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THE WEIGHTED SET RELATION: CHARACTERIZATIONS IN THE CONVEX CASE

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Abstract. Optimization of set-valued mappings has become an important subbranch of optimization. In set optimization, one of the main tools to compare sets is given by set relations, which are binary relations among sets. The main two set relations known in the literature are the upper and lower set relation, which are somewhat counterparts of each other and are used to model robust and optimistic solutions in uncertain (vector) programming, respectively. In this paper, we consider the convex case and propose a new set relation which is able to act as a weighting function between these two important set relations and therefore balances out possible gaps that can occur in modeling set optimization problems.
Keywords. Characterization by separating functionals; Set optimization; Set relations.

1. INTRODUCTION

Set optimization is a modern, and quite dynamic field within applied mathematics that subsumes scalar and vector optimization, therefore providing an important extension of optimization theory in general. Given its large number of applications, such as duality principles in vector optimization, gap functions for vector variational inequalities, inverse problems and variational inequalities, fuzzy optimization, image processing, optimal control problems with differential inclusions, viability theory, medical image registration or in mathematical economics, set optimization has recently expanded as a distinct branch of applied mathematics. As a result, set optimization steadily establishes new links between different areas in optimization.

For an introduction, let us briefly describe how set optimization arises from uncertain multi-objective problems. Many optimization problems are faced with conflicting goals which have to be minimized at the same time. Such problem structures lead to *multi-objective optimization* programs, where different conflicting functions are optimized in parallel, meaning simultaneously. Almost any real-world application of mathematics has conflicting multiple criteria; see, for example, the problem of choosing a portfolio in financial mathematics (compare [1]). Optimal elements of a feasible set are then defined by the concept of *Pareto optimality* (see, for example, [2]). If one expands this concept even further (for instance to infinite dimensional spaces), it is possible to define optimality in more general settings. Then one arrives at *vector optimization*, compare, for example, [3, 4].

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Moreover, most complex multi-objective problems arising in Operational Research are contaminated with uncertain data. The reasons for this can be diverse, and include, among others, rounding errors or numerical inaccuracy, errors in measurements, incomplete information, strongly simplified and/or lumped models or broad estimations leading to contaminated data. For instance, in traffic optimization, uncertain weather conditions, construction works, or traffic jams can highly influence the computed optimal solutions of a train schedule or shortest path problem (compare, for example, [5]). Several examples for uncertain programming can be found in medicine. For instance, in intensity-modulated radiation therapy, Eichfelder and Pilecka [6, 7] explain that for safety purposes one might prefer to do necessary calculations of the optimal radiation dose based on several data sets. Portfolio optimization is subject to uncertainty on account of unreliable predictions, political decisions influencing the markets, etc. Moreover, network flow and network design problems are also heavily faced with uncertainty (see, for instance, [8]).

Uncertainty here means that some parameters are not known. Instead, only an estimated value or a set of possible values can be determined. As inaccurate data can have severe impacts on the model and therefore on the computed solution, it is important to take such uncertainty into account when modeling an optimization problem.

If uncertainty is included in the optimization model, one is left with not only one objective function value, but possibly a whole set of values. Such a situation leads to set optimization problems, where the objective map is set-valued. This non-probabilistic approach gained recognition since the fundamental paper by Ehrgott, Ide and Schöbel [9], who introduced robust solutions for uncertain multiobjective optimization problems, and has since been studied intensively, see, for example, [10, 11] and the references therein.

For instance, several diverse concepts of robustness for dealing with uncertainties in vector optimization can be described using approaches from set-valued optimization (see [11]). The concept of interval arithmetics for computations with strict error bounds [12] is also a special case of dealing with set-valued mappings. An interesting application of set optimization in welfare economics is given in [13]. We refer to [14] for a recent introduction to set optimization and its applications.

An important part of set optimization includes comparing sets by means of set relations, which are binary relations among sets. There is a variety of set relations based on convex cones known in the literature (for an overview, see [14, Chapter 2.6.2]), and several authors have discussed which set relations are appropriate for certain applications (compare [11]).

This work is concerned with combining two of the most prominent set relations, namely, the *upper* and the *lower set relation*. Both relations have thus far mostly been considered fundamentally different, as they represent *robust* and *optimistic* attitude, respectively, of the decision-maker. In a recent paper by Chen et al. [15] (see also Köbis [16]), the authors combined, for the first time, these two relations; in particular, they presented a novel set relation by combining them employing a characterization using nonlinear functionals. In the present paper, we will follow a similar approach by choosing *linear* characterizing functionals, and we show that this ansatz is useful in the convex case and will lead to an easier computation of the involved functionals.

This paper is organized as follows: In Section 2, we recall some preliminary notions and results, present the newly defined set relation and give some examples and properties. Section 3 is dedicated to formulating and investigating set optimization problems where the new set relation is involved. In Section 4, we discuss some existence results, and Section 5 is concerned with a comparison to the approach taken in [15], which affects the nonconvex case. Finally, we summarize our results and give an outlook in Section 6.

2. A Weighted Set Relation

By $\mathscr{P}(Y)$, we denote the power set of the linear space Y (to be specified further in the analysis below) without the empty set.

For two elements A, B of $\mathscr{P}(Y)$, we denote the sum of sets by

$$A+B := \{a+b \mid a\in A,\ b\in B\}$$

and for a given scalar $\alpha\in\mathbb{R}$, the product $\alpha\cdot A$ is defined through $\{\alpha\cdot a | a\in A\}$. The set $D\subseteq Y$ is called a *cone* if for all $d\in D$ and $\lambda\geq 0$, $\lambda d\in D$ holds true. The cone D is *convex* if $D+D\subseteq D$. We say that a set D is *proper* (or *nontrivial*) if $D\neq\{0\}$ and $D\neq Y$. The cone D is *pointed* if $D\cap(-D)=\{0\}$ holds. The dual space of Y shall be indicated by Y^*. We denote the dual cone of D by D^*, which is defined by $D^* := \{\ell\in Y^* | \ell(d)\geq 0 \text{ for all } d\in D\}$.

