma_n||z_n-w_n||||A^*(T_{s_n}^{F_2}-I)Aw_n||\big\},
\end{aligned}$$

which implies that

$$||z_n-p||^2 \le ||w_n-p||^2-||z_n-w_n||^2+2\gamma_n||z_n-w_n||||A^*(T_{s_n}^{F_2}-I)Aw_n||. \tag{3.13}$$

Using (3.10), (3.11) and (3.13), we have

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||\alpha_n(\gamma f(x_n)-Dp)+(I-\alpha_n D)(U_n z_n-p)||^2 \\
&\le (1-\alpha_n\bar{\gamma})^2||U_n z_n-p||^2+2\alpha_n\langle\gamma f(x_n)-Dp,x_{n+1}-p\rangle \\
&\le (1-\alpha_n\bar{\gamma})^2\big\{||x_n-p||^2+3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||-||z_n-w_n||^2 \\
&\quad +2\gamma_n||z_n-w_n||||A^*(T_{s_n}^{F_2}-I)Aw_n||\big\}+2\alpha_n\langle\gamma f(x_n)-Dp,x_{n+1}-p\rangle,
\end{aligned}$$

which is the required inequality. □

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1 and $p\in\Omega$. Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that*

$$\liminf_{k\to\infty}(||x_{n_k+1}-p||-||x_{n_k}-p||)\ge 0.$$

Then $x_{n_k}\rightharpoonup x^*\in\Omega$, i.e., $w_\omega(x_n)\subset\Omega$.

Proof. Let $p \in \Omega$. It follows from Lemma 3.2 that

$$\begin{aligned}
&\frac{(1-\alpha_{n_k}\bar{\gamma})^2}{(1-\alpha_{n_k}\gamma\rho)}\beta_{n_k}(1-\beta_{n_k})||W_{n_k}z_{n_k}-z_{n_k}||^2 \\
&\leq \Big(1-\frac{2\alpha_{n_k}(\bar{\gamma}-\gamma\rho)}{(1-\alpha_{n_k}\gamma\rho)}\Big)||x_{n_k}-p||^2-||x_{n_k+1}-p||^2+\frac{2\alpha_{n_k}(\bar{\gamma}-\gamma\rho)}{(1-\alpha_{n_k}\gamma\rho)}\Big\{\frac{\alpha_{n_k}\bar{\gamma}^2M}{2(\bar{\gamma}-\gamma\rho)} \\
&\quad+\frac{3M_2(1-\alpha_{n_k}\bar{\gamma})^2}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||+\frac{1}{(\bar{\gamma}-\gamma\rho)}\langle\gamma f(p)-Dp,x_{n_k+1}-p\rangle\Big\}.
\end{aligned}$$

From Lemma 3.4 together with the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we obtain

$$\frac{(1-\alpha_{n_k}\bar{\gamma})^2}{(1-\alpha_{n_k}\gamma\rho)}\beta_{n_k}(1-\beta_{n_k})||W_{n_k}z_{n_k}-z_{n_k}||^2\to 0, \quad k\to\infty.$$

Hence, it follows that

$$||W_{n_k}z_{n_k}-z_{n_k}||\to 0, \quad k\to\infty. \tag{3.14}$$

Following a similar argument, we obtain from Lemma 3.2 that

$$\gamma_{n_k}(1-\tau_{n_k})||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||^2\to 0, \quad k\to\infty.$$

From the definition of γ_n, we have

$$\tau_{n_k}(1-\tau_{n_k})\frac{||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||^4}{||A^*(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||^2}\to 0, \quad k\to\infty,$$

which implies that

$$\frac{||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||^2}{||A^*(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||}\to 0, \quad k\to\infty.$$

Since $||A^*(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||$ is bounded, then it follows that

$$||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||\to 0, \quad k\to\infty. \tag{3.15}$$

Hence,

$$||A^*(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||\leq||A^*||||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||=||A||||(T^{F_2}_{s_{n_k}}-I)Aw_{n_k}||\to 0, \quad k\to\infty. \tag{3.16}$$

In view of (3.14), we obtain

$$\begin{aligned}
||U_{n_k}z_{n_k}-z_{n_k}||&=||(1-\beta_{n_k})z_{n_k}+\beta_{n_k}W_{n_k}z_{n_k}-z_{n_k}|| \\
&\leq(1-\beta_{n_k})||z_{n_k}-z_{n_k}||+\beta_{n_k}||W_{n_k}z_{n_k}-z_{n_k}||\to 0, \quad k\to\infty.
\end{aligned} \tag{3.17}$$

From Lemma 3.3, we obtain

$$
\begin{aligned}
&||z_{n_k}-w_{n_k}||^2 \\
&\leq (1-\alpha_{n_k}\bar{\gamma})^2||x_{n_k}-p||^2-||x_{n_k+1}-p||^2+(1-\alpha_{n_k}\bar{\gamma})^2\{3M_2\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}|| \\
&+2\gamma_{n_k}||z_{n_k}-w_{n_k}||||A^*(T_{s_{n_k}}^{F_2}-I)Aw_{n_k}||\}+2\alpha_{n_k}\langle\gamma f(x_{n_k})-Dp,x_{n_k+1}-p\rangle \\
&\leq (1-\alpha_{n_k}\bar{\gamma})^2||x_{n_k}-p||^2-||x_{n_k+1}-p||^2+(1-\alpha_{n_k}\bar{\gamma})^2\{3M_2\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}|| \\
&+2M_3||A^*(T_{s_{n_k}}^{F_2}-I)Aw_{n_k}||\}+2\alpha_{n_k}\langle\gamma f(x_{n_k})-Dp,x_{n_k+1}-p\rangle,
\end{aligned}
$$

where $M_3 := \sup\{\gamma_{n_k}||z_{n_k}-w_{n_k}|| : k\in\mathbb{N}\}$. By applying (3.16), and using $\lim_{k\to\infty}\alpha_{n_k}=0$ together with the hypothesis of Lemma 3.4 and Remark 3.1, we obtain

$$
||z_{n_k}-w_{n_k}||\to 0, \quad k\to\infty. \tag{3.18}
$$

In view of Remark 3.1, we have

$$
\begin{aligned}
||w_{n_k}-x_{n_k}|| &= ||x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1})-x_{n_k}|| \\
&\leq ||x_{n_k}-x_{n_k}||+\theta_{n_k}||x_{n_k}-x_{n_k-1}||\to 0, \quad k\to\infty.
\end{aligned} \tag{3.19}
$$

