J. Nonlinear Var. Anal. 5 (2021), No. 6, pp. 951-964
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.5.2021.6.08

CONVERGENCE RATE OF A GRADIENT PROJECTION METHOD FOR SOLVING VARIATIONAL INEQUALITIES

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Abstract. Under the error bound assumption, we establish the linear convergence rate of a gradient projection method for solving co-coercive variational inequalities. Using this result, we unify and improve several results in variational inequalities, fixed point problems, and convex feasible problems. Numerical experiments are conducted to illustrate the theoretical results.

Keywords. Convex feasible problem; Co-coercivity; Gradient projection method; Error bound; Convergence rate; Variational inequality.

1. Introduction

Variational inequality (VI, for brevity) provides a broad unifying setting for the study of many problems in the field of optimization and equilibrium theory. It also serves as the main computational framework for the practical solutions of a lot of problems in the mathematical sciences. Many solution methods have been proposed for different classes of VIs. Among them, gradient projection methods are simple in form and useful in practice. When the projection is easily computable, the cost of gradient projection methods is cheap and so it can be applied to solution of very large problems.

For strongly monotone (pseudomonotone) VIs, gradient projection methods produce iterative sequences strongly converging to the unique solution of the given VIs with the linear convergence rate (see [1] and thereferences therein). However, when the given VIs are merely monotone, it may generate disconvergent iterative sequences [2, Example 12.1.3]. The class of co-coercive (or inverse strongly monotone) VIs lies between monotone and strongly monotone ones. For co-coercive VIs, the iterative sequences generated by gradient projection methods weakly converge to some solution of the given VIs provided that they are solvable [3, Theorem 2.3]. Based on Hundal's alternating projections counter-example [4], Bauschke et al. [5, Remark 3.8] constructed an example where the iterative sequences converge weakly but not

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Received April 9, 2021; Accepted June 4, 2021.

in norm. Under the weak sharpness assumption, Matsushita et al. [6, Theorem 4.1] showed the finite termination of the gradient projection method for co-coercive VIs. For recent developments on projection methods for solving variational inequalities, the readers are referred to [7, 8, 9, 10, 11, 12].

The purpose of this paper is to establish the convergence rate of a gradient projection method for solving co-coercive VIs. For this purpose, we consider co-coercive VIs satisfying the error bound assumption which is called projection-type error bound. The reader is referred to [2, Chapter 6] for an introduction to the theory of error bounds for VIs, and to [2, Section 12.6] for the application of theory error bounds in rate of convergence analysis. We prove the linear convergence of the iterative sequences and estimate the convergence rate in terms of the co-coercive modulus and the error bound constant. Our proof is based on the general scheme of Tseng in [13] and a sufficient condition [14, Theorem 5.12] for the linear and strong convergence of a sequence in a real Hilbert space. The obtained result allows us to unify and improve several results in variational inequalities, fixed point problems and convex feasible problems.

The paper will proceed as follows. Section 2 introduces the main definitions and preliminary results. Section 3 presents the gradient projection method and some properties of iterative sequences generated by this method. In Section 4, we firstly recall the concept of projection-type error bound and record some examples of VIs satisfying this error bound assumption in infinite dimensional Hilbert spaces. Secondly, we establish the convergence rate of gradient projection method. Finally, this result is then specialized to fixed point problems and convex feasible ones. In the last section, Section 5, we present some numerical experiments to illustrate the obtained theoretical results.

2. Preliminaries

Let H be real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$, and let K be a nonempty closed convex subset of H. For each $u \in H$, there exists a unique point in K (see [15, p. 8]), denoted by $P_K(u)$, such that

$$\|u - P_K(u)\| \leq \|u - v\| \quad \forall v \in K.$$

The distance from the point u to the set K, denoted by $d(u,K)$, is defined by

$$d(u,K) := \|u - P_K(u)\|.$$

Some known properties of the metric projection $P_K : H \to K$ are recalled in the following theorem.

Theorem 2.1. *For every $u, v \in H$, we have*

(a): $\|P_K(u) - P_K(v)\|^2 \leq \langle P_K(u) - P_K(v), u - v \rangle$;

(b): $\|P_K(u) - P_K(v)\|^2 \leq \|u - v\|^2 - \|(u - P_K(u)) - (v - P_K(v))\|^2$.

Let $F : K \to H$ be a mapping. The variational inequality problem VI(K,F) defined by K and F is that of finding a point $u^* \in K$ such that

$$\langle F(u^*), u - u^* \rangle \geq 0 \quad \forall u \in K. \tag{2.1}$$

The solution set of VI(K,F) is abbreviated to Sol(K,F).

Remark 2.1. If $u^* \in \text{Sol}(K,F)$ then $u^* = P_K(u^* - \lambda F(u^*))$ for all $\lambda > 0$.

One often considers (2.1) with F possessing a certain monotonicity property.

Definition 2.1. (See [16, 17]) The mapping $F : K \to H$ is said to be

(a): *strongly monotone* if there exists $\gamma > 0$ such that

$$\langle F(u) - F(v), u - v \rangle \geq \gamma \|u - v\|^2 \quad \forall u, v \in K;$$

(b): *strongly pseudomonotone* if there exists $\gamma > 0$ such that

$$\langle F(u), v - u \rangle \geq 0 \ \Rightarrow \ \langle F(v), v - u \rangle \geq \gamma \|u - v\|^2$$

for every $u, v \in K$;

(c): *co-coercive* if there exists $\mu > 0$ such that

$$\langle F(u) - F(v), u - v \rangle \geq \mu \|F(u) - F(v)\|^2 \quad \forall u, v \in K;$$

(d): *monotone* if

$$\langle F(u) - F(v), u - v \rangle \geq 0 \quad \forall u, v \in K.$$

Remark 2.2.