An important part of set optimization includes comparing sets by means of set relations, which are binary relations among sets or, more precisely, on $\mathscr{P}(Y)$. There is a variety of set relations based on convex cones known in the literature (for an overview, see [14, Chapter 2.6.2]), and several authors have discussed which set relations are appropriate for certain applications (compare [11]). Here, we recall the following set relations, which are among the most used ones in the set optimization community. They have applications in uncertain (vector) programming, where they are associated with robust and optimistic solution concepts, respectively. These were originally introduced by Kuroiwa [17, 18] in case that D in the following definition is a convex cone:

Definition 2.1 (Upper/Lower Set Less Relation)**.** Let Y be a linear space and $D\in\mathscr{P}(Y)$. The *upper set less relation* $\preceq^u_D$ is defined for two sets $A,B\in\mathscr{P}(Y)$ by

$$A\preceq^u_D B :\Longleftrightarrow A\subseteq B-D,$$

and the *lower set less relation* $\preceq^l_D$ is defined by

$$A\preceq^l_D B :\Longleftrightarrow A+D\supseteq B.$$

The above two set relations can be equivalently characterized by means of linear functionals, under some convexity assumptions stated below.

Assumption 2.1. Let Y be a real locally convex space, partially ordered by a convex cone D, and assume that $A,B\subset Y$ are nonempty, $A+D$ and $B-D$ are closed and convex.

Theorem 2.1 ([19])**.** *Let Assumption 2.1 be fulfilled. Then*

$$\begin{aligned} A\preceq^u_D B \quad &\Leftrightarrow \quad \forall\ell\in D^*\setminus\{0\} : \sup_{a\in A}\ell(a)\leq\sup_{b\in B}\ell(b),\\ A\preceq^l_D B \quad &\Leftrightarrow \quad \forall\ell\in D^*\setminus\{0\} : \inf_{a\in A}\ell(a)\leq\inf_{b\in B}\ell(b). \end{aligned}$$

A representation like in Theorem 2.1 is extremely useful (and heavily relies on the strong convexity assumptions). For this case, it provides a way to characterize the set relation by relatively simple algebraic terms. These give rise to introduce a new set relation that represents a balance between these two known relations. Therefore, we will compromise directly using the algebraic characterization given above. This new relation will involve both the upper as well as the lower set relation as special cases, and allows to alternate between the two in a continuous manner.

In the following, we need a stronger assumption than Assumption 2.1, namely, we add the condition that our sets need to be compact and that D has nonempty interior.

Assumption 2.2. Let Y be a real locally convex space, partially ordered by a convex cone D with nonempty interior, and assume that $A, B, C \subset Y$ are nonempty, compact, $A+D$, $B-D$, $C-D$ and $B+D$ are closed and convex.

The set $C \in \mathscr{P}(Y)$ will be used to study transitivity of the relations below. To make the notions fully utilizable for set optimization problems, we will narrow the requirements further and add assumptions on $A \pm D$ for all involved sets in the problem, see Assumption 3.1 below.

Remark 2.1 ([15, Remark 2.6])**.** We assume that D has nonempty interior in Assumption 2.2, because we need this property to show that the set relation $\preceq_D^{\lambda}$, to be introduced below, is reflexive (see Proposition 2.1, (iii)). The Theorem of Weierstrass ensures that an extreme point is attained if the involved sets are nonempty closed bounded subset of a real quasicompact topological linear space Y. This will be needed further below in the proof of Proposition 2.1, (iii).

Remark 2.2. Since we are dealing with linear functionals $\ell \in D^*$, and a simple "less-than-zero"-relation, a re-scaling (with any positive constant $\mu > 0$) of the linearizations is not relevant for the above result. The set of elements of the dual cone can therefore be restricted to $D^* \cap \bar{B} \setminus \{0\} = \{\ell \in D^* : 0 < \|\ell\| < 1\}$, where $\bar{B}$ denotes the unit ball in Y^* equipped with a fixed norm $\|\cdot\|$. That way (and with the compactness of the image sets as stated in Assumption 2.2 in mind), infinite expressions can be avoided in many, yet not necessarily all cases, thereby ruling out edge situations in the analysis. From a more numerical perspective, it is even more beneficial to work with $D^* \cap \mathbb{S} \setminus \{0\} = \{\ell \in D^* : \|\ell\| = 1\}$, where $\mathbb{S}$ denotes the unit sphere in Y^*. For our further investigations throughout this contribution, we will rely on an alternative formulation such that this re-scaling becomes obsolete, at least for the analysis to be carried out here.

A straightforward way to transform the above inequalities to a scheme that compromises between the pessimistic upper set less relation and the optimistic lower set less relation is to simply require that both (or either of) the defining inclusions are fulfilled to declare two sets as to be in relation to each other. This is done for the case of the so-called *set less relation* which goes back to the works of Young [20]. This concept, however, comes with two drawbacks: On the one hand, it often reduces the set of optimal solutions, see Definition 3.1 below, sometimes to the point that it becomes empty [21]. On the other hand, since there is just a simple union of the two other concepts involved, a seamless transition from one of the set relations to the other is ruled out in this case.

Bischoff, Jahn and Köbis [22] propose to use co-called *centered sets*, i.e. special constructions for not dealing with the entire sets A and B in the definition of the set relations but rather just specially defined representatives or (weighted) mean values of points in those sets. Even though this approach is very flexible as it allows for very tailor-made and/or problem specific definitions of the centered sets, it comes with the drawbacks of (a) requiring additional conventions (and computations) with respect to the definition of centered sets and, that way, (b) hampering a direct and general analysis of the set relation.

Another approach is to transform the above inequalities to a scheme that weighs between $\preceq_D^u$ and $\preceq_D^l$ directly. This way, one is weighing out different concepts through a criterion rather than the involved sets themselves.