It follows from (3.14), (3.17), (3.18) and (3.19) that

$$
\lim_{k\to\infty}||W_{n_k}z_{n_k}-w_{n_k}||=0, \quad \lim_{k\to\infty}||W_{n_k}z_{n_k}-x_{n_k}||=0, \quad \lim_{k\to\infty}||U_{n_k}z_{n_k}-w_{n_k}||=0,
$$

and

$$
\lim_{k\to\infty}||U_{n_k}z_{n_k}-x_{n_k}||=0, \quad \lim_{k\to\infty}||z_{n_k}-x_{n_k}||=0. \tag{3.20}
$$

By applying (3.20) and the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we obtain

$$
\begin{aligned}
||x_{n_k+1}-x_{n_k}|| &= ||\alpha_{n_k}\gamma f(x_{n_k})+(1-\alpha_{n_k}D)U_{n_k}z_{n_k}-x_{n_k}|| \\
&\leq \alpha_{n_k}||\gamma f(x_{n_k})-Dx_{n_k}||+(1-\alpha_{n_k}\bar{\gamma})||U_{n_k}z_{n_k}-x_{n_k}||\to 0, \quad k\to\infty.
\end{aligned} \tag{3.21}
$$

Now, we show that $w_\omega(x_n)\subset\cap_{i=1}^{\infty}F(S_i)=F(W)$. Let $x^*\in w_\omega(x_n)$ and suppose that $x^*\notin F(W)$, that is, $Wx^*\neq x^*$. From (3.20), we have that $w_\omega(x_n)=w_\omega(z_n)$. It follows that

$$
\begin{aligned}
\liminf_{k\to\infty}||z_{n_k}-x^*|| &< \liminf_{k\to\infty}|z_{n_k}-Wx^*||| \\
&\leq \liminf_{k\to\infty}\{||z_{n_k}-Wz_{n_k}||+||Wz_{n_k}-Wx^*||\} \\
&\leq \liminf_{k\to\infty}\{||z_{n_k}-Wz_{n_k}||+||z_{n_k}-x^*||\}.
\end{aligned} \tag{3.22}
$$

Since $x_{n_k}\in K$ for all $k\geq 1$ and $\lim_{k\to\infty}||x_{n_k}-z_{n_k}||=0$, we obtain

$$
\begin{aligned}
||Wz_{n_k}-z_{n_k}|| &\leq ||Wz_{n_k}-W_{n_k}z_{n_k}||+||W_{n_k}z_{n_k}-z_{n_k}|| \\
&\leq \sup_{x\in K}||Wx-W_{n_k}x||+||W_{n_k}z_{n_k}-z_{n_k}||.
\end{aligned}
$$

By applying Lemma 2.2 and (3.14), we have $\lim_{k\to\infty}||Wz_{n_k}-z_{n_k}||=0$. Combining this with (3.22) yields

$$
\liminf_{k\to\infty}||z_{n_k}-x^*|| < \liminf_{k\to\infty}||z_{n_k}-x^*||,
$$

which is a contradiction. Hence, $x^*\in F(W)=\cap_{i=1}^{\infty}F(S_i)$, i.e., $w_\omega(x_n)\subset F(W)=\bigcap_{i=1}^{\infty}F(S_i)$.

Next, we show that $w_\omega(x_n) \subset SEP(F_1,F_2)$. First, we show that $w_\omega(x_n) \subset EP(F_1)$. Since $z_{n_k} = T^{F_1}_{r_{n_k}}(w_{n_k} + \gamma_{n_k}A^*(T^{F_2}_{s_{n_k}} - I)Aw_{n_k})$, then

$$F_1(z_{n_k},y) + \frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k} - \gamma_{n_k}A^*(T^{F_2}_{s_{n_k}} - I)Aw_{n_k}\rangle \geq 0, \quad \forall\, y \in C,$$

which implies that

$$F_1(z_{n_k},y) + \frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - z_{n_k}, \gamma_{n_k}A^*(T^{F_2}_{s_{n_k}} - I)Aw_{n_k}\rangle \geq 0, \quad \forall\, y \in C.$$

It follows from the monotonicity of F_1 that

$$\frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - z_{n_k}, \gamma_{n_k}A^*(T^{F_2}_{s_{n_k}} - I)Aw_{n_k}\rangle \geq F_1(y,z_{n_k}), \quad \forall\, y \in C.$$

Since $z_{n_k} \rightharpoonup x^*$, then it follows from (3.8), (3.15), $\liminf_{k\to\infty} r_{n_k} > 0$, and condition (A4) that

$$F_1(y,x^*) \leq 0, \quad \forall\, y \in C. \tag{3.23}$$

Now, for $y \in C$, let $y_t := ty + (1-t)x^*$ for all $t \in (0,1]$. This implies that $y_t \in C$, and it follows from (3.23) that $F_1(y_t,x^*) \leq 0$. From Assumptions (A1)-(A4), we have

$$0 = F_1(y_t,y_t) \leq tF_1(y_t,y) + (1-t)F_1(y_t,x^*) \leq tF_1(y_t,y).$$

Hence, $F_1(y_t,y) \geq 0$, $\forall\, y \in C$. Letting $t \to 0$, and using Condition (A3), we have $F_1(x^*,y) \geq 0$, $\forall\, y \in C$. This implies that $x^* \in EP(F_1)$.