- Strong monotonicity (with modulus γ) implies strong pseudomonotonicity (with modulus γ) and monotonicity;
- Co-coercivity (with modulus μ) implies monotonicity and Lipschitz continuity (with Lipschitz constant $L = 1/\mu$);
- Strong monotonicity (with modulus γ) and Lipschitz continuity (with Lipschitz constant L) imply co-coercivity (with modulus $\mu = \gamma/L^2$)
- Strong pseudomonotonicity does not imply monotonicity [18, Example 3.1].

We recall the following result, which is called Minty lemma [15, Lemma 1.5 on p. 85].

Proposition 2.1. *Consider the problem* VI(K, F) *with K being a nonempty, closed, convex subset of a real Hilbert space H and $F : K \to H$ being monotone and continuous. Then,* Sol(K, F) *is closed convex and u^* is a solution of* VI(K, F) *if and only if*

$$\langle F(u), u - u^* \rangle \geq 0, \quad \forall u \in K.$$

Sufficient conditions for linear and strong convergence of iterative sequences in a real Hilbert space are given in the following proposition.

Proposition 2.2. (See [14, Theorem 5.12]) *Let C be a nonempty, closed, convex subset of a real Hilbert space H. Let $\{u^k\} \subset H$ be such that $\{\|u^k - u\|\}$ is monotonically decreasing for all $u \in C$. Suppose that there exists $q \in [0, 1[$ such that*

$$d(u^{k+1}, C) \leq q\, d(u^k, C) \quad \forall k \in \mathbb{N}.$$

Then, $\{u^k\}$ converges linearly to a point $u^* \in C$. Moreover,

$$\|u^k - u^*\| \leq 2q^k\, d(u^0, C) \quad \forall k \in \mathbb{N}.$$

3. The Gradient Projection Method

We now consider the problem VI(K,F) with K being nonempty, closed, convex and F being co-coercive on K with the modulus $\mu > 0$.

Algorithm 3.1

Data: $u^0 \in K$ and $\lambda \in]0, 2\mu[$.

Step 0: Set $k = 0$.

Step 1: If $u^k = P_K(u^k - \lambda F(u^k))$ then stop.

Step 2: Set $u^{k+1} = P_K(u^k - \lambda F(u^k))$ and replace k by $k+1$; go to **Step 1**.

If the computation terminates at a step k, then one puts $u^{k'} = u^k$ for all $k' \geq k+1$. Thus, for a given stepsize $\lambda \in]0, 2\mu[$, Algorithm 3.1 produces for each initial point $u^0 \in K$ a unique iterative sequence $\{u^k\}$.

We establish some properties of the iterative sequence $\{u^k\}$.

Proposition 3.1. *If u^* is a solution of* VI(K,F) *then, for every $k \in \mathbb{N}$, we have*

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - 2\lambda\mu \left\| F(u^k) - F(u^*) - \frac{u^k - u^{k+1}}{2\mu} \right\|^2 - \left(\frac{2\mu - \lambda}{2\mu} \right) \|u^k - u^{k+1}\|^2 \tag{3.1}$$

Proof. By Theorem 2.1(b), we have

$$\|P_K(u) - P_K(v)\|^2 \leq \|u - v\|^2 - \|(u - P_K(u)) - (v - P_K(v))\|^2 \quad \forall u, v \in H. \tag{3.2}$$

Substituting $u = u^k - \lambda F(u^k)$ and $v = u^* - \lambda F(u^*)$ into (3.2) and using the relation

$$u^{k+1} = P_K(u^k - \lambda F(u^k)), \quad u^* = P_K(u^* - \lambda F(u^*))$$

we obtain

$$\begin{aligned}
&\|u^{k+1} - u^*\|^2 \\
&= \|P_K(u^k - \lambda F(u^k)) - P_K(u^* - \lambda F(u^*))\|^2 \\
&\leq \|(u^k - \lambda F(u^k)) - (u^* - \lambda F(u^*))\|^2 - \|(u^k - \lambda F(u^k) - u^{k+1}) + \lambda F(u^*)\|^2 \\
&= \|(u^k - u^*) - \lambda(F(u^k) - F(u^*))\|^2 - \|(u^k - u^{k+1}) - \lambda(F(u^k) - F(u^*))\|^2 \\
&= \|u^k - u^*\|^2 - 2\lambda\langle F(u^k) - F(u^*), u^k - u^*\rangle - \|u^k - u^{k+1}\|^2 \\
&\quad + 2\lambda\langle F(u^k) - F(u^*), u^k - u^{k+1}\rangle.
\end{aligned}$$

By the co-coercivity of F, we have

$$\langle F(u^k) - F(u^*), u^k - u^*\rangle \geq \mu\|F(u^k) - F(u^*)\|^2.$$

Therefore,

$$\begin{aligned}
&\|u^{k+1}-u^*\|^2\\
&\leq \|u^k-u^*\|^2-2\lambda\mu\|F(u^k)-F(u^*)\|^2-\|u^k-u^{k+1}\|^2\\
&\quad +2\lambda\langle F(u^k)-F(u^*),u^k-u^{k+1}\rangle\\
&=\|u^k-u^*\|^2-2\lambda\mu\left\|F(u^k)-F(u^*)-\frac{u^k-u^{k+1}}{2\mu}\right\|^2-\left(\frac{2\mu-\lambda}{2\mu}\right)\|u^k-u^{k+1}\|^2.
\end{aligned}$$

This completes the proof. □

The following result is a direct corollary of Proposition 3.1.