Before formally introducing the new relation in Definition 2.2 below, we will motivate the concept using the results of Theorem 2.1 directly: Given the convexity prerequisites in Assumption 2.2 and the characterization result using linear functionals $\ell \in D^* \setminus \{0\}$, it is a reasonable idea to use linear weighing by a real factor $\lambda \in [0,1]$:

$$\begin{array}{ccc}
A \preceq_D^u B \Leftrightarrow \sup\limits_{a\in A} \ell(a) \le \sup\limits_{b\in B} \ell(b) & & A \preceq_D^l B \Leftrightarrow \inf\limits_{a\in A} \ell(a) \le \inf\limits_{b\in B} \ell(b) \\
\lambda \cdot (\bullet) \searrow & & \swarrow (1-\lambda)\cdot(\bullet) \\
 & (\bullet) + (\bullet) & \\
\lambda \cdot \sup\limits_{a\in A} \ell(a) + (1-\lambda)\cdot \inf\limits_{a\in A} \ell(a) & \le & \lambda \cdot \sup\limits_{b\in B} \ell(b) + (1-\lambda)\cdot \inf\limits_{b\in B} \ell(b) \\
 & \Downarrow & \\
 & A \preceq_D^\lambda B &
\end{array}$$

Under some stricter assumptions, this procedure is a viable way of defining "$\preceq_D^\lambda$". However, to more directly relate the role of the upper/lower set relation and to be consistent with the literature [15], we apply further term manipulation

$$\begin{array}{lcc}
\forall \ell \in D^* \setminus \{0\}: & \sup\limits_{a\in A} \ell(a) \le \sup\limits_{b\in B} \ell(b) & \inf\limits_{a\in A} \ell(a) \le \inf\limits_{b\in B} \ell(b) \\
 & \sup\limits_{a\in A} \ell(a) - \sup\limits_{b\in B} \ell(b) \le 0 & \inf\limits_{a\in A} \ell(a) - \inf\limits_{b\in B} \ell(b) \le 0 \\
 & \sup\limits_{a\in A} \inf\limits_{b\in B} \ell(a-b) \le 0 & \sup\limits_{b\in B} \inf\limits_{a\in A} \ell(a-b) \le 0
\end{array}$$

which leads to the following equivalent (in the finite case when restricting to uniformly bounded functionals $\ell \in D^*$) characterization:

Definition 2.2 (Weighted Set Relation)**.** Let Assumption 2.2 be satisfied, and let $\lambda \in [0,1]$. The *weighted set relation* $\preceq_D^\lambda$ is defined for two sets $A, B \subseteq Y$ by

$$A \preceq_D^\lambda B :\Longleftrightarrow \forall \ell \in D^* \setminus \{0\}: \ \lambda g_\ell^u(A,B) + (1-\lambda) g_\ell^l(A,B) \le 0,$$

where

$$g_\ell^u(A,B) := \sup_{a\in A} \inf_{b\in B} \ell(a-b),$$
$$g_\ell^l(A,B) := \sup_{b\in B} \inf_{a\in A} \ell(a-b).$$

It is clear that the case $A \preceq_D^u B$ is recovered from Definition 2.2 for $\lambda = 1$ and $A \preceq_D^l B$ is obtained for $\lambda = 0$.

We provide an example below to illustrate the new relation $\preceq_D^\lambda$ and discuss the role of the parameter λ.

Example 2.1. Let Y be the real number line $\mathbb{R}$ (Euclidean space in one dimension with its dual identified by $\mathbb{R}$ itself) and $A := [\underline{a}, \overline{a}]$ and $B := [\underline{b}, \overline{b}]$ be compact sets in $\mathbb{R}$. We choose $D = \mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$ with $D = D^*$. We have

$$g_\ell^u(A,B) = \sup_{a\in A} \inf_{b\in B} \ell(a-b) = \sup_{a\in A} \ell(a) - \sup_{b\in B} \ell(b) = \ell(\overline{a}) - \ell(\overline{b}),$$
$$g_\ell^l(A,B) = \ell(\underline{a}) - \ell(\underline{b}),\; g_\ell^u(B,A) = \ell(\overline{b}) - \ell(\overline{a}),\; g_\ell^l(B,A) = \ell(\underline{b}) - \ell(\underline{a}).$$

Let, for example, $\underline{a} = 5$, $\overline{a} = 10$, $\underline{b} = 0$, $\overline{b} = 11$. Then $B \npreceq_D^u A$, but $B \preceq_D^l A$. Also, $A \preceq_D^u B$, but $A \npreceq_D^l B$. However, we can see that the "amount" of B that is bigger than the supremum of A is very small compared to how the lower bound of B is smaller than the lower bound of A. In that sense, when a decision-maker has no clear understanding of how to choose a set, the new set relation $\preceq_D^\lambda$ can be helpful. We have

$$A \preceq_D^\lambda B \Leftrightarrow \forall \ell > 0 : \lambda \ell(\overline{a} - \overline{b}) + (1-\lambda)\ell(\underline{a} - \underline{b}) \geq 0.$$

This is fulfilled for the above values if $\lambda \geq \frac{5}{6}$. Moreover, we obtain the condition that $\lambda \in \left[0, \frac{5}{6}\right]$ for $B \preceq_D^\lambda A$ to hold true. Figure 1 illustrates this example.

Remark 2.3. This example illustrates also, how the alternative approach of using the set less relation in the sense of Young [20] is potentially of limited use here: If one defines A to be better than B if and only if $A \preceq_D^u B$ and $A \preceq_D^l B$ hold, it is obviously not possible to prefer any of the sets to the other. If, on the other hand, the "new" relation would entail that at least one of $\preceq_D^u$ and $\preceq_D^l$ is fulfilled, we would not be able to make any distinction between the two sets at all.

In $\mathbb{R}$, it is easy to determine minimal/maximal elements (in the sense of vector optimization [2]) of the set A and B and so relatively straightforward to implement centered sets as weighted averages of those. In this example, the weighted set relation as proposed here does coincide with the concept outlined in [22].

The situation in Example 2.1 is in line with [15, Example 2.10], where a weighted set relation is formulated and used in terms of a *nonlinear* functional, see also Section 5 below. This gives rise to examine our set relation $\preceq_D^\lambda$: We will show in the next example that if the considered sets A and B do *not* fulfill Assumption 2.2, then the relation $\preceq_D^\lambda$ fails to be that useful.

