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A NEW ITERATIVE METHOD WITH ALTERNATED INERTIA FOR THE SPLIT FEASIBILITY PROBLEM

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Abstract. In this paper, an iterative method with alternated inertial extrapolation step is proposed to solve the split feasibility problem. The stepsize in the proposed algorithm uses the self adaptive technique, which does not depend on the prior information of the operator norm. The weak convergence is proved under suitable conditions. Finally, a numerical example is given to illustrate the effectiveness of our algorithm.

Keywords. Alternated inertial extrapolation; Split feasibility problem; Self-adaptive; Weak convergence.

1. Introduction

In this paper, we study the split feasibility problem (shortly, SFP), which consists of

$$\text{finding} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{1.1}$$

where C and Q are nonempty closed and convex subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, and $A : \mathbb{R}^N \to \mathbb{R}^M$ is a bounded and linear operator.

In 1994, Censor and Elfving [1] first introduced this problem. Nowadays, the SFP not only plays an important role in phase retrievals, signal processing and other technical fields, but also can be applied to related medical fields and the systems biology. A large number of methods have been proposed by some authors to solve the problem of SFP and its variants, see [2, 3, 4, 5, 6, 7, 8, 9] and the references therein.

Byrne [10, 11] first proposed the so-called CQ algorithm to solve SFP, which generates the sequence $\{x^k\}_{k\in\mathbb{N}}$ through the following iteration:

$$x^{k+1} = P_C(x^k - \lambda_k A^T(I - P_Q)Ax^k), \tag{1.2}$$

where $\lambda_k \in (0, \frac{2}{\|A\|^2})$, $\|A\|^2$ is the spectral radius of the operator $A^T A$, and P_C and P_Q are the metric projections onto C and Q, respectively. Note that the CQ algorithm is a projection method (see, e.g., [12]). In the practical calculation of CQ algorithm (1.2), in order to avoid estimating

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the operator norm $\|A\|$, López et al. [13] introduced a dynamic stepsize selection method where the stepsize λ_k is defined as follows:

$$\lambda_k = \rho_k \frac{\|(I-P_Q)A(x^k)\|^2}{\|A^T(I-P_Q)A(x^k)\|^2},$$

where $\rho_k \in (0,2)$.

In general, the calculation of the projections onto the sets C and Q are difficult when they are general convex sets. To avoid this difficulty, Yang [14] considered the special cases that C and Q are level sets of two convex functions for which he introduced the relaxed CQ algorithm by replacing P_C and P_Q by P_{C_k} and P_{Q_k}, respectively. Note that C_k and Q_k are two half-spaces. Therefore there are explicit expressions for the projections onto C_k and Q_k.

Recently, inspired and motivated by He [15], Dong and Jiang [3] proposed a projection method. Let $F(x^k) = (x^k - P_{C_k}(x^k)) + A^*(I-P_{Q_k})A(x^k))$. The formula of the projection method in [3] is given as follows.

Algorithm 1.1. For any $\sigma > 0$, $\theta \in (0,1)$ and $\mu \in (0,1)$, take arbitrarily $x^0 \in \mathbb{R}^N$ and let

$$y^k = P_X(x^k - \lambda_k F(x^k)),$$

where $\lambda_k = \sigma\rho^{m_k}$ and m_k is the smallest nonnegative integer such that

$$\lambda_k\|F(x^k) - F(y^k)\| \leq \theta\|x^k - y^k\|.$$

Calculate

$$x_{\mathrm{I}}^{k+1} = x^k - \gamma\rho_k d(x^k, y^k),$$

and

$$x_{\mathrm{II}}^{k+1} = P_X(x^k - \gamma\rho_k\lambda_k F(y^k)),$$

where $\gamma \in (0,2)$,

$$d(x^k, y^k) := (x^k - y^k) - \lambda_k(F(x^k) - F(y^k)),$$

and

$$\rho_k := \frac{\langle x^k - y^k, d(x^k, y^k)\rangle + \lambda_k(\|(I-P_{C_k})y^k\|^2 + \|(I-P_{Q_k})Ay^k\|^2)}{\|d(x^k, y^k)\|^2}.$$

Note that the stepsize λ_k in Algorithm 1.1 is given by using the line search technique, which generally costs much since it involves the computation of projections and operations involving the matrices A and A^T.

Recently, to accelerate the iterative algorithms for the SFP, some authors [2, 16] introduced inertial modifications by utilizing the inertial extrapolation in [17]. Generally speaking, the inertial extrapolation step loses the monotonicity of $\{\|x^k - x^*\|\}_{k\in\mathbb{N}}$, where x^* is a solution of the SFP. To take care of this situation, inspired by the work in [18], Shehu, Dong and Liu [19] proposed an alternated inertial CQ algorithm with the self-adaptive stepsize to solve the SFP where the monotonicity of $\{\|x^k - x^*\|\}_{k\in\mathbb{N}}$ was shown.