Corollary 3.1. *If* VI(K,F) *has a solution* u^* *and* $\lambda \in (0,2\mu)$, *then* $\{\|u^{k+1}-u^k\|\}$ *converges to* 0 *and* $\{F(u^k)\}$ *converges in norm to* $F(u^*)$.

Proof. It follows from (3.1) and $\lambda \in (0,2\mu)$ that the sequence $\{\|u^k-u^*\|\}$ is monotonically decreasing and bounded from below by 0. Hence, it is convergent. From (3.1), we have

$$\left(\frac{2\mu-\lambda}{2\mu}\right)\|u^k-u^{k+1}\|^2 \leq \|u^k-u^*\|^2-\|u^{k+1}-u^*\|^2,$$

and

$$2\lambda\mu\left\|F(u^k)-F(u^*)-\frac{u^k-u^{k+1}}{2\mu}\right\|^2 \leq \|u^k-u^*\|^2-\|u^{k+1}-u^*\|^2.$$

Hence, $\{\|u^{k+1}-u^k\|\}$ converges to 0 and $\{F(u^k)\}$ converges to $F(u^*)$ in norm. □

4. Convergence Rate of the Gradient Projection Method

In order to obtain the linear convergence of the iterative sequences generated by Algorithm 3.1, apart from the required co-coercivity of F, we also need an error bound assumption [2, Chapter 6], which is a well-known projection-type error bound.

Definition 4.1. The problem VI(K,F) is said to satisfy the error bound assumption if it has a solution and there exist real numbers $\eta > 0$ such that

$$d(u,\mathrm{Sol}(K,F)) \leq \eta\|u-P_K(u-F(u))\|, \quad \forall u \in K. \tag{4.1}$$

We will record some examples of variational inequalities satisfying the error bound assumption in infinite dimensional Hilbert spaces.

Example 4.1. *Variational inequality.* Consider the problem VI(K,F) with F being Lipschitz continuous on K with constant $L > 0$.

- If F is strongly monotone with modulus $\gamma > 0$ then, from [19, Theorem 3.1], VI(K,F) satisfies the error bound assumption with $\eta = (L+1)/\gamma$;
- If F is strongly pseudomonotone with modulus $\gamma > 0$, then, from [20, Theorem 4.2], VI(K,F) satisfies the error bound assumption with $\eta = (L+1)/\gamma + 1$.

Example 4.2. *Fixed point problem.* Let $T : H \to H$ be a linearly regular mapping, i.e., Fix $T \neq \emptyset$ and there exists $\gamma > 0$ such that

$$d(u,\mathrm{Fix}\ T) \leq \gamma\|u-T(u)\|, \quad \forall u \in H, \tag{4.2}$$

The interested readers are refered to [21] for information on linearly regular mappings.

where Fix T is the set of fixed points of T. Consider VI(K,F) with $K=H$ and $F=I-T$, where $I:H\to H$ is the identity mapping. Then Sol$(K,F)=\text{Fix }T\neq\emptyset$ and VI(K,F) satisfies the error bound assumption with constant $\eta=\gamma$. Indeed, clearly, Fix $T\subset$ Sol(K,F). Let $u^*\in$ Sol(K,F). Then

$$\langle u^*-T(u^*),u-u^*\rangle\geq 0,\quad \forall u\in H.$$

Substituting $u=T(u^*)$ into the above inequality, we obtain $-\|u^*-T(u^*)\|^2=0$ or $u^*=T(u^*)$. Therefore, Sol$(K,F)\subset$ Fix T, and so Sol$(K,F)=$ Fix T. Moreover, it follows from (4.2) that

$$\begin{aligned} d(u,\text{Sol}(K,F)) &= d(u,\text{Fix }T)\\ &\leq \gamma\|u-T(u)\|\\ &= \gamma\|u-(u-F(u))\|\\ &= \gamma\|u-P_K(u-F(u))\|. \end{aligned}$$

Hence, VI(K,F) satisfies the error bound assumption with constant $\eta=\gamma$.

Example 4.3. *Convex feasible problem.* Let $\alpha_1,\alpha_2\in\mathbb{R}$ and $c_1,c_2\in H$ be such that

$$\|c_1\|=\|c_2\|=1,\quad \langle c_1,c_2\rangle\geq 0.$$

Let C_1,C_2 be two closed half-spaces in H given by

$$C_1:=\{u\in H:\langle c_1,u\rangle\leq\alpha_1\},\quad C_2:=\{u\in H:\langle c_2,u\rangle\leq\alpha_2\}.$$

Observe that $C_1\cap C_2\neq\emptyset$ since

$$\bar{u}:=\min\{\alpha_1,\alpha_2\}\frac{c_1+c_2}{1+\langle c_1,c_2\rangle}\in C_1\cap C_2.$$