Example 2.2. Let $Y = \mathbb{R}^2$, $A := \{(1,1)^\top, (-1,-1)^\top\}$ and $B := \{(2,0)^\top, (0,2)^\top\}$. We choose $D = \mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$. Because the set $B + D$ is not convex, Assumption 2.2 is not satisfied and we suspect that the relation $\preceq_D^\lambda$ would not be useful. Indeed, if we set $\lambda = 0$, there exist $\ell \in \mathbb{R}_+^2$ for which

$$g_\ell^l(A,B) = \inf_{b\in B} \ell(a) - \inf_{a\in A} \ell(b) > 0,$$

cf. Figure 2. However, clearly $A \preceq_D^l B$. Therefore, we can see that the relation $\preceq_D^\lambda$ does not reflect this situation well. We conclude that the relation $\preceq_D^\lambda$ is really only useful if the requirements in Assumption 2.2 are met.

In the following proposition, we will study some analytical properties of the set relation $\preceq_D^\lambda$. Note how the analysis is based on the defined functions g_ℓ^u and g_ℓ^l whose properties directly transfer to $\preceq_D^\lambda$ but that we also implicitly often come back to the above requirement of

$$\lambda \cdot \sup_{a\in A} \ell(a) + (1-\lambda) \cdot \inf_{a\in A} \ell(a) \leq \lambda \cdot \sup_{b\in B} \ell(b) + (1-\lambda) \cdot \inf_{b\in B} \ell(b).$$

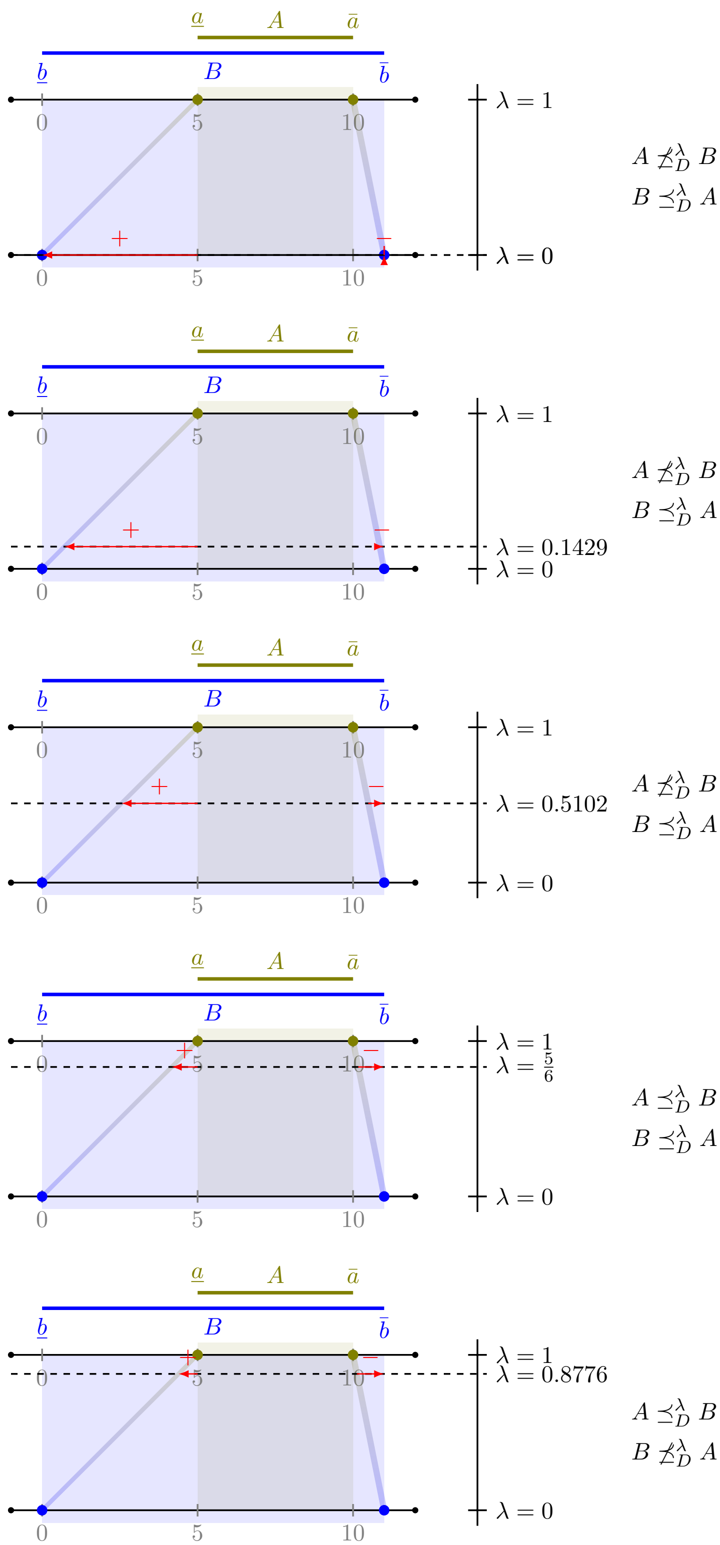

FIGURE 1. Illustration of Example 2.1 with some example values for λ.

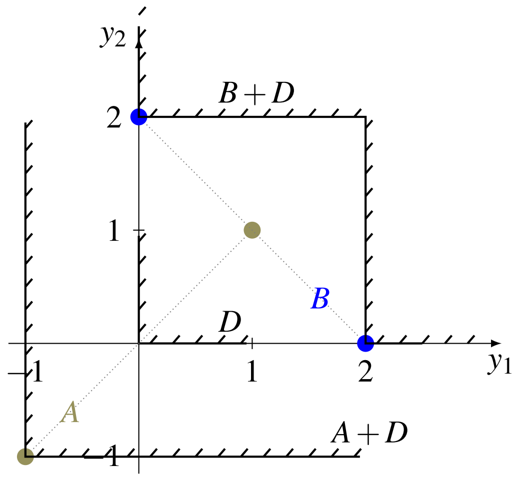

FIGURE 2. Involved sets as described in Example 2.2

Note also, how the simple linear construction allows to effectively study the upper and lower set less relation and combine them at any stage within the proofs.

Proposition 2.1. *Let Assumption 2.2 be satisfied and let $\ell \in D^* \setminus \{0\}$. Then the following assertions hold:*

(i) *We have*

$$g^u_\ell(A,C) \leq g^u_\ell(A,B) + g^u_\ell(B,C),$$
$$g^l_\ell(A,C) \leq g^l_\ell(A,B) + g^l_\ell(B,C).$$

(ii) $g^u_\ell(\alpha A, \alpha B) = \alpha g^u_\ell(A,B)$ *and* $g^l_\ell(\alpha A, \alpha B) = \alpha g^l_\ell(A,B)$ *for any* $\alpha > 0$.