Motivated by the above work, we propose an alternated inertial iterative method with self-adaptive stepsize to solve SFP. The convergence of the iterative algorithm is shown under some mild conditions. Moreover, by comparing with the algorithms in [3], the applicability and efficiency of this method are further illustrated.

The remainder of the paper is organized as follows: In Section 2, we introduce some basic definitions and lemmas that will be used in the process of proof. In Section 3, we propose a new iterative algorithm, followed by the convergence analysis. In Section 4, a numerical experiment is given to illustrate the efficiency of our proposed algorithm. Finally, we end the paper with a conclusion in Section 5.

2. Preliminaries

In this section, we review some definitions and lemmas which are used in the main results.

The following identity will be used for the main results (see Corollary 2.15 of [20]):

$$\|\alpha x+\beta y\|^2=\alpha(\alpha+\beta)\|x\|^2+\beta(\alpha+\beta)\|y\|^2-\alpha\beta\|x-y\|^2, \tag{2.1}$$

for all $\alpha,\beta\in\mathbb{R}$ and $x,y\in\mathbb{R}^N$.

The projection is an important tool for our work in this paper. Let K be a closed convex subset of $\mathbb{R}^N$. Recall that the nearest point or metric projection from $\mathbb{R}^N$ onto K, which is denoted P_K, is defined as follows: for each $x\in\mathbb{R}^N$, P_Kx is the unique point in K such that

$$\|x-P_Kx\|=\min\{\|x-z\|:z\in K\}.$$

The following two lemmas are useful characterizations and properties of projections:

Lemma 2.1. *For any $x\in\mathbb{R}^N$ and $z\in K$, then $z=P_Kx$ if and only if*

$$\langle x-z,y-z\rangle\le 0,\quad \forall y\in K.$$

Lemma 2.2. *For any $x,y\in\mathbb{R}^N$ and $z\in K$, the following assertions hold:*

(i) $\|P_K(x)-P_K(y)\|^2\le\langle P_K(x)-P_K(y),x-y\rangle$;
(ii) $\|P_K(x)-z\|^2\le\|x-z\|^2-\|P_K(x)-x\|^2$;
(iii) $\langle(I-P_K)x-(I-P_K)y,x-y\rangle\ge\|(I-P_K)x-(I-P_K)y\|^2$.

In this paper, we are concerned with the case whenever the involved subsets in the SFP are composed of level sets. Namely, we consider the case whenever C and Q are defined by

$$C=\{x\in\mathbb{R}^N:c(x)\le 0\},\quad Q=\{y\in\mathbb{R}^M:q(y)\le 0\},$$

where $c:\mathbb{R}^N\to\mathbb{R}$ is a convex function, and $q:\mathbb{R}^M\to\mathbb{R}$ is a convex function.

For the functions c and q, we make the following assumptions:

(i) c and q are subdifferentiable on C and Q. (Note that the convex function is subdifferentiable everywhere in $\mathbb{R}^N$.) For any $x\in\mathbb{R}^N$, at least one subgradient $\xi\in\partial c(x)$ can be calculated, where $\partial c(x)$ is defined as follows:

$$\partial c(x)=\{z\in\mathbb{R}^N:c(u)\ge c(x)+\langle u-x,z\rangle,\quad \text{for all } u\in\mathbb{R}^N\}.$$

For any $y\in\mathbb{R}^M$, at least one subgradient $\eta\in\partial q(y)$ can be calculated, where

$$\partial q(x)=\{w\in\mathbb{R}^M:q(v)\ge q(y)+\langle v-y,w\rangle,\quad \text{for all } v\in\mathbb{R}^M\}.$$

(ii) c and q are bounded on bounded sets. (Note that this condition is automatically satisfied if in the finite dimensional spaces.)

Define the halfspaces C_k and Q_k as follows:

$$C_k = \left\{ x \in \mathbb{R}^N : c(w^k) + \langle \xi^k, x - w^k \rangle \leq 0 \right\},$$

where $\xi^k \in \partial c(w^k)$, and

$$Q_k = \left\{ y \in \mathbb{R}^M : q(Aw^k) + \langle \eta^k, y - Aw^k \rangle \leq 0 \right\},$$

where $\eta^k \in \partial q(Aw^k)$.

By the definition of the subgradient, it is clear that $C \subseteq C_k$ and $Q \subseteq Q_k$. The projections onto C_k and Q_k are easy to compute since C_k and Q_k are half-spaces (see [21]).

3. Main Results

In this section, we propose a new projection method with self-adaptive stepsize and alternated inertial term, and prove its convergence under mild conditions.

Throughout this paper, we assume that the solution set of the SFP (1.1), denoted by

$$\Gamma = \{x \,|\, x \in C \quad \text{and} \quad Ax \in Q\},$$

is nonempty.