Finally, we show that $Ax^* \in EP(F_2)$. Since A is a bounded linear operator and $w_\omega(x_n) = w_\omega(w_n)$ by (3.19), then $Aw_{n_k} \rightharpoonup Ax^*$. It follows from (3.15) that

$$T^{F_2}_{s_{n_k}}Aw_{n_k} \rightharpoonup Ax^*, \quad k \to \infty. \tag{3.24}$$

By the definition of $T^{F_2}_{s_{n_k}}Aw_{n_k}$, we have

$$F_2(T^{F_2}_{s_{n_k}}Aw_{n_k},y) + \frac{1}{s_{n_k}}\langle y - T^{F_2}_{s_{n_k}}Aw_{n_k}, T^{F_2}_{s_{n_k}}Aw_{n_k} - Aw_{n_k}\rangle \geq 0, \quad \forall\, y \in Q.$$

Since F_2 is upper semi-continuous in the first argument, it follows from (3.15), (3.24) and $\liminf_{k\to\infty} s_{n_k} > 0$ that $F_2(Ax^*,y) \geq 0$, $\forall y \in Q$. This shows that $Ax^* \in EP(F_2)$. Hence, $w_\omega(x_n) \subset \Omega$ as required. $\square$

Now, we state and prove the strong convergence theorem.

Theorem 3.1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions* (A1)-(A4) *and the fact that F_2 is upper semicontinuous in the first argument. Suppose that $\{W_n\}$ is the sequence defined by* (2.1). *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that conditions* (C1)-(C3) *are satisfied. Then the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega$, where $\hat{x} = P_\Omega(I - D + \gamma f)(\hat{x})$ is a solution of the variational inequality $\langle (D - \gamma f)\hat{x}, \hat{x} - x\rangle \leq 0$, $\forall x \in \Omega$.*

Proof. Let $\hat{x} = P_\Omega(I - D + \gamma f)(\hat{x})$. Then it follows from Lemma 3.2 that

$$||x_{n+1} - \hat{x}||^2 \leq \Big(1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_n\gamma\rho)}\Big)||x_n - \hat{x}||^2 + \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\bar{\gamma}^2 M}{2(\bar{\gamma} - \gamma\rho)} + \frac{3M_2(1 - \alpha_n\bar{\gamma})^2}{2(\bar{\gamma} - \gamma\rho)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| + \frac{1}{(\bar{\gamma} - \gamma\rho)}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n+1} - \hat{x}\rangle\Big\}.$$

Now, we claim that the sequence $\{||x_n - \hat{x}||\}$ converges to zero. In order to establish this, by Lemma 2.4, it suffices to show that $\limsup_{k\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n_k+1} - \hat{x}\rangle \leq 0$ for every subsequence $\{||x_{n_k} - \hat{x}||\}$ of $\{||x_n - \hat{x}||\}$ satisfying $\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0$. Suppose that $\{||x_{n_k} - \hat{x}||\}$ is a subsequence of $\{||x_n - \hat{x}||\}$ such that $\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0$. From Lemma 3.4, we have that $w_\omega\{x_n\} \subset \Omega$. It also follows from (3.20) that $w_\omega\{z_{n_k}\} = w_\omega\{x_{n_k}\}$. From the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\lim_{j\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \limsup_{k\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n_k} - \hat{x}\rangle = \limsup_{k\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, z_{n_k} - \hat{x}\rangle. \tag{3.25}$$

Since $\hat{x} = P_\Omega(I - D + \gamma f)(\hat{x})$, it follows from (3.25) that

$$\limsup_{k\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n_k} - \hat{x}\rangle = \lim_{j\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \langle\gamma f(\hat{x}) - D\hat{x}, x^\dagger - \hat{x}\rangle \leq 0. \tag{3.26}$$

Hence, by (3.21) and (3.26), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n_k+1} - \hat{x}\rangle &\leq \limsup_{k\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n_k+1} - x_{n_k}\rangle + \limsup_{k\to\infty}\langle\gamma f(\hat{x}) - D\hat{x}, x_{n_k} - \hat{x}\rangle \\ &= \langle\gamma f(\hat{x}) - D\hat{x}, x^\dagger - \hat{x}\rangle \leq 0. \end{aligned} \tag{3.27}$$

Using Lemma 2.4, (3.27) together with Remark 3.1, and the condition on α_n, we deduce that $\lim_{n\to\infty}||x_n - \hat{x}|| = 0$ as required. This completes the proof. □

Taking $\gamma = 1$ and $D = I$ in Theorem 3.1, where I is the identity mapping, we obtain the following result.

Corollary 3.1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions* (A1)-(A4) *and the fact that F_2 is upper semicontinuous in the first argument. Suppose that $\{W_n\}$ is the sequence defined by* (2.1)*. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 3.2.

Step 0. Let $x_0, x_1 \in H_1$ be arbitrary and set $n = 1$.
Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\Big\{\theta, \frac{\delta_n}{||x_n - x_{n-1}||}\Big\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.}\end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = T_{r_n}^{F_1}(w_n + \gamma_n A^*(T_{s_n}^{F_2} - I)Aw_n),$$

where

$$\gamma_n := \begin{cases} \tau_n \frac{||(T_{s_n}^{F_2}-I)Aw_n||^2}{||A^*(T_{s_n}^{F_2}-I)Aw_n||^2}, & \text{if } Aw_n \neq T_{s_n}^{F_2}Aw_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases}$$

Step 4. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[(1-\beta_n)z_n + \beta_n W_n z_n].$$

Set $n := n+1$ and return to **Step 1**.

Suppose that conditions (C1)-(C3) *are satisfied. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.2 converges strongly to a point* $\hat{x} \in \Omega$, *where* $\hat{x} = P_\Omega(f)(\hat{x})$ *is a solution of the variational inequality* $\langle (I-f)\hat{x}, \hat{x}-x\rangle \leq 0$, $\forall x \in \Omega$.

Taking $\gamma = 1, D = I$ and $S_n = S$ for all $n \geq 1$ in Theorem 3.1, then we have the following result.