(iii) *For any* $\lambda \in [0,1]$ *the relation* $\preceq^\lambda_D$ *is reflexive and transitive. Hence,* $\preceq^\lambda_D$ *is a preorder.*

(iv) *The relation* $\preceq^\lambda_D$ *is compatible with nonnegative scalar multiplication, i.e., for given* $A, B \in \mathscr{P}(Y)$ *and any* $\alpha \geq 0$*, we have*

$$A \preceq^\lambda_D B \implies \alpha A \preceq^\lambda_D \alpha B.$$

Proof. (i) Choose $\overline{b} \in B$ such that $\sup_{a\in A} \ell(a-\overline{b}) := \sup_{a\in A} \inf_{b\in B} \ell(a-b)$. Such a $\overline{b}$ always exists according to Assumption 2.2 (see also Remark 2.1). Then

$$\begin{aligned}
g^u_\ell(A,C) &= \sup_{a\in A} \inf_{c\in C} \ell(a-c) \\
&= \sup_{a\in A} \inf_{c\in C} \ell(a-\overline{b}+\overline{b}-c) \\
&= \sup_{a\in A} \inf_{c\in C} \ell(a-\overline{b}) + \ell(\overline{b}-c) \\
&= \sup_{a\in A} \ell(a-\overline{b}) + \inf_{c\in C} \ell(\overline{b}-c) \\
&= \sup_{a\in A} \inf_{b\in B} \ell(a-b) + \inf_{c\in C} \ell(\overline{b}-c) \\
&\leq \sup_{a\in A} \inf_{b\in B} \ell(a-b) + \sup_{b\in B} \inf_{c\in C} \ell(b-c).
\end{aligned}$$

The triangle inequality for g^l follows a similar pattern, so the proof is omitted here.

(ii) We have, for any $\alpha > 0$,

$$\begin{aligned}
g_\ell^u(\alpha A, \alpha B) &= \sup_{a\in\alpha A} \inf_{b\in\alpha B} \ell(a-b) \\
&= \sup_{\frac{a}{\alpha}\in A} \inf_{\frac{b}{\alpha}\in B} \ell(a-b) \\
&= \sup_{\tilde a\in A} \inf_{\tilde b\in B} \ell(\tilde a\alpha - \tilde b\alpha) \text{ (with } \tilde a := \frac{a}{\alpha},\ \tilde b := \frac{b}{\alpha}) \\
&= \sup_{\tilde a\in A} \inf_{\tilde b\in B} \alpha\ell(\tilde a - \tilde b) \\
&= \alpha \sup_{\tilde a\in A} \inf_{\tilde b\in B} \ell(\tilde a - \tilde b) = \alpha g_\ell^u(A,B).
\end{aligned}$$

The proof for g_ℓ^l is similar and left out.

(iii) Now we show that $\preceq_D^\lambda$ is reflexive: $\preceq_D^\lambda$ is reflexive if for all $\ell \in D^* \setminus \{0\}$, $\lambda g_\ell^u(A,A) + (1-\lambda)g_\ell^l(A,A) \le 0$. Suppose that this is not the case. Then there exists some $\ell \in D^* \setminus \{0\}$ with $\lambda g_\ell^u(A,A) + (1-\lambda)g_\ell^l(A,A) > 0$. Since $\lambda \ge 0$ and $1-\lambda \ge 0$, we have that $g_\ell^u(A,A) > 0$ or $g_\ell^l(A,A) > 0$. Assume that $g_\ell^u(A,A) > 0$. This means that $g_\ell^u(A,A) = \sup_{a\in A} \inf_{\overline{a}\in A} \ell(a-\overline{a}) > 0$. This implies that $\sup_{a\in A} \ell(a) > \sup_{\overline{a}\in A} \ell(\overline{a})$, a contradiction. A similar contradiction is obtained when assuming $g_\ell^l(A,A) > 0$.

Now we show that $\preceq_D^\lambda$ is transitive for arbitrary $\lambda \in [0,1]$. Let the sets A,B,C be given according to Assumption 2.2, and let $A \preceq_D^\lambda B$ and $B \preceq_D^\lambda C$. Then

$$\begin{aligned}
\lambda g_\ell^u(A,B) + (1-\lambda)g_\ell^l(A,B) &\le 0, \\
\lambda g_\ell^u(B,C) + (1-\lambda)g_\ell^l(B,C) &\le 0.
\end{aligned}$$

It follows that

$$\lambda \left(g_\ell^u(A,B) + g_\ell^u(B,C)\right) + (1-\lambda)\left(g_\ell^l(A,B) + g_\ell^l(B,C)\right) \le 0.$$

Due to the triangle inequality of g_ℓ^u and g_ℓ^l (see (i) above), we immediately obtain

$$\lambda g_\ell^u(A,C) + (1-\lambda)g_\ell^l(A,C) \le 0,$$

which corresponds to $A \preceq_D^\lambda C$. That means that $\preceq_D^\lambda$ is transitive. Hence, $\preceq_D^\lambda$ is a preorder.

(iv) Let $A \preceq_D^\lambda B$. From Definition 2.2, one has

$$\lambda g_\ell^u(A,B) + (1-\lambda)g_\ell^l(A,B) \le 0.$$

This, together with (ii) shows that

$$\lambda g_\ell^u(\alpha A,\alpha B) + (1-\lambda)g_\ell^l(\alpha A,\alpha B) = \alpha\left(\lambda g_\ell^u(A,B) + (1-\lambda)g_\ell^l(A,B)\right) \le 0,$$

for all $\alpha \ge 0$, as required. □

Remark 2.4. Note that the set relation $\preceq_D^\lambda$ is in general not compatible with addition, which means that generally, for A,B,C satisfying Assumption 2.2,

$$A \preceq_D^\lambda B \ \not\Rightarrow\ A + C \preceq_D^\lambda B + C.$$

As a simple example, let $A = \{(0,0)^\top\}$, $B = \{(1,1)^\top\}$, $C = [(1,2)^\top,(2,2)^\top] \cup [(2,2)^\top,(2,1)^\top]$, $D = \mathbb{R}^2_+$ and $\lambda = 0$, where the $[.,.]$-notation indicates the direct connection line between the