Set

$$F(x^k) = (x^k - P_{C_k}(x^k)) + A^*(I - P_{Q_k})A(x^k).$$

Then we give the iterative method:

Algorithm 1 (Alternated Inertial Projection Method with Self-Adaptive Stepsize)

Step 0 Choose the parameters α_k satisfying some assumption, $\mu \in (0,1)$ and $\lambda_1 > 0$. Let $x^0, x^1 \in \mathbb{R}^N$ be given starting points. Set $k := 1$

Step 1 Compute

$$w^k = \begin{cases} x^k, & k = \text{even}, \\ x^k + \alpha_k(x^k - x^{k-1}), & k = \text{odd}, \end{cases} \tag{3.1}$$

and

$$y^k = P_X(w^k - \lambda_k F(w^k)). \tag{3.2}$$

Calculate

$$\lambda_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu \|w^k - y^k\|}{\|F(w^k) - F(y^k)\|}, \lambda_k \right\}, & \text{if } F(w^k) - F(y^k) \neq 0, \\ \lambda_k, & \text{otherwise.} \end{cases} \tag{3.3}$$

If $y^k = w^k$, STOP. Otherwise go to Step 2.

Step 2 Compute

$$x_{\mathrm{I}}^{k+1} = w^k - \gamma \rho_k d(w^k, y^k), \tag{3.4}$$

and

$$x_{\mathrm{II}}^{k+1} = P_X(w^k - \gamma \rho_k \lambda_k F(y^k)), \tag{3.5}$$

where $\gamma \in (0,2)$,

$$d(w^k, y^k) := (w^k - y^k) - \lambda_k (F(w^k) - F(y^k)),$$

and

$$\rho_k := \frac{\langle w^k - y^k, d(w^k, y^k)\rangle + \lambda_k(\|(I - P_{C_k})y^k\|^2 + \|(I - P_{Q_k})Ay^k\|^2)}{\|d(w^k, y^k)\|^2}.$$

Set $k := k+1$ and return to Step 1.

Remark 3.1. Here are some explanations for Algorithm 1.

(i) For convenience, we call the projection algorithms which use update forms (3.4) and (3.5) Algorithm 1 (I) and Algorithm 1 (II), respectively.

(ii) The set X in Algorithm 1 can be chosen variously. It can be chosen to be a simple bounded subset of Hilbert spaces that contains at least one solution of the SFP, it can also be directly chosen as $X = \mathbb{R}^N$. In fact, it can be more generally chosen to be a dynamically changing set X_k provided that $\cap_{k=1}^{\infty} X_k$ contains a solution of the SFP. This does not affect the convergence result (see, e.g., [22]).

Assumption 3.1. We assume that the inertial parameter α_k in Algorithm 1 (I) and Algorithm 1 (II) satisfies one of the following conditions:

(i) $-1+\varepsilon < \alpha_k < \dfrac{2-\gamma}{\gamma} - \varepsilon$,

(ii) $-1+\varepsilon < \alpha_k < -\varepsilon$,

where $\varepsilon > 0$ is a small constant.

From the above assumptions, it is easy to conclude that the inertial parameters α_k can be positive or negative in Algorithm 1 (I), while the inertial parameters α_k must be negative in Algorithm 1 (II).

Remark 3.2. Note that by (3.3), $\lambda_{k+1} \leq \lambda_k$, $\forall k \geq 1$. Also, observe in Algorithm 1 that if $F(w^k) \neq F(y^k)$, then

$$\frac{\mu\|w^k - y^k\|}{\|F(w^k) - F(y^k)\|} \geq \frac{\mu\|w^k - y^k\|}{(1+\|A\|^2)\|w^k - y^k\|} = \frac{\mu}{1+\|A\|^2}$$

which implies that $0 < \min\{\lambda_1, \frac{\mu}{1+\|A\|^2}\} \leq \lambda_k$, $\forall k \geq 1$. This means that $\lim_{k\to\infty}\lambda_k$ exists. Thus, there exists $\lambda > 0$ such that $\lim_{k\to\infty}\lambda_k = \lambda$.

Lemma 3.1. *Let $\{x^k\}_{k\in\mathbb{N}}$ be generated by Algorithm 1. Then there exit a constant $\delta \in (\mu, 1)$ and a positive integer K_0 such that, for any $k \geq K_0$,*

$$\langle w^k - y^k, d(w^k, y^k)\rangle \geq (1-\delta)\|w^k - y^k\|^2,$$

and

$$\rho_k \geq \frac{1-\delta}{1+\delta^2}. \tag{3.6}$$

Proof. From (3.3), we have

$$\begin{aligned}\langle w^k - y^k, d(w^k, y^k)\rangle &= \|w^k - y^k\|^2 - \langle w^k - y^k, \lambda_k(F(w^k) - F(y^k))\rangle \\ &\geq \|w^k - y^k\|^2 - \lambda_k\|w^k - y^k\|\|F(w^k) - F(y^k)\| \\ &\geq (1 - \mu\frac{\lambda_k}{\lambda_{k+1}})\|w^k - y^k\|^2.\end{aligned}$$