Corollary 3.2. *Let* C *and* Q *be nonempty closed convex subsets of real Hilbert spaces* H_1 *and* H_2, *respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator with adjoint* A^*. *Let* $F_1 : C \times C \to \mathbb{R}$ *and* $F_2 : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying conditions* (A1)-(A4) *and the fact that* F_2 *is upper semicontinuous in the first argument. Suppose that* $\{W_n\}$ *is the sequence defined by* (2.1). *Let* $\{x_n\}$ *be a sequence generated as follows:*

Algorithm 3.3.

Step 0. Let $x_0, x_1 \in H_1$ be arbitrary and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\delta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = T_{r_n}^{F_1}(w_n + \gamma_n A^*(T_{s_n}^{F_2} - I)Aw_n),$$

where

$$\gamma_n := \begin{cases} \tau_n \frac{||(T_{s_n}^{F_2}-I)Aw_n||^2}{||A^*(T_{s_n}^{F_2}-I)Aw_n||^2}, & \text{if } Aw_n \neq T_{s_n}^{F_2}Aw_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases}$$

Step 4. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[(1-\beta_n)z_n + \beta_n S z_n].$$

Set $n := n+1$ and return to **Step 1**.

Suppose that conditions (C1)-(C3) *are satisfied. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.3 converges strongly to a point* $\hat{x} \in \Omega$, *where* $\hat{x} = P_\Omega(f)(\hat{x})$ *is a solution of the variational inequality* $\langle (I-f)\hat{x}, \hat{x}-x\rangle \leq 0$, $\forall x \in \Omega$.

4. Applications

In this section, we present some theoretical applications of our results to solve some related problems in nonlinear analysis and optimization.

4.1. **Split equilibrium and zero point problems of maximal monotone operators.** We consider a common solution of split equilibrium and zero point problems for an infinite family of maximal monotone operators. Let $F : H \to H$ be a single-valued nonlinear mapping and let $B : H \to 2^H$ be a multivalued mapping. The problem of finding a zero of the sum of two monotone operators, which is formulated as the following monotone inclusion problem, is to find a point $x \in H$ such that $0 \in (F+B)x$. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. More precisely, some concrete problems in machine learning, image processing and linear inverse problem can be modelled mathematically as this form; see, e.g., [32, 33, 34]. We denote the zero point set $\{x \in H : 0 \in (F+B)x\}$ of $F+B$ by $(F+B)^{-1}0$.

Let H be a real Hilbert space and $B : H \to 2^H$ be a multivalued mapping. The effective domain of B denoted by $D(B)$ is given as $D(B) = \{x \in H : Bx \neq \emptyset\}$.

(1) the *graph* $G(B)$ is defined by

$$G(B) := \{(x,u) \in H \times H : u \in B(x)\};$$

(2) B is said to be monotone if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in D(B), u \in Bx$, and $v \in By$;

(3) B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H;

(4) For a maximal monotone set-valued mapping B on H and $r > 0$, the operator

$$J_r^B := (I + rB)^{-1} : H \to D(B)$$

is called the resolvent of B.

Remark 4.1. In [35], it was shown that $F(J_r^B) = B^{-1}0 \equiv \{x \in H : 0 \in Bx\}$ for all $r > 0$ and J_r^B is singled-valued firmly nonexpansive, that is,

$$||J_r^B x - J_r^B y|| \leq \langle J_r^B x - J_r^B y, x-y\rangle, \quad \text{for all } x, y \in H.$$

The following lemma will be also needed in establishing our results in this section.

Lemma 4.1. [35] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : H \to H$ be a mapping and let$B : H \to 2^H$ be a maximal monotone operator. Then $F(J_r^B(I - rG)) = (G+B)^{-1}(0)$.*

Now, we have the following results.

Theorem 4.1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and suppose that $\{W_n\}$ is the sequence defined by* (2.1). *Let $B_i : H \to 2^H$ be an infinite family of maximal monotone mappings with $D(B_i) \neq \emptyset$ and let $J_{r_i}^{B_i}$ be the resolvent of B_i for each $r_i > 0$. Suppose that $\{x_n\}$ is a sequence generated by Algorithm 3.1 such that conditions* (C1)-(C3) *are satisfied. Then the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega = SEP(F_1, F_2) \cap \bigcap_{i=1}^{\infty}(B_i^{-1}0) \neq \emptyset$, where $\hat{x} = P_\Omega(I - D + \gamma f)(\hat{x})$ is a solution of the variational inequality $\langle (D - \gamma f)\hat{x}, \hat{x} - x\rangle \leq 0$, $\forall x \in \Omega$.*

Proof. Note that $J_{r_i}^{B_i}$ is nonexpansive and $F(J_{r_i}^{B_i}) = B_i^{-1}0$. From Theorem 3.1, taking $J_{r_i}^{B_i} = S_i$ in Definition 2.1, we have the desired conclusion immediately. □

Theorem 4.2. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and suppose that $\{W_n\}$ is the sequence defined by (2.1). Let $B_i : H \to 2^H$ be an infinite family of maximal monotone mappings with $D(B_i) \neq \emptyset$. Let $J_{r_i}^{B_i}$ be the resolvent of B_i for each $r_i \in (0, 2\delta_i)$ and let $G_i : H \to H$ be an infinite family of δ_i-inverse strongly monotone mappings. Suppose that $\{x_n\}$ is a sequence generated by Algorithm 3.1 such that conditions* (C1)-(C3) *are satisfied. Then the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega = SEP(F_1, F_2) \cap \bigcap_{i=1}^{\infty}(B_i + G_i)^{-1}0 \neq \emptyset$, where $\hat{x} = P_\Omega(I - D + \gamma f)(\hat{x})$ is a solution of the variational inequality $\langle (D - \gamma f)\hat{x}, \hat{x} - x\rangle \leq 0$, $\forall x \in \Omega$.*

Proof. Since G_i is δ_i-inverse strongly monotone, we have that $I - r_iG_i$ is nonexpansive. From the nonexpansiveness of $J_{r_i}^{B_i}$, it follows that $J_{r_i}^{B_i}(I - r_iG_i)$ is also nonexpansive. The proof follows from Theorem 3.1 by applying Lemma 4.1 and taking $J_{r_i}^{B_i}(I - r_iG_i) = S_i$ in Definition 2.1. □