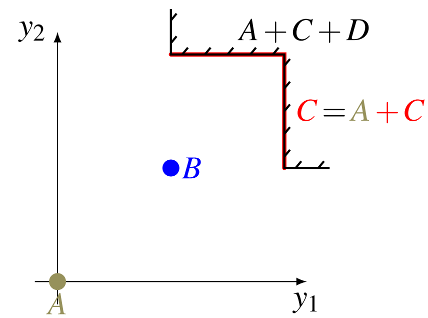

FIGURE 3. Sets in the example of Remark 2.4

points. It clearly holds $A \preceq_D^\lambda B$, but $A+C=C$ already fails to fulfill Assumption 2.2 (in the role of A in that assumption) as $(A+C)+D$ is not convex, see Figure 3

3. FORMULATION OF SET OPTIMIZATION PROBLEMS USING THE WEIGHTED SET RELATION

Let $F: S \rightrightarrows Y$ with $S \subseteq \mathbb{R}^n$. We consider the following set optimization problem

$$\min_{x \in S} F(x). \tag{P}$$

Another benefit of the above algebraic definition of $\preceq_D^\lambda$ comes from the immediate characterization of the set relation through numerical quantities. These can then be used for necessary optimality conditions, see Theorem 3.1 below.

As already indicated below, when fully studying set optimization problems, sets need to be compared both ways, i.e. $A \preceq B$ and $B \preceq A$ need to be checked to characterize efficient solutions. The confined study, cf. Assumption 2.2 above, to only classify $A \preceq B$ from above therefore needs to be broaden in the following sense:

Assumption 3.1. Let Y be a real locally convex space, partially ordered by a convex cone D, $F: S \rightrightarrows Y$, $S \subseteq \mathbb{R}^n$ and assume that for each $x \in S$, $F(x)$ is nonempty, $F(x)+D$ and $F(x)-D$ are closed and convex.

Minimal solutions of the above problem are defined as follows (the so-called *set approach*).

Definition 3.1. Let Assumption 3.1 be fulfilled, $\lambda \in [0,1]$ and D be a closed convex cone in Y. A feasible point $x^* \in S$ is called a *minimal solution* of (P) if

$$x \in S: \; F(x) \preceq_D^\lambda F(x^*) \Longrightarrow F(x^*) \preceq_D^\lambda F(x).$$

By the definition of minimal solutions of the problem (P) w.r.t. the preorder $\preceq_D^\lambda$, we know that if $\bar{x} \in S$ is a minimal solution of the problem (P) w.r.t. the preorder $\preceq_D^\lambda$ and $F(\tilde{x}) \preceq_D^\lambda F(\bar{x})$ for some $\tilde{x} \in S$, then $\tilde{x}$ is a minimal solution of the problem (P) w.r.t. the preorder $\preceq_D^\lambda$. Denote

$$[F(\bar{x})]^{-1} := \left\{ x \in S: \quad F(x) \preceq_D^\lambda F(\bar{x}), \quad F(\bar{x}) \preceq_D^\lambda F(x) \right\}.$$

Let $\lambda \in [0,1]$. We now define (with a slight abuse of notation) a function $g_\ell^\lambda : S \times S \to \mathbb{R} \cup \{\pm\infty\}$ by

$$g_\ell^\lambda(x,y) := \lambda g_\ell^u(F(x),F(y)) + (1-\lambda) g_\ell^l(F(x),F(y)). \tag{3.1}$$

Below, we propose a sufficient and necessary characterization for minimal solutions of the problem (P) w.r.t. the preorder $\preceq_D^\lambda$.

Theorem 3.1. *Let Assumption 3.1 be satisfied and $\bar{x} \in S$. Then $\bar{x}$ is a minimal solution of the problem (P) w.r.t. the preorder $\preceq_D^\lambda$ if and only if*

$$\nexists x \in S \setminus [F(\bar{x})]^{-1} : \quad \forall \ell \in D^* \setminus \{0\} : \quad g_\ell^\lambda(x,\bar{x}) \leq 0. \tag{3.2}$$

Proof. The assertion immediately follows from the fact that $\bar{x} \in S$ is a minimal solution of (P) if there does not exist an $x \in S$ with the properties that $F(x) \preceq_D^\lambda F(\bar{x})$ and $F(\bar{x}) \not\preceq_D^\lambda F(x)$, together with the definition of g_ℓ^λ. □

4. Existence Results

In order to present an existence result for minimal solutions of problem (P) with $\preceq_D^\lambda$, we recall the following definition of semicontinuity of a set-valued map w.r.t. a preorder $\preceq$ (see [21]).

Definition 4.1 (Semicontinuity)**.** Let $S \subseteq \mathbb{R}^n$. The set-valued mapping $F : S \rightrightarrows \mathbb{R}^m$ is called **semicontinuous** at $\bar{x} \in S$ w.r.t. the preorder $\preceq$ if $F(\bar{x}) \in \mathscr{V}$, where $\mathscr{V} := \{T \in \mathscr{P}(\mathbb{R}^m) \mid T \not\preceq V\}$ for some $V \in \mathscr{P}(\mathbb{R}^m)$, implies that there exists a neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ such that $F(x) \in \mathscr{V}$ for all $x \in U$. In other words, F is semicontinuous at $\bar{x}$ if

$$F(\bar{x}) \not\preceq V \text{ for some } V \in \mathscr{P}(\mathbb{R}^m) \implies \exists\, U(\bar{x}) : \; F(x) \not\preceq V \;\forall\, x \in U.$$

F is called semicontinuous w.r.t. $\preceq$ if F is semicontinuous w.r.t. $\preceq$ at every $\bar{x} \in S$.

Remark 4.1. We point out, that the definition of semicontinuity in set optimization problems is a non-trivial task. We refer the reader to [23] for a comprehensive study on various ways of how to study and analyze lower-semicontinuity (based on $\preceq_D^l$) and the relations of these concepts.

We provide the following example for a set-valued mapping that is semicontinuous w.r.t. the weighted set relation $\preceq_D^\lambda$ introduced in Section 2.