By Remark 3.2, we know $\lim_{k\to\infty}\frac{\lambda_k}{\lambda_{k+1}}=1$. Therefore, there exit a constant $\delta\in(\mu,1)$ and a positive integer K_0 such that, for any $k\geq K_0$, $\frac{\lambda_k}{\lambda_{k+1}}<\frac{\delta}{\mu}$. Thus,

$$\langle w^k-y^k,d(w^k,y^k)\rangle\geq(1-\delta)\|w^k-y^k\|^2. \tag{3.7}$$

From Lemma 2.2 (iii), we obtain

$$\begin{aligned}
&-2\lambda_k\langle w^k-y^k,F(w^k)-F(y^k)\rangle\\
&=-2\lambda_k\langle w^k-y^k,(I-P_{C_k})(w^k)-(I-P_{C_k})(y^k)\rangle\\
&\quad-2\lambda_k\langle w^k-y^k,A^*(I-P_{Q_k})A(w^k)-A^*(I-P_{Q_k})A(y^k)\rangle\\
&=-2\lambda_k\langle w^k-y^k,(I-P_{C_k})(w^k)-(I-P_{C_k})(y^k)\rangle\\
&\quad-2\lambda_k\langle Aw^k-Ay^k,(I-P_{Q_k})A(w^k)-(I-P_{Q_k})A(y^k)\rangle\\
&\leq-2\lambda_k\|(I-P_{C_k})(w^k)-(I-P_{C_k})(y^k)\|^2\\
&\quad-2\lambda_k\|(I-P_{Q_k})A(w^k)-(I-P_{Q_k})A(y^k)\|^2\leq 0.
\end{aligned} \tag{3.8}$$

Therefore, using (3.3) and (3.8), we have

$$\begin{aligned}
\|d(w^k,y^k)\|^2&=\|w^k-y^k-\lambda_k(F(w^k)-F(y^k))\|^2\\
&=\|w^k-y^k\|^2+\lambda_k^2\|F(w^k)-F(y^k)\|^2\\
&\quad-2\lambda_k\langle w^k-y^k,F(w^k)-F(y^k)\rangle\\
&\leq(1+\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2})\|w^k-y^k\|^2.
\end{aligned}$$

Similarly, we have

$$\|d(w^k,y^k)\|^2\leq(1+\delta^2)\|w^k-y^k\|^2,\quad\forall k\geq K_0. \tag{3.9}$$

Combining (3.7) and (3.9), we have

$$\rho_k\geq\frac{\langle w^k-y^k,d(w^k,y^k)\rangle}{\|d(w^k,y^k)\|^2}\geq\frac{1-\delta}{1+\delta^2},\quad\forall k\geq K_0.$$

This completes the proof. □

Lemma 3.2. [3, Lemma 3.5] *Let $\{x^k\}_{k\in\mathbb{N}}$ be generated by Algorithm 1. Then*

$$\langle w^k-x^*,d(w^k,y^k)\rangle\geq\rho^k\|d(w^k,y^k)\|^2,\quad\forall x^*\in\Gamma.$$

Lemma 3.3. *Suppose that the Assumption 3.1* (i) *holds. Then the sequence $\{x^k\}_{k\in\mathbb{N}}$ generated by Algorithm* 1 (I) *is bounded.*

Proof. Take k as even and pick any point x^* in Γ. Then

$$\begin{aligned}
\|x_{\mathrm{I}}^{k+2}-x^*\|^2&=\|(w^{k+1}-x^*)-\gamma\rho_{k+1}d(w^{k+1},y^{k+1})\|^2\\
&=\|w^{k+1}-x^*\|^2-2\gamma\rho_{k+1}\langle w^{k+1}-x^*,d(w^{k+1},y^{k+1})\rangle\\
&\quad+\gamma^2\rho_{k+1}^2\|d(w^{k+1},y^{k+1})\|^2.
\end{aligned} \tag{3.10}$$

Combining Lemma 3.2 and (3.10), we have

$$\|x_{\mathrm{I}}^{k+2}-x^*\|^2\leq\|w^{k+1}-x^*\|^2-\gamma(2-\gamma)\rho_{k+1}^2\|d(w^{k+1},y^{k+1})\|^2. \tag{3.11}$$

From the definition of w^k and (2.1), we obtain

$$\begin{aligned}\|w^{k+1}-x^*\|^2 =&\|(1+\alpha_{k+1})(x_{\mathrm{I}}^{k+1}-x^*)-\alpha_{k+1}(x^k-x^*)\|^2\\ =&(1+\alpha_{k+1})\|x_{\mathrm{I}}^{k+1}-x^*\|^2-\alpha_{k+1}\|x^k-x^*\|^2\\ &+\alpha_{k+1}(1+\alpha_{k+1})\|x_{\mathrm{I}}^{k+1}-x^k\|^2.\end{aligned} \tag{3.12}$$

Similar to (3.11), we obtain

$$\|x_{\mathrm{I}}^{k+1}-x^*\|^2 \le \|x^k-x^*\|^2-\gamma(2-\gamma)\rho_k^2\|d(x^k,y^k)\|^2. \tag{3.13}$$