4.2. **Split generalized mixed equilibrium and fixed point problems.** Let $\phi : C \to H$ be a nonlinear mapping and let $\psi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Define

$$G(u, y) = F(u, y) + \langle \phi u, y - u\rangle + \psi(y) - \psi(u) \geq 0, \text{ for all } y \in C.$$

It is known (see [36]) that if $F(u, y)$ satisfies conditions (A1)-(A4), then $G(u, y)$ also satisfies them. Hence, the EP reduces to the problem: Find $\hat{x} \in C$ such that

$$F(\hat{x}, y) + \langle \phi \hat{x}, y - \hat{x}\rangle + \psi(y) - \psi(\hat{x}) \geq 0, \text{ for all } y \in C. \tag{4.1}$$

This problem is called the Generalized Mixed Equilibrium Problem (shortly, (GMEP)). The set of solutions of GMEP is denoted by $GMEP(F, \phi, \psi)$. The GMEP is very general in the sense that it includes as special cases, optimization problems, variational inequality problems, minimization problems, variational inclusion problems, fixed point problems, mathematical programming problems, minimax problems, Nash equilibrium problems in noncooperative games, and many others. Due to its generality, the GMEP has recently attracted attention of many authors; see, e.g., [10, 37] and the references therein. If $\phi = 0$ in (4.1), then the GMEP reduces to the Mixed Equilibrium Problem. If $\psi = 0$ in (4.1), then the GMEP becomes the Generalized Equilibrium Problem. In particular, if $\phi = \psi = 0$ in (4.1), then the GMEP reduces to the EP.

Definition 4.1. Let H_1 and H_2 be Hilbert spaces. Let C and Q be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions. Let $\phi_1 : C \to H_1$, $\phi_2 : Q \to H_2$, be nonlinear mappings, and let $\psi_1 : C \to \mathbb{R} \cup \{+\infty\}$, $\psi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator. The Split Generalised Mixed Equilibrium Problem (shortly, (SGMEP)) (see, for example [38]) is to find a point $\hat{x} \in C$ such that

$$F_1(\hat{x}, x) + \langle \phi_1 \hat{x}, x - \hat{x}\rangle + \psi_1(x) - \psi_1(\hat{x}) \geq 0, \text{ for all } x \in C, \tag{4.2}$$

and $\hat{y} = A\hat{x} \in Q$ solves

$$F_2(\hat{y}, y) + \langle \phi_2 \hat{y}, y - \hat{y}\rangle + \psi_2(y) - \psi_2(\hat{y}) \geq 0, \text{ for all } y \in Q. \tag{4.3}$$

We denote the solution set of (4.2)-(4.3) by $SGMEP(F_1, \phi_1, \psi_1, F_2, \phi_2, \psi_2) = \{\hat{x} \in GMEP(F_1, \phi_1, \psi_1) : A\hat{x} \in GMEP(F_2, \phi_2, \psi_2)\}$.

Taking

$$G_1(u,y) = F_1(u,y) + \langle \phi_1 u, y-u\rangle + \psi_1(y) - \psi_1(u) \geq 0, \text{ for all } y \in C,$$

and

$$G_2(w,z) = F_2(w,z) + \langle \phi_2 w, z-w\rangle + \psi_2(z) - \psi_2(w) \geq 0, \text{ for all } z \in Q,$$

we can directly obtain the following result from Theorem 3.1 when F_1 and F_2 satisfy conditions (A1)-(A4).

Theorem 4.3. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let G_1 and G_2 be as above, and let ϕ_1, ϕ_2, ψ_1 and ψ_2 be the same as in Definition 4.1. Suppose that $\{W_n\}$ is the sequence defined by (2.1). Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 4.1.

Step 0. Let $x_0, x_1 \in H_1$ be arbitrary and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\delta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = T_{r_n}^{G_1}(w_n + \gamma_n A^*(T_{s_n}^{G_2} - I)Aw_n),$$

where

$$\gamma_n := \begin{cases} \tau_n \frac{||(T_{s_n}^{G_2} - I)Aw_n||^2}{||A^*(T_{s_n}^{G_2} - I)Aw_n||^2}, & \text{if } Aw_n \neq T_{s_n}^{G_2}Aw_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases}$$

Step 4. Compute

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n D)[(1-\beta_n)z_n + \beta_n W_n z_n].$$

Set $n := n+1$ and return to **Step 1**.

Suppose that conditions (C1)-(C3) *are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 4.1 converges strongly to a point $\hat{x} \in \Omega = SGMEP(F_1, \phi_1, \psi_1, F_2, \phi_2, \psi_2) \cap \bigcap_{i=1}^{\infty} F(S_i)$, where $\hat{x} = P_\Omega(f)(\hat{x})$ is a solution of the variational inequality $\langle (D - \gamma f)\hat{x}, \hat{x} - x\rangle \leq 0, \forall x \in \Omega$.*

4.3. **Split variational inequality and fixed point problems.** Let C be a nonempty closed convex subset of a real Hilbert space H, and let $B : H \to H$ be a single-valued mapping. The Variational Inequality Problem (shortly, (VIP)) is defined as follows:

$$\text{Find } x^* \in C \text{ such that } \langle y - x^*, Bx^*\rangle \geq 0, \quad \forall\, y \in C.$$

The solution set of the VIP is denoted by $VI(C,B)$. The VIP is a useful mathematical model that unifies many important concepts in applied mathematics, such as, necessary optimality conditions, complementarity problems, network equilibrium problems, and the systems of nonlinear equations. Here, we apply our result to the following Split Variational Inequality Problem (shortly, (SVIP)):

$$\text{Find } x^* \in \bigcap_{n=1}^{\infty} F(S_n) \text{ such that } \langle x - x^*, B_1 x^* \rangle, \quad \forall\, x \in C, \tag{4.4}$$

and

$$y^* = Ax^* \in Q \text{ solves } \langle y - y^*, B_2 y^* \rangle \geq 0, \quad \forall\, y \in Q, \tag{4.5}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $\{S_n\}$ is the sequence of k_n-strict pseudo-contraction mappings in Definition 2.1, $A : H_1 \to H_2$ is a bounded linear operator, and $B_1 : C \to H_1, B_2 : Q \to H_2$ are monotone mappings. We denote the solution set of problem (4.4)-(4.5) by Ω and assume that $\Omega \neq \emptyset$. By taking $F_i(x,y) := \langle y - x, B_i x \rangle, i = 1,2$, the (SVIP) (4.4)-(4.5) becomes the problem of finding a solution of the (SEP) (1.1)-(1.2) which is also a solution of an infinite family of k_n-strict pseudo-contraction mappings $\{S_i\}$. Moreover, all the conditions of Theorem 3.1 are satisfied. Therefore, Theorem 3.1 provides a strong convergence theorem for approximating a common solution of the SVIP and fixed points of an infinite family of k_n-strict pseudo-contraction mappings.