Example 4.1 (See also [15, Example 3.3])**.** We consider the following setting: Let $S = \mathbb{R}$ and consider the set-valued mapping $F : S \rightrightarrows \mathbb{R}^2$ given by

$$F(x) := \left[(1-x,x)^\top, (1,1)^\top\right],$$

where $[(a,b)^\top, (c,d)^\top]$ is the line segment between $(a,b)^\top$ and $(c,d)^\top$, and the preorder $\preceq_D^\lambda$, $D = \mathbb{R}^2_+$. The sets $F(x)$, $x \in S$, are illustrated in Figure 4. Note that this setting was used in [15, Example 3.3] for the nonconvex approach using a set relation that involved *nonlinear* functionals. Since the sets $F(x)$, $x \in S$ are line segments, Assumption 2.2 is naturally fulfilled. We verify that F is semicontinuous w.r.t. the weighted set relation $\preceq_D^\lambda$ for any $\lambda \in [0,1]$: For $\lambda = 1$, we have, for $V = \{(0,0)^\top\}$, $F(x) \not\preceq_D^\lambda V$ for all $x \in \mathbb{R}$. Therefore, for $\lambda = 1$, F is semicontinuous w.r.t. $\preceq_D^\lambda$. Similarly, for $\lambda = 0$ and by choosing $V = \{(0,0)^\top\}$, we get that $F(x) \not\preceq_D^\lambda V$ for all $x \in \mathbb{R}$. That means that for $\lambda = 0$, F is semicontinuous w.r.t. $\preceq_D^\lambda$. Since

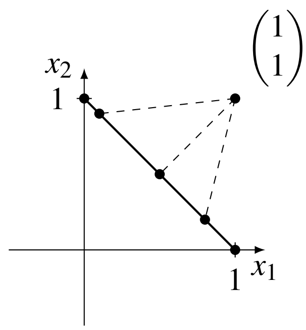

FIGURE 4. Feasible solution sets of F, described in Example 4.1.

$g^u(F(x),V) > 0$ and $g^l(F(x),V) > 0$ for all $x \in \mathbb{R}$, we also obtain that $\lambda g^u(F(x),V) + (1-\lambda)g^l(F(x),V) > 0$ for arbitrary $\lambda \in [0,1]$. Therefore, we can conclude that F is semicontinuous w.r.t. $\preceq_D^\lambda$ for any $\lambda \in [0,1]$.

Now we have the following existence result for problem (P) w.r.t. the new set relation $\preceq_D^\lambda$ introduced in Definition 2.2.

Corollary 4.1 ([15, Corollary 3.7])**.** *Let Assumption 3.1 be satisfied. Suppose that S is compact and that F is semicontinuous w.r.t. the preorder* $\preceq_D^\lambda$ *on S. Then the problem* (P) *has a minimal solution w.r.t. the preorder* $\preceq_D^\lambda$.

Proof. Let us mention that $\preceq_D^\lambda$ is a preorder according to Proposition 2.1. Thus, the result follows immediately by applying [21, Theorem 5.1]. □

Remark 4.2. The proof of [21, Theorem 5.1] is based on the notion of completeness with respect to a preorder and the considerations of coverings of the image sets in the neighborhood U.

Remark 4.3. Again, it becomes apparent how the simplicity (in terms of using linear functionals ℓ and a simple weighting process) in the construction of $\preceq_D^\lambda$ allows to simply translate results from $\preceq_D^l$ and $\preceq_D^u$.

5. RELATIONS TO THE NONCONVEX CASE

In [15], the authors use the nonlinear scalarization functional of Gerstewitz-type [24, 25], i.e.

$$z_D^k\colon Y \to \mathbb{R} \cup \{\pm\infty\} \;:\; z_D^k(y) := \inf\{t \in \mathbb{R} : y \in t \cdot k - D\}$$

with the convention $\inf \emptyset = +\infty$ to implement a compromise between $\preceq_D^u$ and $\preceq_D^l$ and study similar existence results and optimality conditions in the non-convex case. The vector k serves as an 'attachment line' along which the ordering $-D$ slides along until the inclusion $y \in t \cdot k - D$ is fulfilled, see [26] for a comprehensive study. We restrict to a finite-dimensional setting $Y = \mathbb{R}^m$ and a polyhedral ordering structure $D = \{y \in \mathbb{R}^m : \langle w, y\rangle \geq 0 \ \forall w \in \mathscr{W}\}$ with a finite set of support vectors $\mathscr{W} \subset \mathbb{R}^m$ and the vector k being in the interior of D, i.e. $\langle w, k\rangle > 0$ for all $w \in \mathscr{W}$. In this situation, there is an explicit formula for z_D^k:

$$z_D^k(y) = \frac{\max_{w \in \mathscr{W}} \langle w, y\rangle}{\max_{w \in \mathscr{W}} \langle w, k\rangle},$$

cf. [27]. Note how the set $\mathscr{W}$ and the dual cone D^* are related.

The main result, see [28], is that a set relation similar to the one presented here can be achieved using terms of the form $\sup_\bullet \inf_\bullet z_D^k(a-b)$. To keep this outlook section concise, we will restrict the following to $\lambda = 1$; the general case follows equivalently. In this case, the relation $A \preceq_D^u B \leftrightsquigarrow A \preceq_D^{\lambda=1}$ can be qualified through

$$\sup_{a\in A}\inf_{b\in B} z_D^k(a-b) \leq 0.$$

Using the above calculation procedure for z_D^k, this can be written as

$$\sup_{a\in A}\inf_{b\in B} \frac{\max_{w\in\mathscr{W}}\langle w, a-b\rangle}{\max_{w\in\mathscr{W}}\langle w, k\rangle} \leq 0,$$

which, using $k \in \operatorname{int} D$ (i.e. $\langle w,k\rangle > 0$), is the same as

$$\sup_{a\in A}\inf_{b\in B}\max_{w\in\mathscr{W}}\langle w, a-b\rangle \leq 0,$$

such that the resemblance with g_ℓ^u becomes obvious.

The appeal of the approach presented here (rather than directly studying the more complex nonlinear case) is that we solely work with linear functionals from the dual cone of D and can therefore hope for easier translation into actual algorithms of the acquired results.