Substituting (3.13) into (3.12), we arrive at

$$\begin{aligned}\|w^{k+1}-x^*\|^2 \le&\|x^k-x^*\|^2-\gamma(2-\gamma)(1+\alpha_{k+1})\rho_k^2\|d(x^k,y^k)\|^2\\ &+\alpha_{k+1}(1+\alpha_{k+1})\|x_{\mathrm{I}}^{k+1}-x^k\|^2.\end{aligned} \tag{3.14}$$

By (3.1) and (3.4), we have

$$\begin{aligned}\|x_{\mathrm{I}}^{k+1}-x^k\|^2 =&\|w^k-\gamma\rho_k d(w^k,y^k)-x^k\|^2\\ =&\gamma^2\rho_k^2\|d(x^k,y^k)\|^2.\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we have

$$\|w^{k+1}-x^*\|^2 \le\|x^k-x^*\|^2-(1+\alpha_{k+1})\gamma[(2-\gamma)-\gamma\alpha_{k+1}]\rho_k^2\|d(x^k,y^k)\|^2. \tag{3.16}$$

Substituting (3.16) into (3.11), we obtain

$$\begin{aligned}\|x_{\mathrm{I}}^{k+2}-x^*\|^2 \le&\|x^k-x^*\|^2-(1+\alpha_{k+1})\gamma[(2-\gamma)-\gamma\alpha_{k+1}]\rho_k^2\|d(x^k,y^k)\|^2\\ &-\gamma(2-\gamma)\rho_{k+1}^2\|d(w^{k+1},y^{k+1})\|^2.\end{aligned} \tag{3.17}$$

From (3.7), we have

$$\|d(w^k,y^k)\| \ge (1-\delta)\|w^k-y^k\|. \tag{3.18}$$

Combining (3.17), (3.18), and (3.6), we obtain

$$\begin{aligned}\|x_{\mathrm{I}}^{k+2}-x^*\|^2 \le&\|x^k-x^*\|^2-(1+\alpha_{k+1})\gamma[(2-\gamma)-\gamma\alpha_{k+1}]\frac{(1-\delta)^4}{(1+\delta^2)^2}\|x^k-y^k\|^2\\ &-\gamma(2-\gamma)\frac{(1-\delta)^4}{(1+\delta^2)^2}\|w^{k+1}-y^{k+1}\|^2, \quad \forall k\ge K_0.\end{aligned} \tag{3.19}$$

By Assumption 3.1 (i), we obtain from (3.19) that

$$\|x_{\mathrm{I}}^{k+2}-x^*\| \le \|x^k-x^*\|, \quad \forall k\ge K_0. \tag{3.20}$$

This implies that $\{x^{2k}-x^*\}_{k\in\mathbb{N}}$ and $\{x^{2k}\}_{k\in\mathbb{N}}$ are bounded. Furthermore, $\lim_{k\to\infty}\|x^{2k}-x^*\|$ exists. By (3.13), we have

$$\|x_{\mathrm{I}}^{k+1}-x^*\| \le \|x^k-x^*\|, \quad \forall k\ge K_0. \tag{3.21}$$

Therefore, $\{x_{\mathrm{I}}^{2k+1}\}_{k\in\mathbb{N}}$ is bounded. Thus, $\{x^k\}_{k\in\mathbb{N}}$ is bounded. □

Lemma 3.4. *Suppose that the Assumption 3.1* (i) *holds. Let the sequence $\{x^k\}_{k\in\mathbb{N}}$ be generated by Algorithm* 1 (I). *Then*

$$\lim_{k\to\infty}\|w^k-x^k\|=0, \tag{3.22}$$

and

$$\lim_{k\to\infty}\|Aw^k - P_{Q_k}(Aw^k)\| = 0. \tag{3.23}$$

Proof. Take any point x^* in Γ. From (3.17) and the existence of $\lim_{k\to\infty}\|x^{2k}-x^*\|$, we obtain

$$\lim_{k\to\infty}\rho_{2k}^2\|d(x^{2k},y^{2k})\|^2 = 0, \tag{3.24}$$

and

$$\lim_{k\to\infty}\rho_{2k+1}^2\|d(w^{2k+1},y^{2k+1})\|^2 = 0. \tag{3.25}$$

Combining (3.15) and (3.24), we have

$$\lim_{k\to\infty}\|x_{\mathrm{I}}^{2k+1}-x^{2k}\| = 0. \tag{3.26}$$

From the definition of w^k, we have $\|w^{2k+1}-x_{\mathrm{I}}^{2k+1}\| = \alpha_{2k+1}\|x_{\mathrm{I}}^{2k+1}-x^{2k}\|$, which together with (3.26) yields

$$\lim_{k\to\infty}\|w^{2k+1}-x_{\mathrm{I}}^{2k+1}\| = 0. \tag{3.27}$$

Using $w^{2k}=x^{2k}$ and (3.27), we obtain (3.22). On the other hand, from (3.24), (3.25) and the definition of w^k, we have