Consider the problem VI(K,F) with $K=C_1$ and $F=I-P_{C_2}$, where $I:H\to H$ is the identity mapping. Observe that Sol$(K,F)=C_1\cap C_2\neq\emptyset$. Indeed, it is easy to verify that $C_1\cap C_2\subset$ Sol(K,F). Let $u^*\in$ Sol(K,F). Then $u^*\in K=C_1$ and

$$\langle u^*-P_{C_2}(u^*),u-u^*\rangle\geq 0,\quad \forall u\in K=C_1.$$

Moreover, by the variational characterization of the metric projection [15, Theorem 2.3 on p.9],

$$\langle u^*-P_{C_2}(u^*),P_{C_2}(u^*)-u\rangle\geq 0\quad \forall u\in C_2.$$

Substituting $u=\bar{u}\in C_1\cap C_2$ into two above inequalities and adding them, we have $-\|u^*-P_{C_2}(u^*)\|^2\geq 0$ or $u^*=P_{C_2}(u^*)\in C_2$. Therefore, Sol$(K,F)\subset C_1\cap C_2$.

Next, we show that

$$\|u-P_{C_1\cap C_2}(u)\|\leq\|u-P_{C_1}P_{C_2}(u)\|\quad \forall u\in H. \tag{4.3}$$

Indeed, let u be an any point in H. We consider three cases:

Case 1. $u\in C_1\cap C_2$

We have $u=P_{C_1\cap C_2}(u)$ and so (4.3) is satisfied.

Case 2. $u\notin C_2$

We have $P_{C_2}(u)=u-\langle c_2,u\rangle c_2+\alpha_2c_2$. If $P_{C_2}(u)\in C_1$, then $P_{C_1}P_{C_2}(u)=P_{C_2}(u)\in C_1\cap C_2$. So, (4.3) is satisfied. If $P_{C_2}(u)\notin C_1$, then $\langle c_1,u-\langle c_2,u\rangle c_2+\alpha_2c_2\rangle>\alpha_1$ and

$$P_{C_1}P_{C_2}(u)=u-\langle c_2,u\rangle c_2+\alpha_2c_2-\langle c_1,u-\langle c_2,u\rangle c_2+\alpha_2c_2\rangle c_1+\alpha_1c_1\in C_1.$$

It follows from $\langle c_1, c_2\rangle \geq 0$ that

$$\begin{aligned}\langle c_2, P_{C_1}P_{C_2}(u)\rangle &= \langle c_2, u\rangle - \langle c_2, u\rangle\|c_2\|^2 + \alpha_2\|c_2\|^2 \\ &\quad - \langle c_1, u - \langle c_2, u\rangle c_2 + \alpha_2 c_2\rangle\langle c_1, c_2\rangle + \alpha_1\langle c_1, c_2\rangle \\ &\leq \alpha_2 - \alpha_1\langle c_1, c_2\rangle + \alpha_1\langle c_1, c_2\rangle \\ &= \alpha_2.\end{aligned}$$

Hence $P_{C_1}P_{C_2}(u) \in C_1 \cap C_2$. Then (4.3) is satisfied.

Case 3. $u \notin C_1, u \in C_2$

We have $\langle c_1, u\rangle > \alpha_1, \langle c_2, u\rangle \leq \alpha_2$ and $P_{C_1}P_{C_2}(u) = u - \langle c_1, u\rangle c_1 + \alpha_1 c_1 \in C_1$. Then

$$\langle c_2, P_{C_1}P_{C_2}(u)\rangle = \langle c_2, u\rangle - \langle c_1, u\rangle\langle c_1, c_2\rangle + \alpha_1\langle c_1, c_2\rangle \leq \alpha_2$$

It follows that $P_{C_1}P_{C_2}(u) \in C_1 \cap C_2$ and so (4.3) is satisfied. Since

$$d(u, \mathrm{Sol}(K,F)) = \|u - P_{C_1\cap C_2}(u)\|, \quad \|u - P_K(u - F(u))\| = \|u - P_{C_1}P_{C_2}(u)\|$$

for each $u \in H$, (4.3) implies that VI(K,F) satisfies the error bound assumption with $\eta = 1$.

Under error bound assumption (4.1), we establish the linear convergence and estimate the rate of convergence of the iterative sequences generated by Algorithm 3.1. The next proposition [22, Lemma 1] will be used to obtain those results.

Proposition 4.1. *Let $F : K \to H$ be a mapping defined on a nonempty closed convex subset $K \subset H$. For each $u \in K$ and $\alpha > 0$, we have*

$$\min\{1, \alpha\}\|u - P_K(u - F(u))\| \leq \|u - P_K(u - \alpha F(u))\|. \tag{4.4}$$

Theorem 4.1. *Let K be a nonempty, closed, convex subset of a real Hilbert space H and $F : K \to H$ a co-coercive mapping with the modulus μ. Suppose that* VI(K,F) *satisfies the error bound assumption* (4.1) *with constant $\eta > 0$. Let $\lambda \in]0, 2\mu[$ and $\{u^k\}$ be the sequence generated by Algorithm 3.1. Then, $\{u^k\}$ converges linearly to some point $u^* \in \mathrm{Sol}(K,F)$. Moreover,*

$$\|u^k - u^*\| \leq 2q^{k/2}\, d(u^0, \mathrm{Sol}(K,F)), \quad \forall k \in \mathbb{N}, \tag{4.5}$$

where

$$q := \max\left\{0, 1 - \left(\frac{2\mu - \lambda}{2\mu}\right)\left(\frac{\min\{1,\lambda\}}{\eta}\right)^2\right\} \in\,]0,1[. \tag{4.6}$$