There are other functionals (e.g. of Hiriart-Urruty type [29]) being used in the literature to characterize set relations, see [30] for a very general characterization of necessary properties of those functionals for the case of the upper set less relation. A more detailed study on the relationships between scalarization/separation functionals can be found in the recent monograph [26] and [31].

6. Summary and Outlook

In this paper, we proposed a novel set relation which is able to act as a weighted compromise between two important set relations and therefore balances out possible gaps that can occur in modeling set optimization problems. We discussed its properties, formulated a set optimization problem by means of this new set relation and gave an existence theorem.

By restricting the definition to the convex case (i.e. $A+D$ and $B-D$ being convex), we were able to base the characterization solely on linear functionals opening the doorway for efficient algorithms. Note that the prominent pessimistic $\preceq_D^u$ relation and its optimistic counterpart $\preceq_D^l$ are special cases and that all results concerning existence and optimality transfer directly to these situations.

Further research shall include developing numerical methods for obtaining approximations of minimal solutions of set optimization problems involving this new set relation.

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REFERENCES

[1] H. M. Markowitz, Portfolio selection, J. Finance 7 (1952), 77-91.
[2] M. Ehrgott, Multicriteria Optimization, Springer, Berlin, Heidelberg, 2005.
[3] Q. H. Ansari, E. Köbis, J.-C. Yao, Vector Variational Inequalities and Vector Optimization. Theory and Applications. Springer, Berlin, 2018.
[4] J. Jahn, Vector Optimization - Introduction, Theory, and Extensions, Springer, Berlin, Heidelberg, 2011.
[5] M. Goerigk, M. Knoth, M. Müller-Hannemann, M. Schmidt, A. Schöbel, The price of strict and light robustness in timetable information, Transportation Sci. 48 (2014), 225-242.
[6] G. Eichfelder, M. Pilecka, Set approach for set optimization with variable ordering structures part I: Set relations and relationship to vector approach, J. Optim. Theory Appl. 171 (2016), 931-946.
[7] G. Eichfelder, M. Pilecka, Set approach for set optimization with variable ordering structures part II: Scalarization approaches, J. Optim. Theory Appl. 171 (2016), 947-963.
[8] M.-C. Körner, J. A. Mesa, F. Perea, A. Schöbel, D. Scholz, A maximum trip covering location problem with an alternative mode of transportation on tree networks and segments, TOP 22 (2012), 227-253.
[9] M. Ehrgott, J. Ide, A. Schöbel, Minmax robustness for multi-objective optimization problems, Eur. J. Oper. Res. 239 (2014), 17-31.
[10] J. Ide, E. Köbis, Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations, Math. Method. Oper. Res. 80 (2014), 99-127.
[11] J. Ide, E. Köbis, D. Kuroiwa, A. Schöbel, Chr. Tammer, The relationship between multi-objective robustness concepts and set-valued optimization, Fixed Point Theory Appl. 2014 (2014), 83.
[12] R. E. Moore: Interval Analysis. Prentice & Hall, New York, 1966.
[13] B. Mordukhovich: Variational Analysis and Applications, Springer, Cham 2018.
[14] A. Khan, Chr. Tammer, C. Zălinescu: Set-Valued Optimization – An Introduction with Applications, Springer, Berlin, Heidelberg, 2015.
[15] J. Chen, E. Köbis, M. A. Köbis, J.-C. Yao, A new set order relation in set optimization, J. Nonlinear Convex Anal. 18 (2017), 637-649.
[16] E. Köbis, Approaches to Set Optimization Without Convexity Assumptions. Habilitation thesis. Martin-Luther-University Halle-Wittenberg, Germany (2020), https://opendata.uni-halle.de//handle/1981185920/35889http://dx.doi.org/10.25673/35671.
[17] D. Kuroiwa, The natural criteria in set-valued optimization, Sūrikaisekikenkyūsho Kōkyūroku, Research on Nonlinear Analysis and Convex Analysis 1031 (1997), 85-90.
[18] D. Kuroiwa, Some duality theorems of set-valued optimization with natural criteria, in: Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis, T. Tanaka (Edi), pp. 221–228, World Scientific, 1999.
[19] J. Jahn, Vectorization in set optimization, J. Optim. Theory Appl. 167 (2015), 783–795.
[20] R. C. Young (Tanner), The algebra of many-valued quantities, Math. Ann. 104 (1931), 260-290.
[21] J. Jahn, T. X. D. Ha, New order relations in set optimization, J. Optim. Theory Appl. 148 (2011), 209-236.
[22] M. Bischoff, J. Jahn, E. Köbis, Hard uncertainties in multiobjective layout optimization of photovoltaic power plants, Optimization 66 (2017), 361-380.
[23] J. Benoist, N. Popovici, Characterizations of convex and quasiconvex set-valued maps, Math. Meth. Oper. Res. 57 (2003), 427-435.
[24] Chr. Gerstewitz (Tammer), Nichtkonvexe Dualitität in der Vektoroptimierung, Wissenschaftliche Zeitschrift Technische Hochschule Leuna-Merseburg, 25 (1983), 357-364.
[25] Chr. Gerth (Tammer), P. Weidner, Nonconvex separation theorems and some applications in vector optimization, J. Optim. Theory Appl. 67 (1990), 297-320.
[26] Chr. Tammer, P. Weidner, Scalarization and Separation by Translation Invariant Functions. Springer International Publishing, 2020.
[27] Chr. Tammer, K. Winkler, A new scalarization approach and applications in multicriteria d.c. optimization, J. Nonlinear Convex Anal. 4 (2003), 365–380.
[28] E. Köbis, M. A. Köbis, Treatment of set order relations by means of a nonlinear scalarization functional: a full characterization, Optimization, 65 (2016), 1805-1827.

[29] J. B. Hiriart-Urruty, Tangent cones, generalized gradients and mathematical programming in Banach spaces, Mathematics of Operations Research, Institute for Operations Research and the Management Sciences (INFORMS), 4 (1979), 79-97.

[30] E. Köbis, M. Köbis, Characterizing the upper set relation by general functionals, in preparation.

[31] G. Bouza, E. Quintana, Chr. Tammer, A unified characterization of nonlinear scalarizing functionals in optimization, Vietnam J. Math. 47 (2019), 683-713.