$$\lim_{k\to\infty}\rho_k^2\|d(w^k,y^k)\|^2 = 0. \tag{3.28}$$

From the definition of ρ_k, (3.7) and Lemma 3.1, we have

$$\begin{aligned}
\rho_k^2\|d(w^k,y^k)\|^2 &= \rho_k\left[\langle w^k-y^k, d(w^k,y^k)\rangle + \lambda_k(\|(I-P_{C_k})y^k\|^2 + \|(I-P_{Q_k})A(y^k)\|^2)\right]\\
&\geq \rho_k\left[(1-\delta)\|w^k-y^k\|^2 + \lambda_k(\|(I-P_{C_k})y^k\|^2 + \|(I-P_{Q_k})A(y^k)\|^2)\right]\\
&\geq \frac{1-\delta}{1+\delta^2}\left[(1-\delta)\|w^k-y^k\|^2 + \lambda_k(\|y^k-P_{C_k}(y^k)\|^2 + \|Ay^k-P_{Q_k}(Ay^k)\|^2)\right],
\end{aligned}$$

which together with (3.28) and Remark 3.2 yields

$$\lim_{k\to\infty}\|w^k-y^k\| = 0, \tag{3.29}$$

$$\lim_{k\to\infty}\|y^k-P_{C_k}(y^k)\| = 0, \tag{3.30}$$

and

$$\lim_{k\to\infty}\|Ay^k-P_{Q_k}(Ay^k)\| = 0. \tag{3.31}$$

In view of the nonexpansive property of P_{Q_k}, we obtain

$$\begin{aligned}
\|Aw^k-P_{Q_k}(Aw^k)\| &\leq \|Aw^k-Ay^k\| + \|Ay^k-P_{Q_k}(Ay^k)\| + \|P_{Q_k}(Aw^k)-P_{Q_k}(Ay^k)\|\\
&\leq 2\|Aw^k-Ay^k\| + \|Ay^k-P_{Q_k}(Ay^k)\|.
\end{aligned}$$

From (3.29) and (3.31), we obtain (3.23) immediately. □

Theorem 3.1. *Suppose that the Assumption* 3.1 (i) *holds. Then the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *generated by Algorithm* 1 (I) *converges to a point in* Γ.

Proof. Since $\{x^k\}_{k\in\mathbb{N}}$ is bounded, there is a cluster point $\hat{x}$ of $\{x^k\}_{k\in\mathbb{N}}$ with a subsequence $\{x^{k_l}\}_{l\in\mathbb{N}}$ converging to $\hat{x}$. From (3.22), it follows that $\{w^{k_l}\}_{l\in\mathbb{N}}$ also converges to $\hat{x}$.

Next, we show that $\hat{x}\in\Gamma$. In fact, since $P_{C_{k_l}}y^{k_l}\in C_{k_l}$, by the definition of C_{k_l}, we have

$$c(w^{k_l})+\langle \xi^{k_l}, P_{C_{k_l}}y^{k_l}-w^{k_l}\rangle \le 0,$$

where $\xi^{k_l}\in\partial c(w^{k_l})$. By the assumption that ξ^{k_l} is bounded, (3.29), and (3.30), we have

$$c(w^{k_l})\le -\langle \xi^{k_l}, P_{C_{k_l}}y^{k_l}-w^{k_l}\rangle \le \|\xi^{k_l}\|\|w^{k_l}-P_{C_{k_l}}y^{k_l}\|\to 0, \quad l\to\infty,$$

which implies $c(\hat{x})\le 0$, i.e., $\hat{x}\in C$. Since $P_{Q_{k_l}}(Aw^{k_l})\in Q_{k_l}$, we have

$$q(Aw^{k_l})+\langle \eta^{k_l}, P_{Q_{k_l}}(Aw^{k_l})-Aw^{k_l}\rangle \le 0,$$

where $\eta^{k_l}\in\partial q(Aw^{k_l})$. From the boundedness of $\{\eta^{k_l}\}$ and (3.23), one has

$$q(Aw^{k_l})\le \|\eta^{k_l}\|\|Aw^{k_l}-P_{Q_{k_l}}(Aw^{k_l})\|\to 0, \quad l\to\infty,$$

which implies $q(A\hat{x})\le 0$, i.e., $A\hat{x}\in Q$. Therefore, $\hat{x}\in\Gamma$.

Next, we show $\lim_{k\to\infty}\|x^k-\hat{x}\|=0$ via the following two possible cases.

Case 1 Assume that $\{x^{k_l}\}_{l\in\mathbb{N}}$ contains an even subsequence of $\{x^k\}_{k\in\mathbb{N}}$. By replacing x^* with $\hat{x}$ in (3.20), we obtain that $\{\|x^{2k}-\hat{x}\|\}_{k\in\mathbb{N}}$ converges and therefore its limit is zero. From (3.21), it follows $\lim_{k\to\infty}\|x^k-\hat{x}\|=0$.