Proof. Since the co-coercivity implies the monotonicity, we obtain from Proposition 2.1 that $\mathrm{Sol}(K,F)$ is closed and convex. Let $C = \mathrm{Sol}(K,F)$ and $\varphi(u) = [d(u,C)]^2$ for all $u \in H$. Using (3.1) in Proposition 3.1, we have that, for each $u^* \in C$, $\{\|u^k - u^*\|\}$ is monotonically decreasing and

$$\begin{aligned}\varphi(u^{k+1}) &\leq \|u^{k+1} - u^*\|^2 \\ &\leq \|u^k - u^*\|^2 - \left(\frac{2\mu - \lambda}{2\mu}\right)\|u^k - u^{k+1}\|^2.\end{aligned}$$

It follows that

$$\varphi(u^{k+1}) \leq \varphi(u^k) - \left(\frac{2\mu - \lambda}{2\mu}\right)\|u^k - u^{k+1}\|^2. \tag{4.7}$$

Substituting $u = u^k$ and $\alpha = \lambda$ into (4.4) in Proposition 4.1, we obtain

$$\begin{aligned} \|u^k - P_K(u^k - F(u^k))\| &\leq \frac{1}{\min\{1,\lambda\}} \|u^k - P_K(u^k - \lambda F(u^k))\| \\ &= \frac{1}{\min\{1,\lambda\}} \|u^k - u^{k+1}\|. \end{aligned}$$

Combining this and the error bound assumption (4.1), we obtain

$$\begin{aligned} \varphi(u^k) = [d(u^k, C)]^2 &\leq \eta^2 \|u^k - P_K(u^k - F(u^k))\|^2 \\ &\leq \left(\frac{\eta}{\min\{1,\lambda\}}\right)^2 \|u^k - u^{k+1}\|^2. \end{aligned}$$

This inequality and (4.7) imply

$$\begin{aligned} \varphi(u^{k+1}) &\leq \varphi(u^k) - \left(\frac{2\mu - \lambda}{2\mu}\right)\left(\frac{\min\{1,\lambda\}}{\eta}\right)^2 \varphi(u^k) \\ &= \left(1 - \left(\frac{2\mu - \lambda}{2\mu}\right)\left(\frac{\min\{1,\lambda\}}{\eta}\right)^2\right) \varphi(u^k) \\ &\leq q\, \varphi(u^k). \end{aligned}$$

Hence,

$$d(u^{k+1}, C) \leq q^{1/2}\, d(u^k, C), \quad \forall k \in \mathbb{N}.$$

Applying Proposition 2.2, we deduce that $\{u^k\}$ converges linearly to a solution u^* of VI(K,F), and

$$\|u^k - u^*\| \leq 2q^{k/2} d(u^0, \mathrm{Sol}(K,F)), \quad \forall k \in \mathbb{N}.$$

This completes the proof. □

Remark 4.1. The value q in (4.6) can be regarded as a function $q = q(\lambda)$ of the variable $\lambda \in]0, 2\mu[$. Consider the function $f(\lambda) = (2\mu - \lambda)(\min\{1,\lambda\})^2$ with $\lambda \in]0, 2\mu[$. By some calculations, we obtain

$$\max_{\lambda \in]0,2\mu[} f(\lambda) = \begin{cases} 2\mu - 1 & \text{if} \quad \mu \geq \frac{3}{4}, \\ \frac{32\mu^3}{27} & \text{if} \quad \mu < \frac{3}{4}. \end{cases}$$

Observe that, for $\lambda \in]0, 2\mu[$, we have

$$q(\lambda) = \max\left\{0, 1 - \frac{f(\lambda)}{2\mu\eta^2}\right\}.$$

It follows that

$$\min_{\lambda \in]0,2\mu[} q(\lambda) = q^* := \begin{cases} \max\left\{0, 1 - \frac{2\mu - 1}{2\mu\eta^2}\right\} & \text{if} \quad \mu \geq \frac{3}{4}, \\ \max\left\{0, 1 - \frac{16\mu^2}{27\eta^2}\right\} & \text{if} \quad \mu < \frac{3}{4}. \end{cases}$$

Therefore, the best value of q in (4.6) is q^*.

We firstly recover results on the linear convergence of gradient projection method for strongly monotone (pseudomonotone) variational inequalities from Theorem 4.1.

Corollary 4.1. *Let K be a nonempty closed convex set in real Hilbert space H and let F be a mapping from K into H that is strongly monotone on K with the modulus γ and Lipschitz continuous on K with the constant L. Let $\{u^k\}$ be a sequence given by*

$$\begin{cases} u^0 \in K, \\ u^{k+1} = P_K(u^k - \lambda F(u^k)), \quad \forall k \in \mathbb{N}, \end{cases} \tag{4.8}$$

where $\lambda \in \left]0, 2\gamma/L^2\right[$. Then $\{u^k\}$ converges linearly to the unique solution u^* of VI(K,F) *and satisfies* (4.5) *with*

$$q = 1 - \left(\frac{2\gamma/L^2 - \lambda}{2\gamma/L^2}\right)\left(\frac{\gamma \min\{1,\lambda\}}{L+1}\right)^2 \in]0,1[. \tag{4.9}$$