5. Numerical Examples

In this section, we present some numerical experiments to demonstrate the efficiency of our algorithm in comparison with Algorithm 1 proposed in [20] and Algorithm 3.3 in [39]. We plot the graph of errors against the number of iterations in each case. All numerical computations were carried out using Matlab 2019(b).

Example 5.1. Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = \mathbb{R}$. Define $A : \mathbb{R} \to \mathbb{R}$ by $Ax = \frac{x}{2}$ and $A^* y = \frac{y}{2}$. Clearly, A is a bounded linear operator. Define $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ by $F_1(x,y) = -11x^2 + xy + 10y^2$ and $F_2(x,y) = -15x^2 + xy + 14y^2$. It is easily verified that F_1 and F_2 satisfy conditions $(A_1) - (A4)$. Using Lemma 2.6, we obtain

$$T_r^{F_1}(u) = \frac{u}{21r+1}, \quad \forall\, x \in C,$$

and

$$T_s^{F_2}(v) = \frac{v}{29s+1}, \quad \forall\, v \in Q.$$

Define an infinite family of mappings $S_n : \mathbb{R} \to \mathbb{R}$ by

$$S_n x := -\frac{2}{n} x \quad \text{for all } x \in \mathbb{R}.$$

It can easily be verified that S_n is k_n-strict pseudo-contractive for each $n \in \mathbb{N}$. Define $S_n' = t_n I + (1 - t_n) S_n, t_n \in [k_n, 1)$. Let $\{\zeta_n\}$ be a sequence of nonnegative real numbers defined by $\zeta_n = \{\frac{n}{3n-1}\}$ for all $n \in \mathbb{N}$ and W_n be generated by $\{S_n\}, \{\zeta_n\}$ and $\{t_n\}$. Let $f(x) = \frac{1}{5}x$, then $\rho = \frac{1}{5}$ is the Lipschitz constant for f. Let $D(x) = \frac{x}{3}$ with constant $\bar{\gamma} = \frac{1}{3}$. Then we take $\gamma = 1$, which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Choose $\tau_n = 0.8, \theta = 0.9, \alpha_n = \frac{1}{n+3}, \delta_n = \frac{1}{(n+3)^2}, \beta_n = \frac{n+1}{2n+3}, r_n =$

$s_n = \frac{n+2}{n+3}$, and $t_n = \frac{1}{n+3}$. It can easily be checked that all the conditions on the control sequences in Theorem 3.1 are satisfied. Then Algorithm 3.1 becomes

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \quad 0 \le \theta_n \le \hat{\theta}_n \\ z_n = \frac{1}{21r_n+1} w_n - \gamma_n \frac{29s_n}{4(21r_n+1)(29s_n+1)} w_n \\ x_{n+1} = \frac{1}{5n+15} x_n + \left(\frac{3n+8}{3n+9}\right)\left(\frac{n+2}{2n+3} z_n + \frac{n+1}{2n+3} W_n z_n\right), \end{cases}$$

where

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\delta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise,} \end{cases}$$

and

$$\gamma_n := \begin{cases} \tau_n \frac{||(T_{s_n}^{F_2} - I)Aw_n||^2}{||A^*(T_{s_n}^{F_2} - I)Aw_n||^2}, & \text{if } Aw_n \neq T_{s_n}^{F_2} Aw_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases}$$

We use the stopping criterion $||x_{n+1} - x_n|| < 10^{-5}$ and choose four different initial values as follows:

(I) $x_0 = -50$ and $x_1 = \frac{27}{89}$;
(II) $x_0 = 200$ and $x_1 = 13.732$;
(III) $x_0 = \frac{17}{19}$ and $x_1 = -20$;
(IV) $x_0 = -0.95$ and $x_1 = -300$.

The numerical results are presented in Figure 1 and Table 1.

TABLE 1. **Numerical results for Example 5.1**

		Alg. 1	Alg. 3.3 in [39]	Alg. 3.1
Case I	CPU time (sec)	0.010	0.7233	0.0085
	No of Iter.	4	14	5
Case II	CPU time (sec)	0.0109	0.7008	0.0110
	No. of Iter.	5	19	6
Case III	CPU time (sec)	0.0113	0.7138	0.0108
	No of Iter.	5	19	6
Case IV	CPU time (sec)	0.0101	7.4439	0.0101
	No of Iter.	6	36	8