Case 2 Assume that $\{x^{k_l}\}_{l\in\mathbb{N}}$ contains an odd subsequence of $\{x^k\}_{k\in\mathbb{N}}$. From (3.26), it concludes that $\{x^k\}_{k\in\mathbb{N}}$ has an even subsequence, which converges to $\hat{x}$. By using Case 1, we have $\lim_{k\to\infty}\|x^k-\hat{x}\|=0$. □

Theorem 3.2. *Suppose that the Assumption* 3.1 (ii) *holds, Then the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *generated by Algorithm* 1 (II) *converges to a point in* Γ.

Proof. Take k as even and pick any point x^* in Γ. Following the line of the proof of Theorem 3.2 in [3], we have

$$\|x_{\mathrm{II}}^{k+2}-x^*\|^2\le\|w^{k+1}-x^*\|^2-\gamma(2-\gamma)\rho_{k+1}^2\|d(w^{k+1},y^{k+1})\|^2-\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2. \tag{3.32}$$

Similar to (3.12), we have

$$\begin{aligned}\|w^{k+1}-x^*\|^2 =&(1+\alpha_{k+1})\|x_{\mathrm{II}}^{k+1}-x^*\|^2-\alpha_{k+1}\|x^k-x^*\|^2\\ &+\alpha_{k+1}(1+\alpha_{k+1})\|x_{\mathrm{II}}^{k+1}-x^k\|^2.\end{aligned} \tag{3.33}$$

Using similar arguments in obtaining (3.32), we obtain

$$\|x_{\mathrm{II}}^{k+1}-x^*\|^2\le\|x^k-x^*\|^2-\gamma(2-\gamma)\rho_k^2\|d(x^k,y^k)\|^2-\|x_{\mathrm{I}}^{k+1}-x_{\mathrm{II}}^{k+1}\|^2. \tag{3.34}$$

Combining (3.33) and (3.34), we have

$$\begin{aligned}\|w^{k+1}-x^*\|^2 =&\|x^k-x^*\|^2-(1+\alpha_{k+1})\gamma(2-\gamma)\rho_k^2\|d(x^k,y^k)\|^2\\ &-(1+\alpha_{k+1})\|x_{\mathrm{I}}^{k+1}-x_{\mathrm{II}}^{k+1}\|^2+\alpha_{k+1}(1+\alpha_{k+1})\|x_{\mathrm{II}}^{k+1}-x^k\|^2.\end{aligned} \tag{3.35}$$

Substituting (3.35) into (3.32), we arrive at

$$\begin{aligned}\|x_{\mathrm{II}}^{k+2}-x^*\|^2 \le & \|x^k-x^*\|^2-(1+\alpha_{k+1})\gamma(2-\gamma)\rho_k^2\|d(x^k,y^k)\|^2\\ & -\gamma(2-\gamma)\rho_{k+1}^2\|d(w^{k+1},y^{k+1})\|^2-(1+\alpha_{k+1})\|x_{\mathrm{I}}^{k+1}-x_{\mathrm{II}}^{k+1}\|^2\\ & -\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2+\alpha_{k+1}(1+\alpha_{k+1})\|x_{\mathrm{II}}^{k+1}-x^k\|^2.\end{aligned} \tag{3.36}$$

Using Lemma 3.1 and (3.36), we obtain

$$\begin{aligned}\|x_{\mathrm{II}}^{k+2}-x^*\|^2 \le & \|x^k-x^*\|^2-(1+\alpha_{k+1})\gamma(2-\gamma)\frac{(1-\delta)^4}{(1+\delta^2)^2}\|x^k-y^k\|^2\\ & -\gamma(2-\gamma)\frac{(1-\delta)^4}{(1+\delta^2)^2}\|w^{k+1}-y^{k+1}\|^2\\ & -(1+\alpha_{k+1})\|x_{\mathrm{I}}^{k+1}-x_{\mathrm{II}}^{k+1}\|^2-\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2\\ & +\alpha_{k+1}(1+\alpha_{k+1})\|x_{\mathrm{II}}^{k+1}-x^k\|^2, \quad \forall k\ge K_0.\end{aligned} \tag{3.37}$$

By Assumption 3.1 (ii), we obtain from (3.37) that

$$\|x_{\mathrm{II}}^{k+2}-x^*\| \le \|x^k-x^*\|.$$

Following the lines of the proof of Lemma 3.4 and Theorem 3.1, we can show that $\{x^k\}_{k\in\mathbb{N}}$ converges to a solution of the SFP (1.1). This completes the proof. □

4. Numerical Experiments

In this section, we present a numerical example to compare Algorithm 1 with the Algorithm 3.1 in [3]. All codes were written in MATLAB R2016a and performed on a PC Desktop Intel(R) Pentium(R) CPU N3540 @ 2.16GHz 2.16 GHz, RAM 4.00 GB.