Proof. Under our assumptions, VI(K,F) has a unique solution (see, e.g. [23] for a proof). Moreover, F is co-coercive with the modulus $\mu = \gamma/L^2$ by Remark 2.2 and satisfies the error bound assumption with constant $\eta = (L+1)/\gamma$ by Example 4.1. Hence, $\{u^k\}$ defined by (4.8) is the sequence generated by Algorithm 3.1. Applying Theorem 4.1 and observing that $\gamma \leq L$, we deduce that $\{u^k\}$ converges linearly to the unique solution of VI(K,F) and satisfies (4.5) with constant q given in (4.9). □

Corollary 4.2. *Let K be a nonempty closed convex set in real Hilbert space H and let F be a mapping from K into H that is strongly pseudomonotone on K with the modulus γ and co-coercive on K with the constant μ. Let $\lambda \in]0, 2\mu[$ and $\{u^k\}$ be a sequence generated by Algorithm 3.1. Then $\{u^k\}$ converges linearly to the unique solution u^* of* VI(K,F) *and satisfies* (4.5) *with*

$$q = 1 - \left(\frac{2\mu - \lambda}{2\mu}\right)\left(\frac{\gamma \min\{1,\lambda\}}{L+\gamma+1}\right)^2 \in]0,1[. \tag{4.10}$$

Proof. Under our assumptions, VI(K,F) has a unique solution [20, Theorem 2.1]. Moreover, by Example 4.1, VI(K,F) satisfies the error bound assumption with constant $\eta = (L+1)/\gamma + 1$. Applying Theorem 4.1, we deduce that $\{u^k\}$ converges linearly to the unique solution of VI(K,F) and satisfies (4.5) with constant q given in (4.10). □

Remark 4.2. Since the strong monotonicity and the Lipschitz continuity imply the strong pseudomonotonicity and co-coercivity, Corollary 4.1 follows directly from Corollary 4.2. The next example constructs a class of strongly pseudomonotone and co-coercive variational inequalities but none of them is strongly monotone.

Example 4.4. Let H be any nontrivial real Hilbert space and $\alpha, \beta \in \mathbb{R}$ such that $\alpha > \beta > 0$. Let

$$K_\alpha = \{u \in H : \|u\| \leq \alpha\}, \quad F_\beta = \frac{\beta}{\max\{\|u\|, \beta\}} u.$$

Then VI(K_α, F_β) is strongly pseudomonotone and co-coercive. Indeed, suppose that $u, v \in K_\alpha$ such that $\langle F_\beta(u), v-u\rangle \geq 0$. Then $\langle u, v-u\rangle \geq 0$. Since $\|v\| \leq \alpha$, we have

$$\begin{aligned}\langle F_\beta(v), v-u\rangle &= \frac{\beta}{\max\{\|v\|,\beta\}}\langle v, v-u\rangle \\ &\geq \frac{\beta}{\max\{\alpha,\beta\}}(\langle v, v-u\rangle - \langle u, v-u\rangle) \\ &= \frac{\beta}{\alpha}\|v-u\|^2.\end{aligned}$$

Hence, F_β is strongly pseudomontone with constant $\gamma = \beta/\alpha$. Observe that $F_\beta = P_{K_\beta}$, where

$$K_\beta = \{u \in H : \|u\| \leq \beta\}.$$

It follows from Theorem 2.1(b) that F_β is co-coercive with modulus $\mu = 1$. Let $e \in H$ such that $\|e\| = 1$ and setting $u = \alpha e$ and $v = \beta e$. Then $u \neq v$ and

$$F_\beta(u) = \frac{\beta}{\max\{\|\alpha e\|,\beta\}}\alpha e = \frac{\beta}{\max\{\alpha,\beta\}}\alpha e = \beta e,$$

$$F_\beta(v) = \frac{\beta}{\max\{\|\beta e\|,\beta\}}\beta e = \frac{\beta}{\max\{\beta,\beta\}}\beta e = \beta e.$$

Hence, $\langle F_\beta(u) - F_\beta(v), u-v\rangle = 0$, and so VI($K_\alpha, F_\beta$) is not strongly monotone.

By using Theorem 4.1 and Example 4.2, we could establish the linear convergence of Mann iteration method for nonexpansive and linearly regular mappings.

Corollary 4.3. *Let $T : H \to H$ be a nonexpansive and linearly regular mapping with constant $\gamma > 0$. Let $\lambda \in]0,1[$ and $\{u^k\}$ be a sequence given by*

$$\begin{cases} u^0 \in H, \\ u^{k+1} = (1-\lambda)u^k + \lambda T(u^k) \quad \forall k \in \mathbb{N}. \end{cases} \tag{4.11}$$

Then $\{u^k\}$ converges linearly to some point $u^* \in \operatorname{Fix} T$ and

$$\|u^k - u^*\| \leq 2q^{k/2} d(u^0, \operatorname{Fix} T), \tag{4.12}$$

where

$$q = \max\left\{0, 1 - \frac{(1-\lambda)\lambda^2}{\gamma^2}\right\} \in]0,1[. \tag{4.13}$$