Example 5.2. We consider the second example in the infinite dimensional Hilbert space $H = L^2([0,1])$ with the inner product defined by

$$\langle x, y \rangle := \int_0^1 x(t)y(t)dt \quad \text{for all } x, y \in H$$

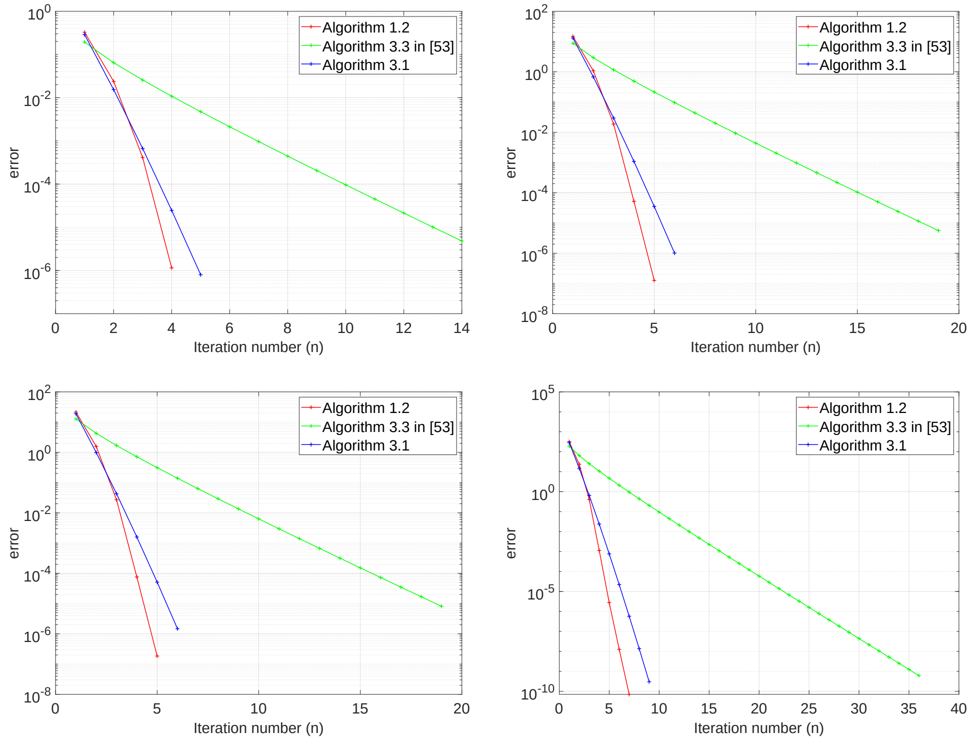

FIGURE 1. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

and the induced norm by

$$||x|| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}} \quad \text{for all } x \in H.$$

We define $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ by $F_1(x,y) = \langle L_1 x, y - x\rangle$ and $F_2(x,y) = \langle L_2 x, y - x\rangle$, where $L_1 x(t) = \frac{x(t)}{3}$ and $L_2 x(t) = \frac{x(t)}{2}$. It can easily be checked that F_1 and F_2 satisfy conditions (A1)-(A4). Let $A : L_2([0,1]) \to L_2([0,1])$ be defined by $Ax(t) = \frac{x(t)}{3}$ and $A^* y(t) = \frac{y(t)}{3}$. Then, A is a bounded linear operator. From Lemma 2.6, we obtain

$$T_r^{F_1}(u) = \frac{3u}{r+3}, \quad \forall\, u \in C,$$

and

$$T_s^{F_2}(v) = \frac{2v}{s+2}, \quad \forall\, v \in Q.$$

Let $f(x) = \frac{x(t)}{3}$. Then $\rho = \frac{1}{3}$. Take $D(x) = \frac{x(t)}{2}$ with constant $\bar{\gamma} = \frac{1}{2}$. Then, we take $\gamma = 1$, which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Define the sets $C := \{x \in H : ||x|| \leq 1\}$ and $Q := \{y \in H : ||y|| \leq 1\}$, and define an infinite family of mappings $S_n : L^2([0,1]) \to L^2([0,1])$ by

$$(S_n x)(t) = \int_0^1 t^n x(s) ds \quad \text{for all } t \in [0,1].$$

It can easily be verified that S_n is nonexpansive for each $n \in \mathbb{N}$, and hence 0-strict pseudo-contractive. Define $S_n' = \theta_n I + (1-\theta_n)S_n,\ \theta_n \in [0,1)$. Let $\{\zeta_n\}$ be a sequence of nonnegative real numbers defined by $\zeta_n = \{\frac{n}{2n+1}\}$ for all $n \in \mathbb{N}$ and let W_n be generated by $\{S_n\}, \{\zeta_n\}$ and $\{\theta_n\}$. Choose $\tau_n = 0.7, \theta = 0.8, \alpha_n = \frac{1}{n+1}, \delta_n = \frac{1}{(n+1)^2}, \beta_n = \frac{n}{2n+1}, r_n = s_n = \frac{n+1}{n+3}, t_n = \frac{1}{n+3}$. It can easily be checked that all the conditions on the control sequences in Theorem 3.1 are satisfied. Then Algorithm 3.1 becomes

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \quad 0 \le \theta_n \le \hat{\theta}_n \\ z_n = \frac{3}{r_n+3} w_n - \gamma_n \frac{s_n}{3(r_n+3)(s_n+2)} w_n \\ x_{n+1} = \frac{1}{3n+3} x_n + \left(\frac{2n+1}{2n+2}\right)\left(\frac{n+1}{2n+1} z_n + \frac{n}{2n+1} W_n z_n\right), \end{cases}$$

where

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta,\ \frac{\delta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}, \end{cases}$$

and

$$\gamma_n := \begin{cases} \tau_n \frac{||(T_{s_n}^{F_2} - I)Aw_n||^2}{||A^*(T_{s_n}^{F_2} - I)Aw_n||^2}, & \text{if } Aw_n \neq T_{s_n}^{F_2} Aw_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases}$$

We choose two different initial values as follows and plot the graph of errors against the number of iterations for three different stopping criterion:
(I) $x_0 = \frac{9}{10} t^6$ and $x_1 = \frac{2}{5} t^8$;
(II) $x_0 = \frac{9}{10} t^5$ and $x_1 = \frac{2}{5} t^6$;
The numerical results are reported in Figures 2, 3 and Table 2.

6. Conclusion

We studied the SEP and the FPP of an infinite family of strict pseudo-contractions. We proposed a new inertial iterative scheme with the self-adaptive step size for approximating a common solution of the problems. Under mild conditions on the control sequences, we prove a strong convergence theorem in Hilbert spaces. Our proposed algorithm is simple and easy to implement. We applied our results to some optimization problems. Finally, we presented some numerical experiments to demonstrate the efficiency of the proposed algorithm in comparison with some recent results in the literature.