For convenience, we denote the vector with all elements 0 by e_0 and the vector with all elements 1 by e_1 in what follows. In the numerical results listed in the following table, 'Iter.' and 'CPU time' denote the number of iterations and the cpu time in seconds, respectively.

Example 4.1. [23] Consider the SFP, where $A=(a_{ij})_{M\times N}\in\mathbb{R}^{M\times N}$ and $a_{ij}\in(0,1)$ generated randomly and

$$C=\{x\in\mathbb{R}^N|c(x)\le 0\} \quad \text{where} \quad c(x)=-x_1+x_2^2+\cdots+x_N^2,$$

and

$$Q=\{y\in\mathbb{R}^M|q(y)\le 0\} \quad \text{where} \quad q(x)=y_1+20y_2^2+\cdots+20y_M^2-1.$$

Note that C is the set above the function $x_1=x_2^2+\cdots+x_N^2$ and Q is the set below the function $y_1=-20y_2^2-\cdots-20y_M^2+1$. The initial point $x^0\in(0,100e_1)$ is randomly chosen. In the numerical experiment, we took the objective function value $p(x^k)=\frac{1}{2}\|x^k-P_{C_k}(x^k)\|^2+\frac{1}{2}\|Ax^k-P_{Q_k}(Ax^k)\|^2<\varepsilon$ as the stopping criterion and $\varepsilon=10^{-3}$.

We took $\mu=0.7$, $\gamma=0.8$, $\alpha_k=1.45$, and $\lambda_1=0.01$ in Algorithm 1(I), $\mu=0.05$, $\gamma=1.8$, $\alpha_k=-0.15$, and $\lambda_1=0.01$ in Algorithm 1(II), $\theta=0.85$, $\gamma=1.9$, $\rho=0.05$, and $\sigma=0.02$ in Algorithm 3.1(I) of [3] and $\theta=0.9$, $\gamma=1.9$, $\rho=0.1$, and $\sigma=0.02$ in Algorithm 3.1(II) of [3]. Note that the set X of (3.2) and (3.5) in the algorithm can be selected as the dynamic set X_k by Remark 3.1 (ii). Here we select X_k as a projection onto C_k. Although we need to calculate the projection here, its performance is better than other selection results. The statistical data are

TABLE 1. Computational results of four algorithms for Example 4.1.

Problem size		Iter				CPU time			
m	n	Alg 1(I)	Alg 1(II)	Alg 3.1(I) in [3]	Alg 3.1(II) in [3]	Alg 1(I)	Alg 1(II)	Alg 3.1(I) in [3]	Alg 3.1(II) in [3]
100	100	320	334	398	699	0.4334	0.4954	0.5971	1.0513
150	100	276	308	363	540	0.4953	0.5403	0.6297	0.9521
200	250	742	788	1108	2284	2.0677	2.3518	3.1463	6.4041
300	300	700	957	1684	2306	2.5858	3.9326	6.4978	8.8949

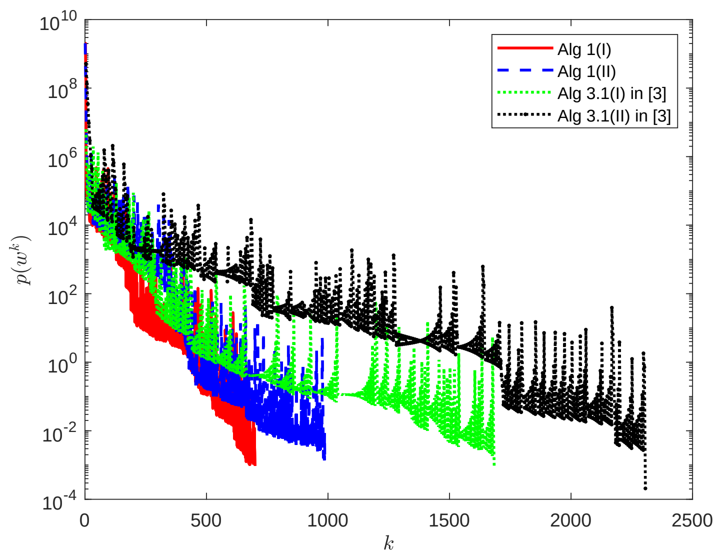

FIGURE 1. Computational results of four algorithms for Example 4.1 with $m = 300, n = 300$.

obtained by averaging the number of iterations and CPU costs from 30 independent trials. The corresponding results reported in Table 1 and Figure 1 illustrate that Algorithm 1(I) performs better than Algorithm 1(II) and they are both better than Algorithm 3.1 of [3] from the iteration numbers and CPU time.

5. CONCLUSION

In this paper, we proposed a new iterative algorithm for solving the split feasibility problem. For our algorithm, there are two main highlights. One is the use of the alternated inertial method, and the other is the stepsize selection does not need any prior information about the operator norm. The weak convergence of the algorithm was established under suitable conditions. Numerical examples illustrated the advantages of the proposed algorithm.

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