Proof. Consider VI(K, F) as given in Example 4.2. Then $\operatorname{Sol}(K,F) = \operatorname{Fix} T \neq \emptyset$ and VI(K,F) satisfies the error bound assumption with $\eta = \gamma$. Moreover, for every $u, v \in H$, by the Cauchy-Schwarz inequality and the nonexpansiveness of T, we have

$$\begin{aligned}\|F(u) - F(v)\|^2 &= \|(u - T(u)) - (v - T(v))\|^2 \\ &= \|u-v\|^2 - 2\langle T(u) - T(v), u-v\rangle + \|T(u) - T(v)\|^2 \\ &\leq \|u-v\|^2 - 2\langle T(u) - T(v), u-v\rangle + \|u-v\|^2 \\ &= 2\langle (u - T(u)) - (v - T(v)), u-v\rangle \\ &= 2\langle F(u) - F(v), u-v\rangle.\end{aligned}$$

Hence, F is co-coercive with the modulus $\mu = 1/2$ and $\lambda \in]0, 2\mu[$. The sequence $\{u^k\}$ defined by (4.11) is rewritten as

$$\begin{aligned} u^{k+1} &= (1-\lambda)u^k + \lambda T(u^k) \\ &= u^k - \lambda(u^k - T(u^k)) \\ &= u^k - \lambda F(u^k) \\ &= P_K(u^k - \lambda F(u^k)). \end{aligned}$$

Hence, $\{u^k\}$ is the sequence generated by Algorithm 3.1. Applying Theorem 4.1, we deduce that $\{u^k\}$ converges linearly to $u^* \in$VI(K,F)= Fix T and satisfies (4.12) with constant q given in (4.13). □

Remark 4.3. We can use the theory of averaged nonexpansive operators to deduce the linear convergence of the sequence $\{u^k\}$ defined by (4.11). Let $\lambda \in]0, 1[$ and $T : H \to H$ be nonexpansive and linearly regular. Consider the mapping $G : H \to H$ given by $G(u) = (1-\lambda)u + \lambda T(u)$ for every $u \in H$. Then, Fix(G)=Fix(T), and G is averaged nonexpansive and linearly regular. It follows from [21, Corollary 5.4] that the sequence $\{u^k\}$ defined by (4.11) converges linearly to some point $u^* \in$ Fix(T).

By using Theorem 4.1 and Example 4.3, we deduce a generalization of von-Neumann alternating projection method for finding the intersection of two closed half-spaces.

Corollary 4.4. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $c_1, c_2 \in H$ be such that $\|c_1\| = \|c_2\| = 1, \langle c_1, c_2\rangle \geq 0$. Let C_1, C_2 be two closed half-spaces in H given by*

$$C_1 := \{u \in H : \langle c_1, u\rangle \leq \alpha_1\}, \quad C_2 := \{u \in H : \langle c_2, u\rangle \leq \alpha_2\}.$$

Let $\lambda \in]0, 2[$ and $\{u^k\}$ be a sequence given by

$$\begin{cases} u^0 \in C_1, \\ u^{k+1} = P_{C_1}((1-\lambda)u^k + \lambda P_{C_2}(u^k)). \end{cases} \tag{4.14}$$

Then $\{u^k\}$ converges linearly to some point $u^* \in C_1 \cap C_2$ and satisfies

$$\|u^k - u^*\| \leq 2q^{k/2} d(u^0, C_1 \cap C_2), \tag{4.15}$$

where

$$q = 1 - \left(\frac{2-\lambda}{2}\right)(\min\{1, \lambda\})^2 \in]0, 1[. \tag{4.16}$$

Proof. Consider VI(K,F) as given in Example 4.3. Then Sol$(K,F) = C_1 \cap C_2 \neq \emptyset$ and VI(K,F) satisfied the error bound assumption with constant $\eta = 1$. Let $u, v \in K$, it follows from Theorem 2.1(a) that

$$\begin{aligned} \langle F(u) - F(v), u - v\rangle &= \langle u - P_{C_2}(u) - (v - P_{C_2}(v)), u - v\rangle \\ &= \|u - v\|^2 - \langle P_{C_2}(u) - P_{C_2}(v), u - v\rangle \\ &\geq \|u - v\|^2 - 2\langle P_{C_2}(u) - P_{C_2}(v), u - v\rangle + \|P_{C_2}(u) - P_{C_2}(v)\|^2 \\ &= \|(u - P_{C_2}(u)) - (v - P_{C_2}(v))\|^2 \\ &= \|F(u) - F(v)\|^2. \end{aligned}$$

The interested readers are refered to [24] for information on averaged nonexpansive operators.

Hence, F is co-coercive with the modulus $\mu = 1$. The sequence $\{u^k\}$ is rewritten as

$$\begin{aligned} u^{k+1} &= P_{C_1}((1-\lambda)u^k + \lambda P_{C_2}(u^k)) \\ &= P_K(u^k - \lambda(u^k - P_{C_2}(u^k))) \\ &= P_K(u^k - \lambda F(u^k))). \end{aligned}$$

Hence, $\{u^k\}$ is the sequence generated by Algorithm 3.1. Applying Theorem 4.1, we deduce that $\{u^k\}$ converges linearly to some point $u^* \in \mathrm{VI}(K,F) = C_1 \cap C_2$ and satisfies (4.15) with constant q given in (4.16). □

Remark 4.4. Choose the stepsize $\lambda = 1$. The sequence $\{u^k\}$ defined by (4.14) is given by

$$\begin{cases} u^0 \in C_1, \\ u^{k+1} = (P_{C_1} P_{C_2})(u^k). \end{cases}$$

It is indeed the iterative sequence generated by von-Neumann alternating projection method. Moreover, $\{u^k\}$ converges linearly to some point $u^* \in C_1 \cap C_2$ and satisfies

$$\|u^k - u^*\| \leq 2\left(\frac{1}{\sqrt{2}}\right)^k d(u^0, C_1 \cap C_2).$$

5. Numerical Illustrations

In this section, we present some numerical experiments to illustrate the linear convergence of the gradient projection method. Codes are implemented in MATLAB 2019b running on a Macbook Pro laptop with an Intel core CPU i7 at 2.6 GHz and 16 GB memory.