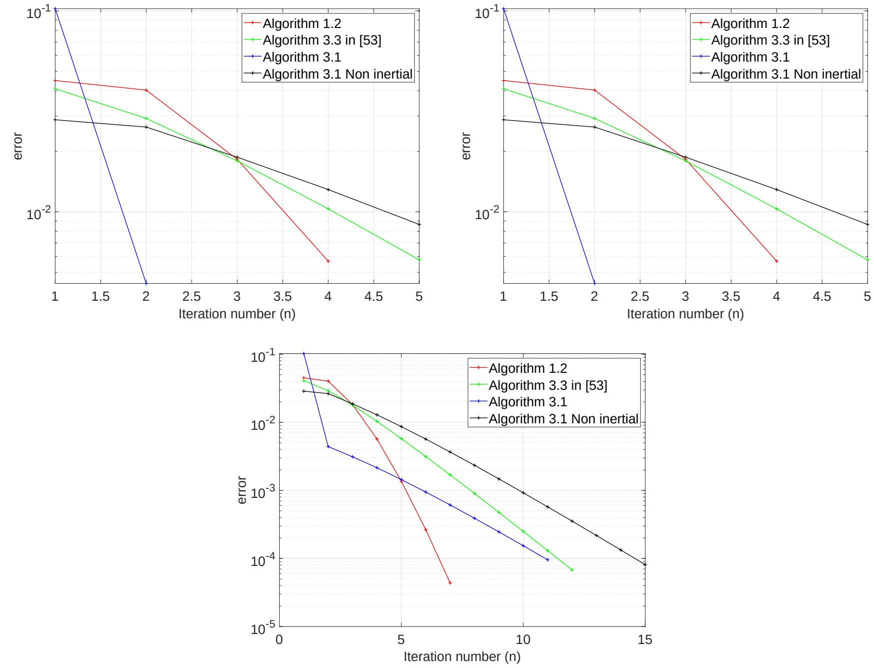

FIGURE 2. Top left: Case I, $\varepsilon = 10^{-2}$; Top right: Case I, $\varepsilon = 10^{-3}$; Bottom : Case I, $\varepsilon = 10^{-4}$.

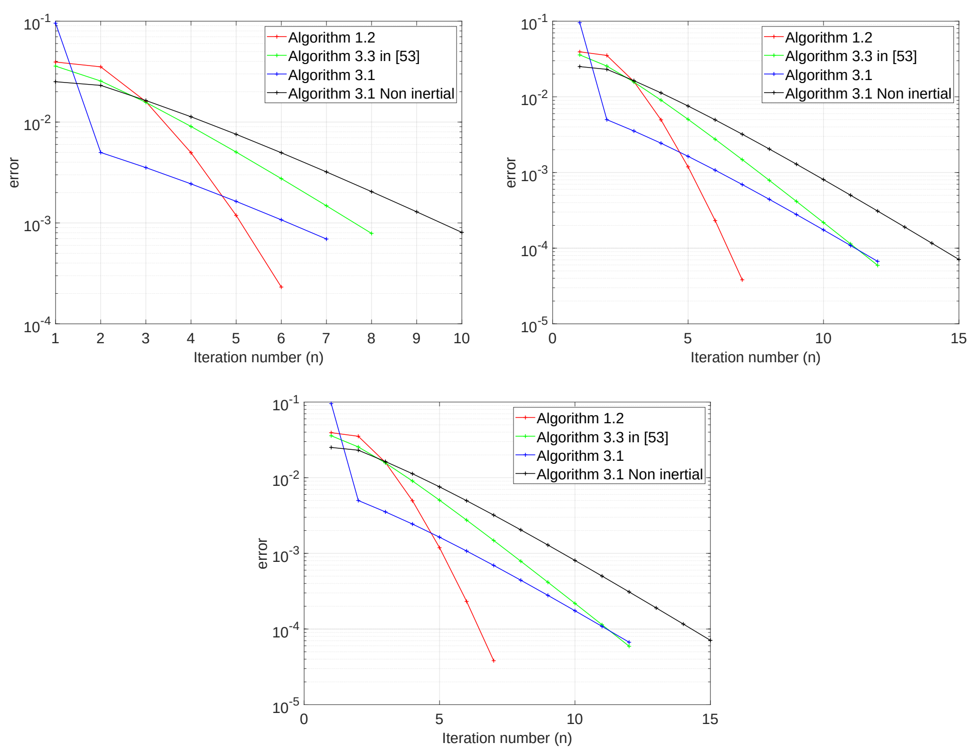

FIGURE 3. Top left: Case II, $\varepsilon = 10^{-2}$; Top right: Case II, $\varepsilon = 10^{-3}$; Bottom : Case II, $\varepsilon = 10^{-4}$.

TABLE 2. **Numerical results for Example 5.2**

		Alg. 1	Alg. 3.3 in [39]	Alg. 3.1	Alg. 3.1 $\theta_n = 0$
Case I with $\varepsilon = 10^{-2}$	CPU time (sec)	0.5043	0.3078	0.4134	0.66576
	No of Iter.	4	5	2	5
Case I with $\varepsilon = 10^{-3}$	CPU time (sec)	0.4239	0.2934	0.3810	0.6456
	No. of Iter.	4	5	2	5
Case I with $\varepsilon = 10^{-4}$	CPU time (sec)	0.5458	0.5318	1.4584	1.6223
	No of Iter.	7	12	11	15
Case II with $\varepsilon = 10^{-2}$	CPU time (sec)	0.5360	0.3857	0.9671	1.1362
	No of Iter.	6	8	7	10
Case II with $\varepsilon = 10^{-3}$	CPU time (sec)	0.5666	0.5500	1.5777	1.6738
	No of Iter.	7	12	12	15
Case II with $\varepsilon = 10^{-4}$	CPU time (sec)	0.5444	0.5320	1.5385	1.6465
	No of Iter.	7	12	12	15

Acknowledgments

The third author was supported by the National Research Foundation (NRF) of South Africa, Incentive Funding for Rated Researchers (Grant Number 119903). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the NRF.

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