Example 5.1. As in [25], we consider the following VI with $F(u) = Mu + q$, where the matrix M randomly generated as: $M = A^T A + B + D$. Here every entry of the $n \times n$ matrix A and of the $n \times n$ skew-symmetric matrix B is uniformly generated from $(-2;2)$, and every diagonal entry of the $n \times n$ diagonal D is uniformly generated from (1; 3) (so M is positive defnite), with every entry of q uniformly generated from $(0;3)$. The feasible set K is the simplex

$$K = \{u \in \mathbb{R}^n_+, \sum_{i=1}^n u_i = n\}.$$

The projecttion onto K is computed by *quadprog* from Matlab. In this example, F is strongly monotone with modulus $\gamma = \lambda_{\min}$, the smallest eigen value of M and Lipschitz continuous with $L = \|M\|$. Hence, it follows from Example 4.1 that the error bound assumption (4.1) is satisfied. We choose $\lambda = \gamma/L^2 \in]0, 2\gamma/L^2[$ and let $\{u^k\}$ be the iterative sequence generated by Algorithm 3.1. It follows from Theorem 4.1 that the sequence $\{u^k\}$ converges linearly to the unique solution u^* of the problem $\mathrm{VI}(K,F)$. The performance of the projection gradient method is displayed in Figure 5.1.

Example 5.2. We consider the variational inequality in Example 4.4 with $H = \mathbb{R}^{1000}$ and various values of α and β. Since F_β is strongly pseudomonotone and Lipschitz continuous, it follows from Example 4.1 that the error bound assumption (4.1) holds. Since K_α is a ball, the projection onto K is given by the following explicit formula

$$P_{K_\alpha}(u) = \frac{\min\{\alpha, \|u\|\}}{\|u\|} u.$$

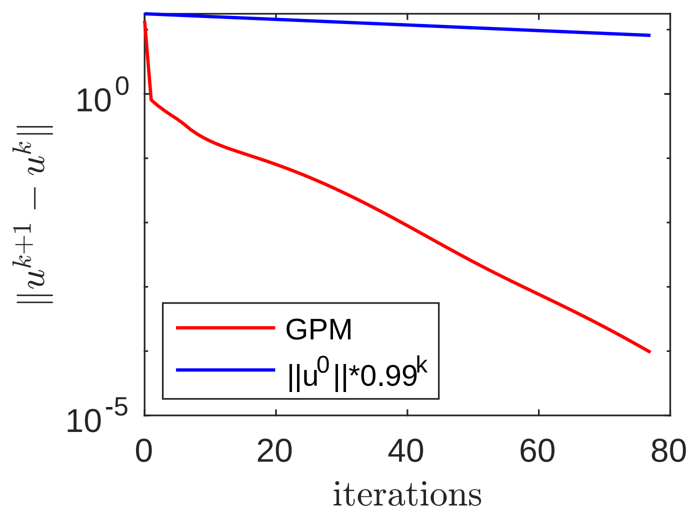

FIGURE 1. Performance of the Gradient Projection Method (GPM) for Example 5.1 with $n = 10$.

We choose $\lambda = 0.5 \in]0,2[$ and let $\{u^k\}$ be the iterative sequence generated by Algorithm 3.1. It follows from Theorem 4.1 that the sequence $\{u^k\}$ converges linearly to the unique solution $u^* = 0$ of the problem $\mathrm{VI}(K_\alpha, F_\beta)$. The performance of the projection gradient method is displayed in Figure 5.2.

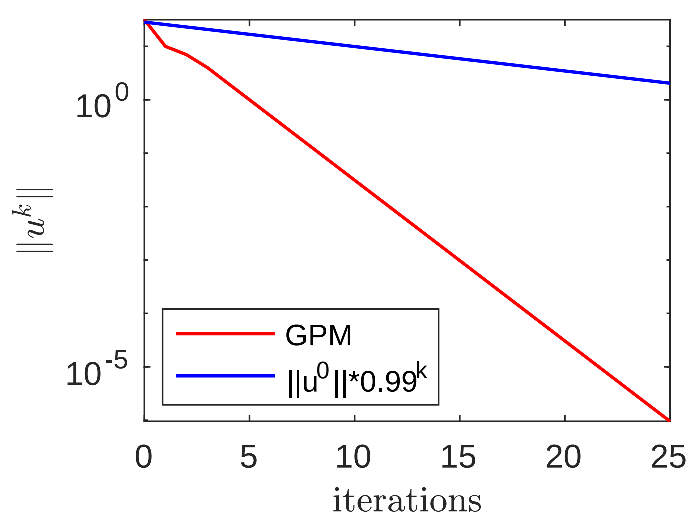

FIGURE 2. Performance of the Gradient Projection Method (GPM) for Example 5.2 with $\alpha = 10, \beta = 6$.

Acknowledgements
We are grateful to the Handing Editor and the referees for their constructive comments. The third author was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) project 101.01-2019.320.

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